Question 3

15 points total

(a) 2 points

i. 1 point
For a graph that starts at $q = +Q$
For a horizontal line (at $q = +Q$)

ii. 3 points
For a graph that starts at $q = 0$ for $r = 0$
For a concave upward curve in the 4th quadrant between $r = 0$ and $r = \alpha$
For a graph that approaches $-Q$ at $r = \alpha$ and equals $-Q$ beyond that point

(b) 2 points

For a graph that is decreasing and between the dashed curve and the x-axis
For a graph that goes to zero at $r = \alpha$ and is zero beyond that point
(c)

i. 3 points

For stating and using Gauss’s law in any form
\[ \oint E \cdot dA = \frac{q_{enc}}{\varepsilon_0} \]
1 point

For indicating that \( q_{enc} = 0 \)
1 point

For correctly stating that the electric field is zero
1 point

ii. 4 points

For indicating the need to integrate with respect to volume to find the negative charge enclosed
1 point

\[ q_{neg} = \int \rho(r) dV \]
Substitute and integrate with appropriate limits

\[ q_{neg} = \int_0^r -\frac{\beta}{r^2} e^{-r^2/a} (4\pi r^2) dr = 4\pi \beta \int_0^r e^{-r^2/a} dr \]
1 point

\[ q_{neg} = -4\pi \beta a \left[ e^{-r^2/a} \right]_0^r \]
For a correct expression for negative charge as a function of distance \( r \)
1 point

\( q_{neg} = -4\pi \beta a \left( 1 - e^{-r^2/a} \right) \) or \( q_{neg} = 4\pi \beta a \left( e^{-r^2/a} - 1 \right) \)

For including the \( +Q \) when substituting for \( q_{enc} \)
1 point

For correct substitution for the surface area of a sphere
1 point

\[ E \left( 4\pi r^2 \right) = \frac{q_{enc}}{\varepsilon_0} = \frac{Q - 4\pi \beta a \left( 1 - e^{-r^2/a} \right)}{\varepsilon_0} \]

\[ E = \frac{1}{4\pi \varepsilon_0 r^2} \left[ Q - 4\pi \beta a \left( 1 - e^{-r^2/a} \right) \right] \]

(d) 1 point

For correctly stating or implying that \( a \) is the radius of the atom or the radius of the electron cloud
1 point
E&M.3.

A scientist describes an electrically neutral atom with a model that consists of a nucleus that is a point particle with positive charge $+Q$ at the center of the atom and an electron volume charge density of the form

$$\rho(r) = \begin{cases} \frac{-\beta}{r^2} e^{-\alpha/r} & r < \alpha \\ 0 & r > \alpha \end{cases}$$

where $\alpha$ and $\beta$ are positive constants and $r$ is the distance from the center of the atom.

(a) On the axes below, let $r$ stand for the radius of a Gaussian sphere. Sketch the graph for each of the following charges enclosed by the Gaussian sphere as a function of $r$. Explicitly label any intercepts, asymptotes, maxima, or minima with numerical values or algebraic expressions, as appropriate.

i. The nuclear charge only

![Graph of nuclear charge only](image)

ii. The electron charge only

![Graph of electron charge only](image)
(b) The dashed curve on the graph below represents the electric field as a function of distance \( r \) due to the positive nucleus of the atom without any electrons. The nucleus is modeled as a point particle of charge \( +Q \).

On the same graph, sketch the electric field as a function of distance \( r \) for the neutral atom as defined by the scientist's model, which includes the nucleus and the negative electrons surrounding it.

![Graph showing electric field as a function of distance.]

\[ \text{for } r > \alpha, \quad E = 0 \]

(c) Use Gauss's law to derive an expression for the electric field strength due to the neutral atom for the following positions in terms of \( Q, \alpha, \beta, r, \) and fundamental constants, as appropriate.

i. \( r > \alpha \)

\[ \int \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\varepsilon_0} = 0 \quad (\text{Because } r > \alpha \text{ is outside of the electrically neutral atom, so } Q_{\text{enc}} = 0) \]

ii. \( r < \alpha \)

\[ \int \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\varepsilon_0} = \frac{Q - 4\pi J \beta \alpha (1 - e^{-r/\alpha})}{\varepsilon_0} \]

\[ E = \frac{Q - 4\pi J \beta \alpha (1 - e^{-r/\alpha})}{4\pi \varepsilon_0 r^2} \]

(d) Based on the model proposed by the scientist, what is the physical meaning of the constant \( \alpha \)?

\( \alpha \) is the edge of the electron cloud, after which the charge density is 0.
E&M.3.

A scientist describes an electrically neutral atom with a model that consists of a nucleus that is a point particle with positive charge $+Q$ at the center of the atom and an electron volume charge density of the form

$$
\rho(r) = \begin{cases} 
-\frac{\beta}{r^2} e^{-r/\alpha} & r < \alpha \\
0 & r > \alpha
\end{cases}
$$

where $\alpha$ and $\beta$ are positive constants and $r$ is the distance from the center of the atom.

(a) On the axes below, let $r$ stand for the radius of a Gaussian sphere. Sketch the graph for each of the following charges enclosed by the Gaussian sphere as a function of $r$. Explicitly label any intercepts, asymptotes, maxima, or minima with numerical values or algebraic expressions, as appropriate.

i. The nuclear charge only

\[ q \]

\[ +Q \]

\[ -Q \]

\[ r \]

ii. The electron charge only

\[ q \]

\[ +Q \]

\[ -Q \]

\[ r \]
(b) The dashed curve on the graph below represents the electric field as a function of distance $r$ due to the positive nucleus of the atom without any electrons. The nucleus is modeled as a point particle of charge $+Q$. On the same graph, sketch the electric field as a function of distance $r$ for the neutral atom as defined by the scientist’s model, which includes the nucleus and the negative electrons surrounding it.

