Question 4

Intent of Question

The primary goals of this question were to assess a student's ability to (1) calculate a probability using basic probability rules or the geometric distribution; (2) recognize that a probability calculation for independent events does not depend on the previous outcomes of those events; and (3) assess whether a claim about the probability of a single event is reasonable based on a calculated probability of a series of those events.

Solution

Part (a):

If the failure rate for the super igniters is 15 percent, then the probability that each igniter fails is 0.15, and the probability that it does not fail is 0.85. Therefore the probability that the first 30 igniters tested do not fail is $(0.85)^{30} \approx 0.0076$. The solution can also be written as $(1 - 0.15)^{30} \approx 0.0076$.

Part (b):

Given that there are no failures in the first 30 trials, the probability that the first failure occurs on the 31^{st} trial is 0.15, and the probability that it does not occur on the 31^{st} but occurs on the 32^{nd} trial is $(0.85)(0.15) = 0.1275$. Therefore the probability that the first failure occurs on the 31^{st} or 32^{nd} super igniter tested is $0.15 + 0.1275 = 0.2775$.

Note that this is equivalent to asking for the probability that the first failure occurs on the first or second trial, which is $0.15 + (0.85)(0.15) = 0.2775$.

Part (c):

The result of the probability calculation in part (a) provides a reason to believe that the failure rate of the super igniters is less than 15 percent. The calculated probability of 0.0076 shows that there is less than a 1 percent chance that 30 or more igniters in a row would not fail if the failure rate was 15 percent. This probability is smaller than conventional significance levels such as $\alpha = 0.05$ or $\alpha = 0.01$, and thus is small enough to make it reasonable to believe that the failure rate of the super igniters is less than 15 percent.

Scoring

Parts (a), (b), and (c) are scored as essentially correct (E), partially correct (P), or incorrect (I).

Part (a) is scored as follows:

Essentially correct (E) if the response gives the correct probability *AND* correct justification.

Partially correct (P) if the response correctly notes that the answer is the probability that there will be 30 successes in 30 attempts, but does not carry out a correct probability calculation;
OR

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if the response defines the random variable X as the trial with the first failure, identifies X as having a geometric distribution with $p = 0.15$, and writes the desired probability as $P(X > 30)$, but does not carry out a correct probability calculation;

OR

if the response defines the random variable X as the number of failures in the first 30 attempts, identifies X as a binomial random variable with $p = 0.15$ and $n = 30$, and writes the desired probability as $P(X = 0)$, but does not carry out a correct probability calculation;

OR

if the response gives the correct probability but, in specifying a geometric or binomial distribution, has an incorrect or incomplete definition of parameters or value(s) of the random variable.

Incorrect (I) if the response does not meet the criteria for E or P.

Note: Justification can be given using the multiplication rule; *OR* by defining X to be the trial with the first failure, recognizing that X has a geometric distribution, and using that information to find $P(X > 30)$; *OR* by defining X to be the number of failures in the first 30 attempts, and then finding $P(X = 0)$ using either probability rules or the binomial distribution with $n = 30$ and $p = 0.15$.

Part (b) is scored as follows:

Essentially correct (E) if the response gives the correct probability AND correct justification.

Partially correct (P) if the response makes a reasonable attempt to calculate a geometric, binomial, or conditional probability, but does not successfully carry out the calculation;

OR

if the response gives the correct probability but, in specifying a geometric or binomial distribution, has an incorrect or incomplete definition of parameters or value(s) of the random variable.

Incorrect (I) if the response finds an incorrect probability resulting from an unreasonable attempt to calculate a geometric, binomial, or conditional probability or otherwise does not meet the criteria for E or P.

Note: Similar to part (a) justification can be given using probability rules; *OR* by stating that X is geometric where X is the trial with the first failure, then finding $P(X = 1 \text{ or } X = 2)$; *OR* by stating that X is the number of failures in two trials and finding $1 - P(X = 0)$ or $P(X = 1 \text{ or } X = 2)$ using the binomial distribution.

Part (c) is scored as follows:

Essentially correct (E) if the response states that it is reasonable to believe that the failure rate is less than 15 percent *AND* bases this decision on the fact that the probability of 30 consecutive successful launches with a failure rate of 15 percent (that is, answer from part (a)) is small *AND* does so in the context of the situation.

Partially correct (P) if the response otherwise satisfies the criteria for an (E) but does so without any context;

OR

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if the response states a significance level and makes a decision in a context that is appropriate to the given probability in part (a) and the stated significance level but does not explicitly compare the probability and the significance level (no linkage).

Incorrect (I) if the response does not explicitly make a decision about whether it is reasonable to conclude that the failure rate is less than 15 percent (For example: "As seen in Part (a), if the failure rate is 15 percent then the probability of 30 successful launches in a row is very small.");

OR

if the response otherwise does not meet the criteria for E or P.

