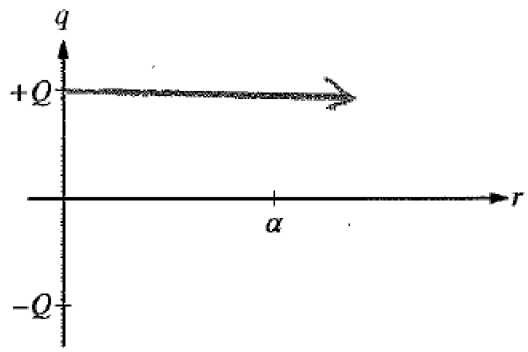
Question 3

15 points total | **Distribution of points**

(a)

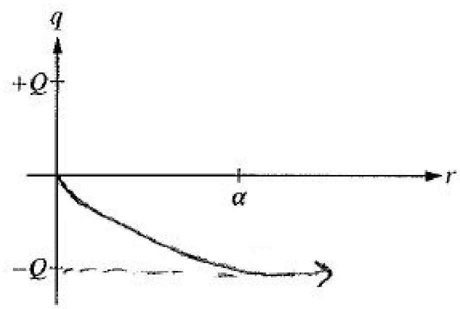
i. 2 points



For a graph that starts at $q = +Q$ — 1 point

For a horizontal line (at $q = +Q$) — 1 point

ii. 3 points

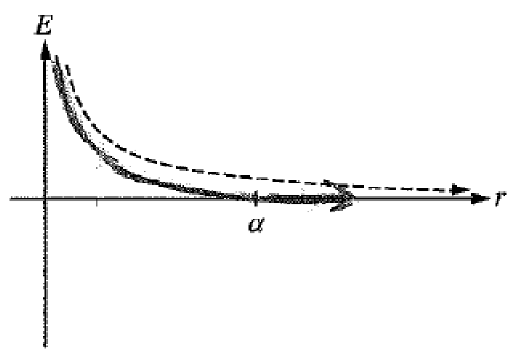


For a graph that starts at $q = 0$ for $r = 0$ — 1 point

For a concave upward curve in the 4th quadrant between $r = 0$ and $r = \alpha$ — 1 point

For a graph that approaches $-Q$ at $r = \alpha$ and equals $-Q$ beyond that point — 1 point

(b) 2 points



For a graph that is decreasing and between the dashed curve and the x-axis — 1 point

For a graph that goes to zero at $r = \alpha$ and is zero beyond that point — 1 point

Question 3 (continued)

	Distribution of points
(c)	
i. 3 points	
For stating and using Gauss's law in any form	1 point
$\oint E \bullet dA = \frac{q_{enc}}{\varepsilon_0}$	
For indicating that $q_{enc} = 0$	1 point
For correctly stating that the electric field is zero	1 point
ii. 4 points	
For indicating the need to integrate with respect to volume to find the negative charge enclosed	1 point
$q_{neg} = \int \rho(r)\,dV$	
Substitute and integrate with appropriate limits	
$q_{neg} = \int_0^r -\frac{\beta}{r^2} e^{-r/\alpha}\left(4\pi r^2\right)dr = 4\pi\beta \int_0^r -e^{-r/\alpha}\,dr$	
$q_{neg} = -4\pi\beta\alpha\left[-e^{-r/\alpha}\right]_0^r$	
For a correct expression for negative charge as a function of distance r	1 point
$q_{neg} = -4\pi\beta\alpha\left(1 - e^{-r/\alpha}\right)$ or $q_{neg} = 4\pi\beta\alpha\left(e^{-r/\alpha} - 1\right)$	
For including the $+Q$ when substituting for q_{enc}	1 point
For correct substitution for the surface area of a sphere	1 point
$E\left(4\pi r^2\right) = \frac{q_{enc}}{\varepsilon_0} = \frac{Q - 4\pi\beta\alpha\left(1 - e^{-r/\alpha}\right)}{\varepsilon_0}$	
$E = \frac{1}{4\pi\varepsilon_0 r^2}\left[Q - 4\pi\beta\alpha\left(1 - e^{-r/\alpha}\right)\right]$	
(d) 1 point	
For correctly stating or implying that α is the radius of the atom or the radius of the electron cloud	1 point

E&M.3.

