Question 5

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

The twice-differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table above.

(a) Find the x-coordinate of each relative minimum of f on the interval $[-2, 3]$. Justify your answers.

(b) Explain why there must be a value c, for $-1 < c < 1$, such that $f''(c) = 0$.

(c) The function h is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.

(d) Evaluate $\int_{-2}^{3} f'(g(x))g'(x)\,dx$.

(a) $x = 1$ is the only critical point at which f' changes sign from negative to positive. Therefore, f has a relative minimum at $x = 1$.

1 : answer with justification

(b) f' is differentiable $\Rightarrow$ f' is continuous on the interval $-1 \le x \le 1$

$$\frac{f'(1) - f'(-1)}{1 - (-1)} = \frac{0 - 0}{2} = 0$$

Therefore, by the Mean Value Theorem, there is at least one value c, $-1 < c < 1$, such that $f''(c) = 0$.

$2 : \begin{cases} 1 : f'(1) - f'(-1) = 0 \\ 1 : \text{explanation, using Mean Value Theorem} \end{cases}$

(c) $h'(x) = \frac{1}{f(x)} \cdot f'(x)$

$$h'(3) = \frac{1}{f(3)} \cdot f'(3) = \frac{1}{7} \cdot \frac{1}{2} = \frac{1}{14}$$

$3 : \begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$

(d)
$$\begin{aligned} \int_{-2}^{3} f'(g(x))g'(x)\,dx &= \left[f(g(x)) \right]_{x=-2}^{x=3} \\ &= f(g(3)) - f(g(-2)) \\ &= f(1) - f(-1) \\ &= 2 - 8 = -6 \end{aligned}$$

$3 : \begin{cases} 2 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

5. The twice-differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table above.

(a) Find the x-coordinate of each relative minimum of f on the interval $[-2, 3]$. Justify your answers.

Critical numbers: $x = -1, 1$

$\longleftarrow - \underset{-1}{|} - \underset{1}{|} + \longrightarrow$ f'

f has a rel. min. at $x = 1$ because $f'(1) = 0$ and f' switches sign from negative to positive there.

(b) Explain why there must be a value c, for $-1 < c < 1$, such that $f''(c) = 0$.

$f'(-1) = 0$ and $f'(1) = 0$, and $f'(x)$ is differentiable and continuous on the interval so by Rolle's Theorem there is some value c where $f''(c) = 0$.

Do not write beyond this border.

Continue problem 5 on page 19.

NO CALCULATOR ALLOWED

(c) The function h is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.

$$h'(x) = \frac{f'(x)}{f(x)}$$

$$h'(3) = \frac{f'(3)}{f(3)}$$

$$h'(3) = \frac{\frac{1}{2}}{7}$$

$$h'(3) = \frac{1}{2} \cdot \frac{1}{7}$$

$$\boxed{h'(3) = \frac{1}{14}}$$

Do not write beyond this border.

(d) Evaluate $\int_{-2}^{3} f'(g(x))g'(x)\,dx$.

$u = g(x)$
$du = g'(x)\,dx$

$$\int f'(u)\,du$$

$$\Big[f(g(x))\Big]_{-2}^{3}$$

$$f(g(3)) - f(g(-2))$$

$$f(1) - f(-1)$$

$$2 - 8$$

$$\boxed{-6}$$

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

5. The twice-differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table above.

(a) Find the x-coordinate of each relative minimum of f on the interval $[-2, 3]$. Justify your answers.

On the interval [-2,3], the x-coordinate 0 is a relative minimum of f because f'(x) changes from negative to positive.

(b) Explain why there must be a value c, for $-1 < c < 1$, such that $f''(c) = 0$.

There must be a value c because the Mean Value Theorem states that on a closed interval, if the function is differentiable, there must be a value c.

Continue problem 5 on page 19.

NO CALCULATOR ALLOWED

(c) The function h is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.

$$h'(x) = \frac{f'(x)}{f(x)}$$

$$h'(3) = \frac{f'(3)}{f(3)}$$

$$= \frac{\frac{1}{2}}{7}$$

$$\boxed{= \frac{1}{14}}$$

Do not write beyond this border.

