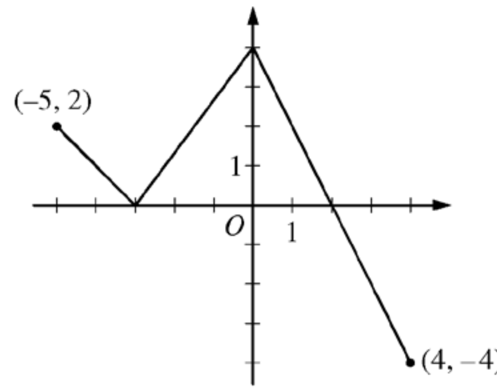
Question 3

The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above.



Graph of f

Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

(a) Find $g(3)$.

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

(c) The function h is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.

(d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

(a) $g(3) = \int_{-3}^{3} f(t)\,dt = 6 + 4 - 1 = 9$

1 : answer

(b) $g'(x) = f(x)$

The graph of g is increasing and concave down on the intervals $-5 < x < -3$ and $0 < x < 2$ because $g' = f$ is positive and decreasing on these intervals.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

(c) $h'(x) = \dfrac{5xg'(x) - g(x)5}{(5x)^2} = \dfrac{5xg'(x) - 5g(x)}{25x^2}$

$$h'(3) = \frac{(5)(3)g'(3) - 5g(3)}{25 \cdot 3^2} = \frac{15(-2) - 5(9)}{225} = \frac{-75}{225} = -\frac{1}{3}$$

$3 : \begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$

(d) $p'(x) = f'\left(x^2 - x\right)(2x - 1)$

$p'(-1) = f'(2)(-3) = (-2)(-3) = 6$

$3 : \begin{cases} 2 : p'(x) \\ 1 : \text{answer} \end{cases}$

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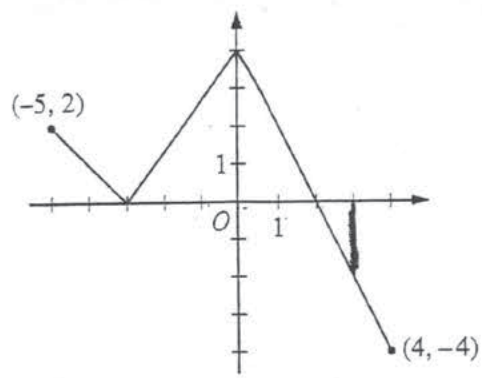

Graph of f

3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

(a) Find $g(3)$.

$$g(3) = \int_{-3}^{3} f(t)\,dt$$

$$= \tfrac{1}{2}(5)(4) - \tfrac{1}{2}(1)(2)$$

$$= 10 - 1 = \boxed{9}$$

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

$$g'(x) > 0 \Leftrightarrow f(x) > 0$$

$$g''(x) < 0 \Leftrightarrow f'(x) < 0$$

$$(-5, -3), (0, 2)$$

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NO CALCULATOR ALLOWED

(c) The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.

$$h(x) = \frac{g(x)}{5x} \qquad h'(x) = \frac{(5x)(g'(x)) - g(x)\cdot 5}{25x^2}$$

$$h'(3) = \frac{(15)(f(3)) - g(3)\cdot 5}{9\cdot 25} = \boxed{\frac{(15)(-2) - (9)(5)}{9\cdot 25}}$$

(d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

$$p(x) = f(x^2 - x)$$

$$p'(x) = f'(x^2 - x)\cdot(2x-1)$$

$$p'(-1) = f'(1+1)\cdot(-2-1) = f'(2)\cdot -3$$

$$f'(2) = \frac{-4-4}{4-0} = \frac{-8}{4} = -2$$

$$= (-2)(-3)$$

$$= \boxed{6}$$

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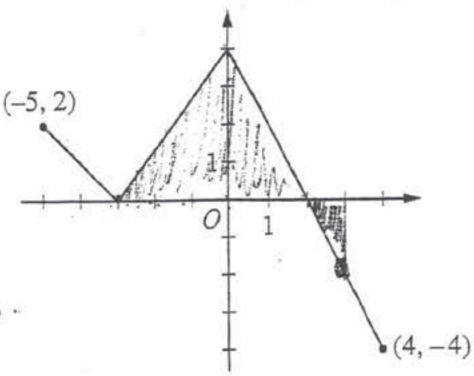

Graph of f

3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

(a) Find $g(3)$.

$$\int_{-3}^{3} f(t)\,dt.$$

$$5 \cdot 4 = \frac{20}{2} = 10 + \frac{1(2)}{2}$$

$$= 11$$

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

$$g'(x) = f(x)$$

$(0, 2)$

because g' is both positive and decreasing

when g' is positive g is incr.
when g' is dec g is concave down

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NO CALCULATOR ALLOWED

(c) The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.