Notes:

- Justification based on the probability can come by stating a significance level and noting that the probability is smaller than the significance level *OR* by simply stating that the probability of 0.0076 is small *OR* by referring to the expected number of failures (4.5) as being very unlikely because zero failures is more than two standard deviations below 4.5.
- If the response bases the decision on the expected number of failures (4.5) for $n = 30$ and $p = 0.15$ without referencing why zero failures would be considered to be too far below 4.5 to give reason to doubt the stated 15 percent failure rate, the response is scored P.
- If the calculation in part (a) is incorrect, the answer in part (c) needs to be consistent with the answer in part (a), unless the value is recalculated in part (c).

4 Complete Response

Three parts essentially correct

3 Substantial Response

Two parts essentially correct and one part partially correct

2 Developing Response

Two parts essentially correct and no parts partially correct

OR

One part essentially correct and one or two parts partially correct

OR

Three parts partially correct

1 Minimal Response

One part essentially correct

OR

No parts essentially correct and two parts partially correct

4. A company manufactures model rockets that require igniters to launch. Once an igniter is used to launch a rocket, the igniter cannot be reused. Sometimes an igniter fails to operate correctly, and the rocket does not launch. The company estimates that the overall failure rate, defined as the percent of all igniters that fail to operate correctly, is 15 percent.

A company engineer develops a new igniter, called the super igniter, with the intent of lowering the failure rate. To test the performance of the super igniters, the engineer uses the following process.

Step 1: One super igniter is selected at random and used in a rocket.

Step 2: If the rocket launches, another super igniter is selected at random and used in a rocket.

Step 2 is repeated until the process stops. The process stops when a super igniter fails to operate correctly or 32 super igniters have successfully launched rockets, whichever comes first. Assume that super igniter failures are independent.

(a) If the failure rate of the super igniters is 15 percent, what is the probability that the first 30 super igniters selected using the testing process successfully launch rockets?

$p(\text{first 30 are successful}) = [p(\text{success})]^{30} = 0.85^{30} = \boxed{0.0076}$

(b) Given that the first 30 super igniters successfully launch rockets, what is the probability that the first failure occurs on the thirty-first or the thirty-second super igniter tested if the failure rate of the super igniters is 15 percent?

Failures are independent, so $p(\text{failure} \mid 30\text{ success}) = p(\text{failure})$

$p(31^{st} \text{ or } 32^{nd} \text{ fails}) = p(\text{failure}) + p(\text{success, then failure})$

$= 0.15 + (0.85)(0.15)$

$= \boxed{0.2775}$

(c) Given that the first 30 super igniters successfully launch rockets, is it reasonable to believe that the failure rate of the super igniters is less than 15 percent? Explain.

Yes. If the failure rate is actually 15 percent, there is a 0.0076 probability (because $.85^{30} = 0.0076$) of 30 launches in a row being successful, which is extremely unlikely. 0.0076 is less than our typical α value of 0.05, so we can conclude at the $\alpha = 0.05$ level that if 30 super igniters successfully launch rockets, the failure rate of super igniters is less than 15 percent.

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4. A company manufactures model rockets that require igniters to launch. Once an igniter is used to launch a rocket, the igniter cannot be reused. Sometimes an igniter fails to operate correctly, and the rocket does not launch. The company estimates that the overall failure rate, defined as the percent of all igniters that fail to operate correctly, is 15 percent. p=.15

A company engineer develops a new igniter, called the super igniter, with the intent of lowering the failure rate. To test the performance of the super igniters, the engineer uses the following process.

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Step 2 is repeated until the process stops. The process stops when a super igniter fails to operate correctly or 32 super igniters have successfully launched rockets, whichever comes first. Assume that super igniter failures are independent.

(a) If the failure rate of the super igniters is 15 percent, what is the probability that the first 30 super igniters selected using the testing process successfully launch rockets? p=.15 ; x=30

~~geometric~~ ~~geometcdf(.85, 30)~~

1-geometcdf(.15,30)

=.00763

(b) Given that the first 30 super igniters successfully launch rockets, what is the probability that the first failure occurs on the thirty-first or the thirty-second super igniter tested if the failure rate of the super igniters is 15 percent?

geometpdf + geometpdf
p: .15 p: .15
x: 31 x: 32

.0011 + .00097 = .00212

(c) Given that the first 30 super igniters successfully launch rockets, is it reasonable to believe that the failure rate of the super igniters is less than 15 percent? Explain.

Yes, there is only a .00763 probability of the first 30 rockets launching successfully, and since this chance is so small, it is highly likely that the failure rate is much lower than .15 because it is unlikely we obtained these results by chance.

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4. A company manufactures model rockets that require igniters to launch. Once an igniter is used to launch a rocket, the igniter cannot be reused. Sometimes an igniter fails to operate correctly, and the rocket does not launch. The company estimates that the overall failure rate, defined as the percent of all igniters that fail to operate correctly, is 15 percent. .15

A company engineer develops a new igniter, called the super igniter, with the intent of lowering the failure rate. To test the performance of the super igniters, the engineer uses the following process.

Step 1: One super igniter is selected at random and used in a rocket.