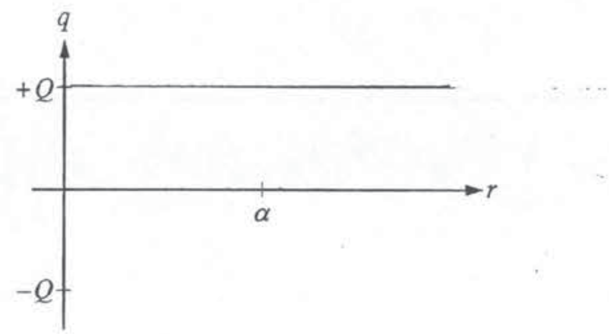
A scientist describes an electrically neutral atom with a model that consists of a nucleus that is a point particle with positive charge $+Q$ at the center of the atom and an electron volume charge density of the form

$$\rho(r) = \begin{cases} -\dfrac{\beta}{r^2} e^{-r/\alpha} & r < \alpha \\ 0 & r > \alpha \end{cases}$$

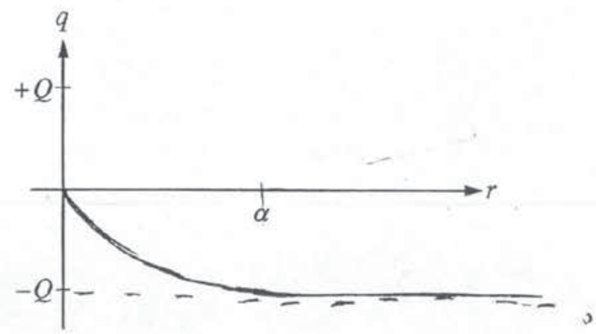
where α and β are positive constants and r is the distance from the center of the atom.

(a) On the axes below, let r stand for the radius of a Gaussian sphere. Sketch the graph for each of the following charges enclosed by the Gaussian sphere as a function of r. Explicitly label any intercepts, asymptotes, maxima, or minima with numerical values or algebraic expressions, as appropriate.

i. The nuclear charge only



ii. The electron charge only



$Q_{enc} = \int \rho \, dV = 4\pi \int \rho \, r^2 \, dr$

$dV = 4\pi r^2 dr$

$4\pi \int -\frac{b}{r^2} e^{-r/\alpha} r^2 \, dr$

$-b 4\pi \int_0^r e^{-r/\alpha} \, dr$

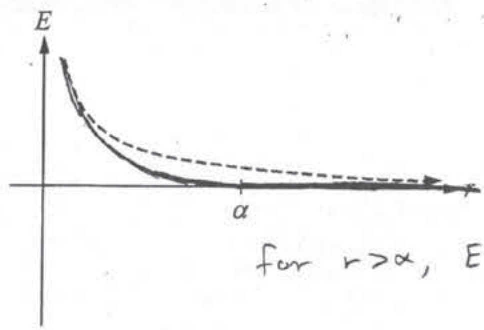
$\left(-\alpha e^{-r/\alpha}\right)\Big|_0$ $e^{-r/\alpha} \cdot \frac{1}{\alpha}$

$-b 4\pi\left(-\alpha e^{-r/\alpha} + 1\right)$

$-b 4\pi\left(1 - \alpha e^{-r/\alpha}\right)$

GO ON TO THE NEXT PAGE.

(b) The dashed curve on the graph below represents the electric field as a function of distance r due to the positive nucleus of the atom without any electrons. The nucleus is modeled as a point particle of charge $+Q$. On the same graph, sketch the electric field as a function of distance r for the neutral atom as defined by the scientist's model, which includes the nucleus and the negative electrons surrounding it.



for $r > \alpha$, $E = 0$

(c) Use Gauss's law to derive an expression for the electric field strength due to the neutral atom for the following positions in terms of $Q, \alpha, \beta, r,$ and fundamental constants, as appropriate.

i. $r > \alpha$

For $r > \alpha$ $\quad Q_{enc} = +Q - (Q) = 0$ (Because $r > \alpha$ is outside of the electrically neutral atom, so $Q_{enc} = 0$)

$$\int \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} = 0 \quad \boxed{E = 0}$$

ii. $r < \alpha$

For $r < \alpha$ $\quad Q_{enc} = Q - 4\pi\beta\alpha(1 - e^{-r/\alpha})$

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

$$E(4\pi r^2) = \frac{Q - 4\pi\beta\alpha(1 - e^{-r/\alpha})}{\epsilon_0}$$

$$\boxed{E = \frac{Q - 4\pi\beta\alpha(1 - e^{-r/\alpha})}{4\pi\epsilon_0 r^2}}$$

$$(\text{cloud}) \Rightarrow Q_{enc} = \int_0^r \rho\, dV = 4\pi \int_0^r \rho r^2 dr = 4\pi \int_0^r -\beta e^{-r/\alpha} dr \qquad dV = 4\pi r^2 dr$$

$$-4\pi\beta \int_0^r e^{-r/\alpha} dr$$

$$-4\pi\beta \left(-\alpha e^{-r/\alpha}\right)_0^r = -4\pi\beta\alpha(-e^{-r/\alpha} + 1) = -4\pi\beta\alpha(1 - e^{-r/\alpha})$$

(d) Based on the model proposed by the scientist, what is the physical meaning of the constant α ?