$g'(3) = -2$
$g(3) = 11$

$$h'(x) = \frac{5x(g'(x)) - g(x) \cdot 5}{25x^2}$$

$$h'(3) = \frac{5(3)(g'(3)) - g(3) \cdot 5}{25(9)}$$

$$h'(3) = \frac{15(-2) - 11(5)}{225}$$

(d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

$$p'(x) = f'(x^2 - x) \cdot (2x - 1)$$

$$p'(-1) = f'(0)(-3)$$

$$p'(-1) = -18$$

$f'(0) = g(0)$
$g(0) = 6$
$f'(0) = 6$

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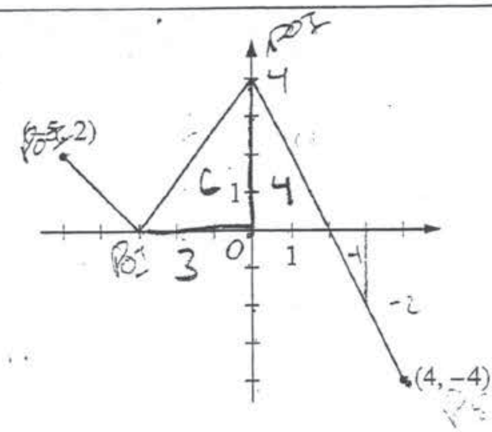
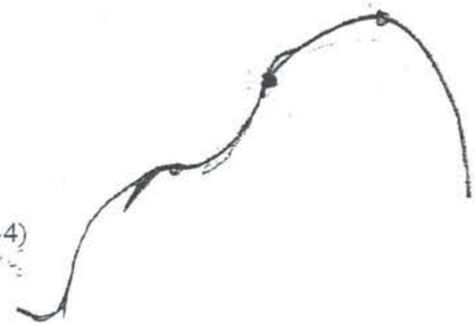


Graph of f



3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

(a) Find $g(3)$.

$g(3) = 10 + \int_{2}^{3} -2x+4\,dx \quad -x^2+4x \Big|_2^3 = (-9+12)-(-4+8)$

$= 3-4 = -1$

$g(3) = 9$

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

$(-5,-2)\ (0,2)$

In these x-values f(x) is positive and the slopes of f(x) show that they are concave down because they are positive

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(c) The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.

$$h'(x) = \frac{5x f(x) - 5g(x)}{25x^2}$$

$$h'(3) = \frac{15f(3) - 5(g(3))}{25 \cdot 9} = \frac{-30 - 40}{25 \cdot 9}$$

$$h'(3) = \boxed{\frac{-70}{225}}$$

(d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

$$f(x) = \frac{4}{3}x + 4$$

$$f(x^2 - x) = \frac{4}{3}(x^2 - x) + 4$$

$$p(x) = \frac{4}{3}(x^2 - x) + 4$$

$$p'(x) = \frac{4}{3}(2x - 1)$$

$$m_{T\,x=-1} = \frac{4}{3}(2(-1) - 1) = \boxed{-4}$$

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Question 3

Overview

In this problem students were given the graph of a piecewise continuous function f defined on the closed interval $[-5, 4]$. The graph of f consists of line segments whose slopes can be determined precisely. A second function g is defined by $g(x) = \int_{-3}^{x} f(t)\,dt$. In part (a) students must calculate $g(3) = \int_{-3}^{3} f(t)\,dt$ by using a decomposition of $\int_{-3}^{3} f(t)\,dt$, such as $\int_{-3}^{3} f(t)\,dt = \int_{-3}^{2} f(t)\,dt + \int_{2}^{3} f(t)\,dt$, and by applying the relationship between the definite integral of a continuous function and the area of the region between the graph of that function and the x-axis. In part (b) students were expected to apply the Fundamental Theorem of Calculus to conclude that $g'(x) = f(x)$ on the interval $[-5, 4]$. Students were to then conclude that $g''(x) = f'(x)$ wherever $f'(x)$ is defined on $[-5, 4]$. Students needed to explain that the intervals $(-5, -3)$ and $(0, 2)$ are the only open intervals where both $g'(x) = f(x)$ is positive and decreasing. In part (c) students were expected to apply the quotient rule to find $h'(3)$ using the result from part (a) and the value $g'(3) = f(3)$ from the graph of f. In part (d) students were expected to apply the chain rule to find $p'(-1)$. This required finding $f'(2)$ from the graph of f.

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

The student earned 6 points: no points in part (a), 1 point in part (b), 3 points in part (c), and 2 points in part (d). In part (a) the student reports an incorrect value for $g(3)$. In part (b) the student gives an incomplete answer, but the student is eligible for and earned the reason point. In part (c) the student's work is correct based on the imported incorrect value for $g(3)$. In part (d) the student earned both derivative points but reports an incorrect value of $p'(-1)$.

Sample: 3C
Score: 3

The student earned 3 points: 1 point in part (a), no points in part (b), 2 points in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student provides values outside of the given intervals, so the student is not eligible for the reason point. In part (c) the student's derivative is correct, but the answer is incorrect. In part (d) the student presents an incorrect expression for $p(x)$ near $x = -1$, so the student is not eligible for any points.