Question 5

Consider the function $f(x) = \dfrac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by

$$f'(x) = \frac{k - 2x}{\left(x^2 - kx\right)^2}.$$

(a) Let $k = 3$, so that $f(x) = \dfrac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

(b) Let $k = 4$, so that $f(x) = \dfrac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

(c) Find the value of k for which f has a critical point at $x = -5$.

(d) Let $k = 6$, so that $f(x) = \dfrac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f. Find $\int f(x)\,dx$.

(a) $f(4) = \dfrac{1}{4^2 - 3 \cdot 4} = \dfrac{1}{4}$ $\qquad f'(4) = \dfrac{3 - 2 \cdot 4}{\left(4^2 - 3 \cdot 4\right)^2} = -\dfrac{5}{16}$

An equation for the line tangent to the graph of f at the point whose x-coordinate is 4 is $y = -\dfrac{5}{16}(x - 4) + \dfrac{1}{4}$.

$2 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line equation} \end{cases}$

(b) $f'(x) = \dfrac{4 - 2x}{\left(x^2 - 4x\right)^2}$ $\qquad f'(2) = \dfrac{4 - 2 \cdot 2}{\left(2^2 - 4 \cdot 2\right)^2} = 0$

$f'(x)$ changes sign from positive to negative at $x = 2$.
Therefore, f has a relative maximum at $x = 2$.

$2 : \begin{cases} 1 : \text{considers } f'(2) \\ 1 : \text{answer with justification} \end{cases}$

(c) $f'(-5) = \dfrac{k - 2 \cdot (-5)}{\left((-5)^2 - k \cdot (-5)\right)^2} = 0 \Rightarrow k = -10$

1 : answer

(d) $\dfrac{1}{x^2 - 6x} = \dfrac{1}{x(x - 6)} = \dfrac{A}{x} + \dfrac{B}{x - 6} \Rightarrow 1 = A(x - 6) + Bx$

$x = 0 \Rightarrow 1 = A \cdot (-6) \Rightarrow A = -\dfrac{1}{6}$

$x = 6 \Rightarrow 1 = B \cdot (6) \Rightarrow B = \dfrac{1}{6}$

$\dfrac{1}{x(x - 6)} = \dfrac{-1/6}{x} + \dfrac{1/6}{x - 6}$

$$\int f(x)\,dx = \int \left(\frac{-1/6}{x} + \frac{1/6}{x - 6} \right) dx$$
$$= -\frac{1}{6}\ln|x| + \frac{1}{6}\ln|x - 6| + C = \frac{1}{6}\ln\left|\frac{x - 6}{x}\right| + C$$

$4 : \begin{cases} 2 : \text{partial fraction decomposition} \\ 2 : \text{general antiderivative} \end{cases}$

5A

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NO CALCULATOR ALLOWED

5. Consider the function $f(x) = \frac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by $f'(x) = \frac{k - 2x}{\left(x^2 - kx\right)^2}$.

(a) Let $k = 3$, so that $f(x) = \frac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

$$f'(4) = \frac{3-2(4)}{((4)^2-3(4))^2} = \frac{3-8}{(16-12)^2} = \frac{-5}{(4)^2} = -\frac{5}{16}$$

$$f(4) = \frac{1}{(4)^2-3(4)} = \frac{1}{16-12} = \frac{1}{4}$$

$$y - \frac{1}{4} = -\frac{5}{16}(x-4)$$

$$y = -\frac{5}{16}(x-4) + \frac{1}{4}$$

(b) Let $k = 4$, so that $f(x) = \frac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

$$f'(2) = \frac{4-2(2)}{(4-4(2))^2} = \frac{0}{16} = 0$$

By 1st Derivative test,
there is a relative maximum
@ $x = 2$ because the derivative
value changes from positive to negative.

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NO CALCULATOR ALLOWED

(c) Find the value of k for which f has a critical point at $x = -5$.

$$f'(5) = 0 = \frac{k - 2(-5)}{(25 - k(-5))^2}$$

$$0 = \frac{k+10}{(25+5k)^2}$$

$$k = -10$$

(d) Let $k = 6$, so that $f(x) = \dfrac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f.