![Graph showing electric field as a function of distance](image)

(c) Use Gauss’s law to derive an expression for the electric field strength due to the neutral atom for the following positions in terms of $Q$, $\alpha$, $\beta$, $r$, and fundamental constants, as appropriate.

\[ \text{i. } r > \alpha \quad \mathcal{E}_\text{enc} = \frac{Q_{\text{enc}}}{E_0} \]

\[ \mathcal{E}_\text{enc} = 0 \quad \Rightarrow \quad \mathcal{E} = 0 \quad \text{N/C} \]

\[ \text{ii. } r < \alpha \quad \mathcal{E}_\text{enc} = \frac{Q_{\text{enc}}}{E_0} \]

\[ \mathcal{E}_\text{enc} = \rho(r) \quad V = \left( -\frac{\beta}{x^2} e^{-r/\alpha} \right) \left( \frac{1}{\alpha} \right) \quad \Rightarrow \quad \mathcal{E} = \frac{-\beta}{3} \frac{1}{\beta} e^{-r/\alpha} \]

\[ E(\mathcal{E}_\text{enc}) = \left( \frac{1}{E_0} \right) \left( -\frac{\beta}{3} \frac{1}{\beta} e^{-r/\alpha} \right) \]

\[ 4\mathcal{E} = -\frac{\beta e^{-r/\alpha}}{3 E_0 r} \]

\[ \mathcal{E} = -\frac{\beta e^{-r/\alpha}}{12 E_0 r} \]

(d) Based on the model proposed by the scientist, what is the physical meaning of the constant $\alpha$?

$\alpha$ is the radius of the atom.
A scientist describes an electrically neutral atom with a model that consists of a nucleus that is a point particle with positive charge +Q at the center of the atom and an electron volume charge density of the form

\[ \rho(r) = \begin{cases} \frac{-\beta}{r^2} e^{-r/\alpha} & r < \alpha \\ 0 & r > \alpha \end{cases} \]

where \( \alpha \) and \( \beta \) are positive constants and \( r \) is the distance from the center of the atom.

(a) On the axes below, let \( r \) stand for the radius of a Gaussian sphere. Sketch the graph for each of the following charges enclosed by the Gaussian sphere as a function of \( r \). Explicitly label any intercepts, asymptotes, maxima, or minima with numerical values or algebraic expressions, as appropriate.

i. The nuclear charge only

\[
\begin{array}{c}
\text{q} \\
+Q \\
\text{a} \\
0 \\
-Q \\
\text{r}
\end{array}
\]

ii. The electron charge only

\[
\begin{array}{c}
\text{q} \\
+Q \\
\text{a} \\
0 \\
-Q \\
\text{r}
\end{array}
\]
(b) The dashed curve on the graph below represents the electric field as a function of distance \( r \) due to the positive nucleus of the atom without any electrons. The nucleus is modeled as a point particle of charge \( +Q \). On the same graph, sketch the electric field as a function of distance \( r \) for the neutral atom as defined by the scientist's model, which includes the nucleus and the negative electrons surrounding it.

![Graph with dashed and solid curves]

(c) Use Gauss's law to derive an expression for the electric field strength due to the neutral atom for the following positions in terms of \( Q, \alpha, \beta, r, \) and fundamental constants, as appropriate.

i. \( r > \alpha \)

\[
E = \frac{q}{\varepsilon_0(4\pi \alpha^2)} \quad q = 0 \quad \text{E-field} = 0
\]

ii. \( r < \alpha \)

\[
E = \frac{Q}{\varepsilon_0(4\pi r^2)} \quad q = \frac{4\pi r^3}{3} = \frac{Q}{r^2} e^{-\frac{r}{\alpha}} 4\pi \alpha^2
\]

\[
E = \frac{-4\pi e^{-\frac{r}{\alpha}}}{3 \varepsilon_0 4\pi r^2} = \frac{-B e^{-\frac{r}{\alpha}}}{3 \varepsilon_0 r} = E
\]

(d) Based on the model proposed by the scientist, what is the physical meaning of the constant \( \alpha \)?

*Radius of the electron (cloud) volume.*
Question 3

Overview

This question dealt with Gauss’ law. It was not at all like a standard problem with an insulating or conducting sphere. Instead the students were told that they had a nucleus with a positive charge surrounded by a negative charge distributed nonuniformly. Students had to sketch graphs of positive and negative charge and electric field as a function of \( r \). Then they were told to use Gauss’ law to solve for the electric field outside and inside the boundary of the charge at \( r = a \).

Sample: E3 A  
Score: 15

This response earned full credit for all parts. The work in part (a)(ii) to determine the negative charge enclosed within \( a \) was performed completely and correctly in part (c)(ii).

Sample: E3 B  
Score: 9

Part (a)(i) earned 1 point for starting the graph at \(+Q\). Part (a)(ii) also earned 1 point for starting the graph at zero. Parts (b) and (c)(i) earned full credit. Part (c)(ii) earned 1 point for recognizing that the Gaussian surface was a sphere. However, there was no integration and no recognition that both positive and negative charge must be included. Part (d) earned 1 point.

Sample: E3 C  
Score: 6

Part (a)(i) earned 1 point for starting the graph at \(+Q\), and part (a)(ii) earned no credit. Part (b) also earned nothing since the curve is always above the dashed line. Part (c)(i) earned full credit. The solution in part (c)(ii) begins with Gauss’ Law for a sphere, earning 1 point, but does not correctly determine the negative charge enclosed or recognize that \(+Q\) was also enclosed. Part (d) earned 1 point.