Step 2: If the rocket launches, another super igniter is selected at random and used in a rocket.

Step 2 is repeated until the process stops. The process stops when a super igniter fails to operate correctly or 32 super igniters have successfully launched rockets, whichever comes first. Assume that super igniter failures are independent.

(a) If the failure rate of the super igniters is 15 percent, what is the probability that the first 30 super igniters selected using the testing process successfully launch rockets?

$1 - .15 = .85$ P(successful launch) = .85

$.85^{30} = .0076$

(b) Given that the first 30 super igniters successfully launch rockets, what is the probability that the first failure occurs on the thirty-first or the thirty-second super igniter tested if the failure rate of the super igniters is 15 percent?

P(fail on 31 or 32 | 30 successful launches) = **.1671**

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.00127}{.0076}$

$.85^{30}(.15) = .0011$

$.85^{30}(.15^2) = .00017$

$.0011 + .00017 = .00127$

(c) Given that the first 30 super igniters successfully launch rockets, is it reasonable to believe that the failure rate of the super igniters is less than 15 percent? Explain.

No because each launch is independent; the success of the previous launch has no effect on the next launch

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Question 4

Overview

The primary goals of this question were to assess a student's ability to (1) calculate a probability using basic probability rules or the geometric distribution; (2) recognize that a probability calculation for independent events does not depend on the previous outcomes of those events; and (3) assess whether a claim about the probability of a single event is reasonable based on a calculated probability of a series of those events.

Sample: 4A
Score: 4

In part (a) the response uses probability rules to demonstrate an understanding of statistical independence by indicating that the probability that the first 30 trials are successful is equal to the probability of success on a single trial raised to the power of 30. The response completes the justification with the expression 0.85^{30} and reports the correct probability. Because the correct probability is reported and correct justification is provided, part (a) was scored as essentially correct. In part (b) the response again demonstrates an understanding of statistical independence with the conditional probability statement "$P(\text{failure} \mid 30 \text{ successes}) = P(\text{failure})$." The statement communicates an understanding that a probability calculation for independent events does not depend on the previous outcomes of those events. The response completes the justification with the correct probability calculation and reports the correct probability. Because the correct probability is reported and correct justification is provided, part (b) was scored as essentially correct. In part (c) the response correctly answers yes and gives a clear explanation that the decision is based on the fact that if the failure rate is 15 percent, the probability of 30 consecutive successful launches is small, or "extremely unlikely." Additionally, the response quotes the probability from part (a) and correctly compares that probability to a 0.05 level of significance to further justify the decision. Context is provided with the inclusion of "igniters," "launches," and "rockets" in the response. Because the correct decision is given in context and is based on the fact that the probability from part (a) is small, part (c) was scored as essentially correct. Because all three parts were scored as essentially correct, the response earned a score of 4.

Sample: 4B
Score: 3

In part (a) the response demonstrates an understanding of the testing process described in the statement of the question with the use of the geometric distribution. The parameter value and value of the geometric random variable are given, and the geometric cumulative distribution function is used to calculate the probability that the first 30 randomly selected super igniters successful launch rockets. Because the correct probability is reported and correct justification is provided, part (a) was scored as essentially correct. In part (b) the response again uses the geometric distribution and specifies the parameter value and values of the random variables. The probability that the first failure occurs on the 31st trial and the probability that the first failure occurs on the 32nd trial are calculated and added. However, the response does not account for the given condition that the first 30 igniters successfully launched rockets. Although incorrect, this result is accepted as a reasonable attempt to calculate the conditional probability. Because a reasonable, but unsuccessful, attempt to calculate the correct probability is given, part (b) was scored as partially correct. In part (c) the response correctly answers yes, quotes the probability from part (a), and gives a clear explanation that the decision is based on the fact that the "probability of the first 30 rockets launching successfully" is "so small." Context is provided with the inclusion of "rockets launching" in the response. Because the correct decision is given in context and is based on the fact that the probability from part (a) is small, part (c) was

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scored as essentially correct. Because two parts were scored as essentially correct, and one part was scored as partially correct, the response earned a score of 3.

Sample: 4C
Score: 2

In part (a) the response indicates that the probability of a successful launch is 0.85 and uses probability rules for independent events to calculate the correct probability. Because the correct probability is reported and correct justification is given, part (a) was scored as essentially correct. In part (b) the response uses the conditional probability formula to compute the answer. The probability from part (a) is correctly used as the denominator. However, the numerator is a reasonable, but incorrect, attempt to calculate the sum of the probabilities that the first failure occurs on the 31^{st} or 32^{nd} trial. Because a reasonable, but unsuccessful, attempt to calculate the correct probability is given, part (b) was scored as partially correct. In part (c) the response incorrectly answers no. The incorrect decision is based on not understanding that a claim about the probability of a single event can be based on a calculated probability of a series of those events. Because the incorrect answer is given, part (c) was scored as incorrect. Because one part was scored as essentially correct, one part was scored as partially correct, and one part was scored as incorrect, the response earned a score of 2.