Find $\int f(x)\,dx$.

$$f(x) = \frac{1}{x(x-6)}$$

$$\int f(x)dx = \int \frac{dx}{x(x-6)} = \int \frac{A}{x}dx + \int \frac{B\,dx}{x-6}$$

$1 = A(x-6) + Bx$ let $x = 0$

$\hookrightarrow 1 = -6A$ $A = -\frac{1}{6}$

let $x = 6$

$\hookrightarrow 1 = 6B$ $B = \frac{1}{6}$

$$\int f(x)dx = -\frac{1}{6}\int \frac{dx}{x} + \frac{1}{6}\int \frac{dx}{x-6}$$

$$\int f(x)dx = -\frac{1}{6}\ln|x| + \frac{1}{6}\ln|x-6| + C$$

$$\int f(x)dx = \frac{1}{6}(\ln|x-6| - \ln|x|) + C$$

$$\int f(x)dx = \frac{1}{6}\ln\left|\frac{x-6}{x}\right| + C$$

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5B
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5. Consider the function $f(x) = \dfrac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by

$$f'(x) = \frac{k - 2x}{\left(x^2 - kx\right)^2}.$$

(a) Let $k = 3$, so that $f(x) = \dfrac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

$$f'(x) = \frac{3-2x}{(x^2-3x)^2}$$

$x = 4$,

$$f(4) = \frac{1}{16-12} = \frac{1}{4}$$

$$f'(4) = \frac{3-8}{(16-12)^2} = \frac{5}{16}$$

$\therefore\ y - \frac{1}{4} = \frac{5}{16}(x-4)$

$y = \frac{5}{16}x - 1$

(b) Let $k = 4$, so that $f(x) = \dfrac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

$$f'(x) = \frac{4-2x}{(x^2-4x)^2}$$

$f'(2) = 0$

$$f''(x) = \frac{-2(x^2-4x)^2 - (4-2x)\cdot(x^2-4x)\cdot(2x-4)}{(x^2-4x)^4}$$

$f''(2) = -\frac{1}{2} < 0$

$\therefore$ f has a relative maximum at $x=2$

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(c) Find the value of k for which f has a critical point at $x = -5$.

$f'(5) = 0$

$f'(5) = \dfrac{k-10}{(25-5k)^2} = 0$

$k = 10$

(d) Let $k = 6$, so that $f(x) = \dfrac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f.

Find $\int f(x)\,dx$.

$\int f(x)\,dx = \int \dfrac{1}{x^2-6x}\,dx$

$\dfrac{1}{x^2-6x} = \dfrac{1}{x(x-6)}$

$= \dfrac{A}{x} + \dfrac{B}{x-6}$

$\therefore A(x-6) + Bx = 1$

$\begin{cases} A+B=0 \\ -6A = 1 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{6} \\ B = \frac{1}{6} \end{cases}$

$\int f(x)\,dx = -\frac{1}{6}\int \frac{1}{x}\,dx + \frac{1}{6}\int \frac{1}{x-6}\,dx$

$= -\frac{1}{6}\ln|x| + \frac{1}{6}\ln|x-6| + C$

$= \frac{1}{6}\ln\left|\frac{x-6}{x}\right| + C$

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5. Consider the function $f(x) = \dfrac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by

$$f'(x) = \frac{k - 2x}{\left(x^2 - kx\right)^2}.$$

(a) Let $k = 3$, so that $f(x) = \dfrac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

$f(4) = \dfrac{1}{16 - 3(4)} = \dfrac{1}{16-12} = \dfrac{1}{4}$

$k=3$ $\quad x=4 \quad y=\dfrac{1}{4}$

$\dfrac{3-2(4)}{(4^2-3(4))^2} = \dfrac{-5}{(16-12)^2}$

$$\boxed{y - \frac{1}{4} = -\frac{5}{16}(x-4)}$$

$\dfrac{-5}{4^2} = -\dfrac{5}{16}$

(b) Let $k = 4$, so that $f(x) = \dfrac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

$4-6$

$+ \quad | \quad -$
2

$f'(x) = \dfrac{4-2x}{(x^2-4(2))^2} = \dfrac{4-2x}{(x^2-8)^2}$

$2-8$

$\dfrac{4-2(2)}{(4-8)^2} = \dfrac{0}{(-4)^2} = \dfrac{0}{16} = 0$

maximum at $x=2$ because the graph increases up to $x=2$ and then decreases.