α is the edge of the electron cloud, after which the charge density is 0.

GO ON TO THE NEXT PAGE.

E&M.3.

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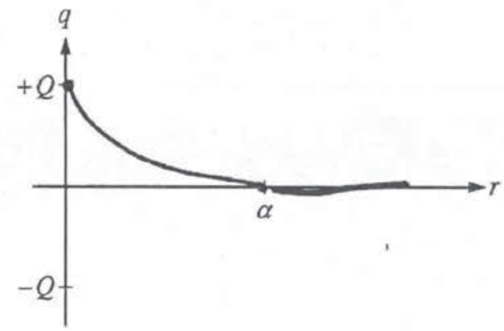

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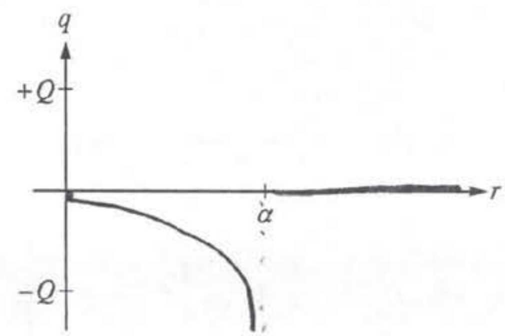

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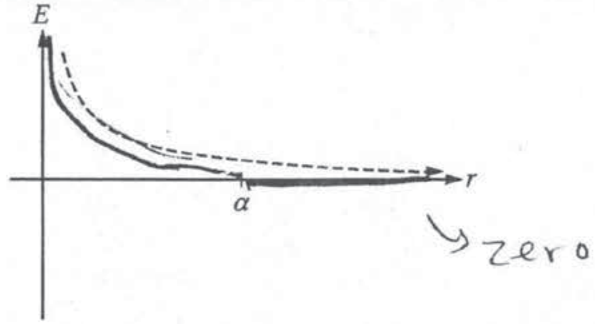
(b) The dashed curve on the graph below represents the electric field as a function of distance r due to the positive nucleus of the atom without any electrons. The nucleus is modeled as a point particle of charge $+Q$ On the same graph, sketch the electric field as a function of distance r for the neutral atom as defined by the scientist's model, which includes the nucleus and the negative electrons surrounding it.



(c) Use Gauss's law to derive an expression for the electric field strength due to the neutral atom for the following positions in terms of $Q, \alpha, \beta, r,$ and fundamental constants, as appropriate.

i. $r > \alpha$ $\quad \int E \cdot dA = \frac{Q_{enc}}{\varepsilon_0}$

$Q_{enc} = \rho V = 0$

$\int E \cdot dA = 0 \Rightarrow \boxed{E = 0 \text{ N/C}}$

ii. $r < \alpha$

$\int E \cdot dA = \frac{Q_{enc}}{\varepsilon_0}$

$Q_{enc} = \rho(r) V = \left(-\frac{\beta}{\cancel{r}} e^{-r/a}\right)\left(\frac{4}{3}\pi r^{\cancel{3}}\right) = -\frac{\beta}{3}\pi r^2 e^{-r/\alpha}$

$E(4\cancel{\pi} r^{\cancel{2}}) = \left(\frac{1}{\varepsilon_0}\right)\left(-\frac{\beta}{3}\cancel{\pi}\cancel{r} e^{-r/\alpha}\right)$

$4E = \frac{-\beta e^{-r/\alpha}}{3\varepsilon_0 r}$

$\boxed{E = \frac{-\beta e^{-r/\alpha}}{12\varepsilon_0 r}}$

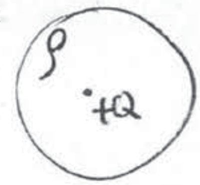
(d) Based on the model proposed by the scientist, what is the physical meaning of the constant α ?

α is the radius of the atom.

E&M.3.

A scientist describes an electrically neutral atom with a model that consists of a nucleus that is a point particle with positive charge $+Q$ at the center of the atom and an electron volume charge density of the form

$$\rho(r) = \begin{cases} -\dfrac{\beta}{r^2} e^{-r/\alpha} & r < \alpha \\ 0 & r > \alpha \end{cases}$$



where α and β are positive constants and r is the distance from the center of the atom.