(d) Evaluate $\int_{-2}^{3} f'(g(x))g'(x)\,dx$.

$$= \int_{-2}^{3} f'(g(x))g'(x)\,dx$$

$$= f(g(x))\Big|_{-2}^{3}$$

$$= f(g(3)) - f(g(-2))$$

$$= f(1) - f(-1)$$

$$= 2 - 8$$

$$\boxed{= -6}$$

GO ON TO THE NEXT PAGE

NO CALCULATOR ALLOWED

inc-dec
max

dec-inc
min

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative increasing	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

5. The twice-differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table above.

(a) Find the x-coordinate of each relative minimum of f on the interval $[-2, 3]$. Justify your answers.

-relative min when $f'(x)$ changes from decreasing to increasing

between $(-1,1)$, there is a minimum

(b) Explain why there must be a value c, for $-1 < c < 1$, such that $f''(c) = 0$.

Rolle's Theorem states that on an open interval $x_1 < c < x_2$, there must be a value such that $f''(c)=0$.

Do not write beyond this border.

Continue problem 5 on page 19.

NO CALCULATOR ALLOWED

(c) The function h is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.

$$h(x) = \ln(f(x))$$
$$h'(x) = \frac{1}{f(x)} \cdot f'(x)$$
$$h'(3) = \frac{1}{f(3)} \cdot f'(3)$$
$$= \frac{1}{7} \cdot \frac{1}{2}$$
$$= \frac{1}{14}$$

Do not write beyond this border.

(d) Evaluate $\int_{-2}^{3} f'(g(x))g'(x)\,dx.$ $= f'(g(x))g'(x)\Big]_{-2}^{3}$

$$f'(g(3))g'(3) - f'(g(-2))g'(-2)$$
$$f'(1)\cdot -2 - f'(-1)\cdot 2$$
$$[\cancel{0\cdot -2}] - [\cancel{0\cdot 2}]$$
$$0$$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Question 5

Overview

In this problem students were provided with a table giving values of two twice-differentiable functions f and g at various values of x. Part (a) asked students to find the x-coordinate of each relative minimum of f on the given interval. Students should have determined that $x = 1$ is a critical point and that f' changes sign from negative to positive at that point. In part (b) students had to explain why there is a value c, for $-1 < c < 1$, such that $f''(c) = 0$. Because the function is twice differentiable, f' is continuous on the interval $-1 \le x \le 1$, and because $f'(1) = f'(-1) = 0$, the Mean Value Theorem guarantees that there is at least one value c, $-1 < c < 1$, such that $f''(c) = 0$. In part (c) students needed to differentiate $h(x)$ using the chain rule to get $h'(x) = \frac{1}{f(x)} \cdot f'(x)$. Using values from the table, $h'(3) = \frac{1}{14}$. Part (d) required students to find the antiderivative of the integrand to get $f(g(3)) - f(g(-2))$. Using values from the table, the result is -6.

Sample: 5A
Score: 9

The student earned all 9 points.

Sample: 5B
Score: 6

The student earned 6 points: no points in part (a), no points in part (b), 3 points in part (c), and 3 points in part (d). In part (a) the student provides a seemingly correct justification but gives an incorrect answer. There is not a relative minimum at $x = 0$. In part (b) the student does not communicate that $f'(1) - f'(-1) = 0$. The student names the Mean Value Theorem but does not connect it to the question asked. The student's explanation is not complete. In parts (c) and (d), the student's work is correct.

Sample: 5C
Score: 3

The student earned 3 points: no points in part (a), no points in part (b), 3 points in part (c), and no points in part (d). In part (a) the student refers to "$f'(x)$ changes from decreasing to increasing." To earn the point, the student needs to communicate that f' changes sign from negative to positive. In part (b) the student does not communicate that $f'(1) - f'(-1) = 0$. The student names Rolle's Theorem but does not connect it to the question asked. The student's explanation is not complete. In part (c) the student's work is correct. In part (d) the student evaluates the integrand at the limits of integration without first finding an antiderivative. The student does not earn any points for use of the Fundamental Theorem of Calculus; therefore, the student is not eligible for the answer point.