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(c) Find the value of k for which f has a critical point at $x = -5$.

$$f'(-5) = \frac{k-2x}{(x^2-kx)^2} = 0$$

$$\left(\frac{k-2(-5)}{25+5k}\right)^2$$

$$\frac{k+10}{(25+5k)^2} = 0$$

$125 + 25k^2$

$k+10\ (125^{-1}) +$

(d) Let $k = 6$, so that $f(x) = \dfrac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f.

Find $\int f(x)\,dx$.

$$\int f(x)\,dx$$

$$\int \frac{1}{x^2-6x}\,dx$$

$$\int (x^2-6x)^{-1}\,dx$$

$$u^{-1} \cdot \frac{1}{2x-6}\,du$$

$$u^0\ \frac{1}{2x-6} + C$$

$$\boxed{\frac{1}{2x-6} + C}$$

$u = x^2 - 6x$

$du = 2x - 6\,dx$

$dx = \dfrac{1}{2x-6}\,du$

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Question 5

Overview

In this problem students were given $f(x) = \dfrac{1}{x^2 - kx}$ and $f'(x) = \dfrac{k - 2x}{\left(x^2 - kx\right)^2}$, where the parameter, k, is a nonzero constant. In part (a) for $k = 3$, students were asked to write an equation for the line tangent to the graph of f at the point with $x = 4$. Students needed to compute $f(4)$ and $f'(4)$ and then use those values to produce an equation. In part (b) for $k = 4$, students needed to determine whether f had a relative minimum, a relative maximum, or neither at $x = 2$. Students were expected to confirm that $f'(2) = 0$ and apply the First Derivative Test. Since f' changes sign from positive to negative at $x = 2$, students should have concluded that f has a relative maximum at $x = 2$. In part (c) students had to find the value of k for which f has a critical point at $x = -5$. Students were expected to solve $f'(-5) = 0$ to determine that $k = -10$. In part (d) students were expected to use partial fraction decomposition to rewrite $f(x)$ as a sum of rational expressions. The result is used to find $\int f(x)\,dx$. The partial fraction decomposition yields $\dfrac{1}{x(x-6)} = \dfrac{A}{x} + \dfrac{B}{x-6} = \dfrac{-1/6}{x} + \dfrac{1/6}{x-6}$, and the general antiderivative is $-\dfrac{1}{6}\ln|x| + \dfrac{1}{6}\ln|x-6| + C$.

Sample: 5A
Score: 9

The response earned all 9 points.

Sample: 5B
Score: 6

The response earned 6 points: 1 point in part (a), 1 point in part (b), no points in part (c), and 4 points in part (d). In part (a) the student uses f' to determine the slope but makes an arithmetic error, so the student did not earn the first point. The student uses the slope to present a line that passes through the given point, so the second point was earned. In part (b) the student attempts to use the Second Derivative Test. The student shows that $f'(2) = 0$, so the first point was earned. The student makes an error in the computation of $f''(2)$, so the second point was not earned. In part (c) the student considers $x = 5$ instead of $x = -5$. In part (d) the student's work is correct.

Sample: 5C
Score: 3

The response earned 3 points: 2 points in part (a), 1 point in part (b), no points in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student earned the first point. Because the student's justification does not refer to the sign of the first derivative of f, the student did not earn the second point. In part (c) the student does not report a value for k. In part (d) the student does not find a partial fraction decomposition, and the antiderivative is incorrect.