(a) On the axes below, let r stand for the radius of a Gaussian sphere. Sketch the graph for each of the following charges enclosed by the Gaussian sphere as a function of r. Explicitly label any intercepts, asymptotes, maxima, or minima with numerical values or algebraic expressions, as appropriate.

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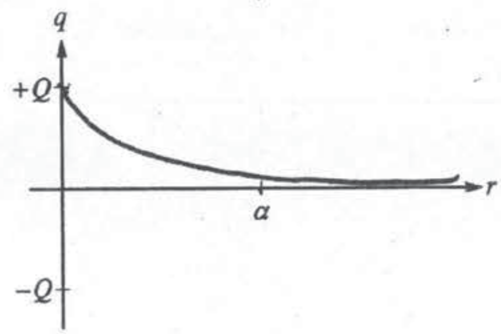
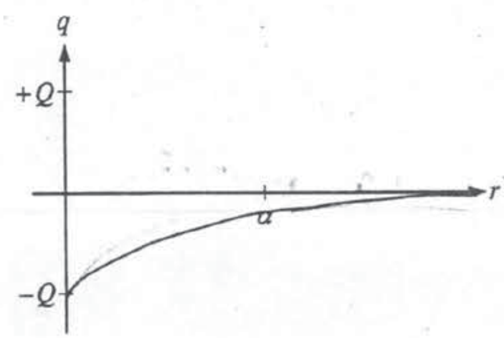


ii. The electron charge only



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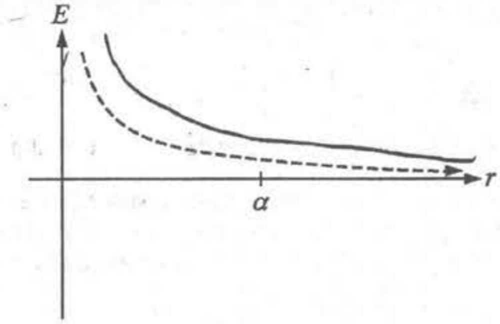

(c) Use Gauss's law to derive an expression for the electric field strength due to the neutral atom for the following positions in terms of Q, α, β, r, and fundamental constants, as appropriate.

i. $r > \alpha$

$E = \frac{q}{\varepsilon_0 (4\pi r^2)}$ $\quad q = 0 \quad$ $\boxed{E\text{ field} = 0}$

ii. $r < \alpha$

$E = \frac{q}{\varepsilon_0 (4\pi r^2)}$ $\quad q = \rho \frac{4\pi r^3}{3} = \frac{-\beta}{\cancel{r^2}} e^{\frac{-r}{\alpha}} \frac{4\pi r^{\cancel{3}}}{3}$

$E = \frac{-4\pi r \beta e^{\frac{-r}{\alpha}}}{3\varepsilon_0 \cancel{4\pi} r^2} = \boxed{\frac{-\beta e^{-\frac{r}{\alpha}}}{3\varepsilon_0 r} = E}$

(d) Based on the model proposed by the scientist, what is the physical meaning of the constant α ?

Radius of the electron (cloud) volume.

Question 3

Overview

This question dealt with Gauss' law. It was not at all like a standard problem with an insulating or conducting sphere. Instead the students were told that they had a nucleus with a positive charge surrounded by a negative charge distributed nonuniformly. Students had to sketch graphs of positive and negative charge and electric field as a function of r. Then they were told to use Gauss' law to solve for the electric field outside and inside the boundary of the charge at $r = \alpha$.

Sample: E3 A
Score: 15

This response earned full credit for all parts. The work in part (a)(ii) to determine the negative charge enclosed within α was performed completely and correctly in part (c)(ii).

Sample: E3 B
Score: 9

Part (a)(i) earned 1 point for starting the graph at $+Q$. Part (a)(ii) also earned 1 point for starting the graph at zero. Parts (b) and (c)(i) earned full credit. Part (c)(ii) earned 1 point for recognizing that the Gaussian surface was a sphere. However, there was no integration and no recognition that both positive and negative charge must be included. Part (d) earned 1 point.

Sample: E3 C
Score: 6

Part (a)(i) earned 1 point for starting the graph at $+Q$, and part (a)(ii) earned no credit. Part (b) also earned nothing since the curve is always above the dashed line. Part (c)(i) earned full credit. The solution in part (c)(ii) begins with Gauss' Law for a sphere, earning 1 point, but does not correctly determine the negative charge enclosed or recognize that $+Q$ was also enclosed. Part (d) earned 1 point.