Question 6

A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

(a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

(b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

(c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

(a) $P_1(x) = f(0) + f'(0)x = -4 + f'(0)x$

$P_1\left(\frac{1}{2}\right) = -4 + f'(0) \cdot \frac{1}{2} = -3$

$f'(0) \cdot \frac{1}{2} = 1$

$f'(0) = 2$

$2 : \begin{cases} 1 : \text{uses } P_1(x) \\ 1 : \text{verifies } f'(0) = 2 \end{cases}$

(b) $P_3(x) = -4 + 2x + \left(-\frac{2}{3}\right) \cdot \frac{x^2}{2!} + \frac{1}{3} \cdot \frac{x^3}{3!}$

$= -4 + 2x - \frac{1}{3}x^2 + \frac{1}{18}x^3$

$3 : \begin{cases} 1 : \text{first two terms} \\ 1 : \text{third term} \\ 1 : \text{fourth term} \end{cases}$

(c) Let $Q_n(x)$ denote the Taylor polynomial of degree n for h about $x = 0$.

$h'(x) = f(2x) \Rightarrow Q_3'(x) = -4 + 2(2x) - \frac{1}{3}(2x)^2$

$Q_3(x) = -4x + 4 \cdot \frac{x^2}{2} - \frac{4}{3} \cdot \frac{x^3}{3} + C;\ C = Q_3(0) = h(0) = 7$

$Q_3(x) = 7 - 4x + 2x^2 - \frac{4}{9}x^3$

OR

$h'(x) = f(2x),\ h''(x) = 2f'(2x),\ h'''(x) = 4f''(2x)$

$h'(0) = f(0) = -4,\ h''(0) = 2f'(0) = 4,\ h'''(0) = 4f''(0) = -\frac{8}{3}$

$Q_3(x) = 7 - 4x + 4 \cdot \frac{x^2}{2!} - \frac{8}{3} \cdot \frac{x^3}{3!} = 7 - 4x + 2x^2 - \frac{4}{9}x^3$

$4 : \begin{cases} 2 : \text{applies } h'(x) = f(2x) \\ 1 : \text{constant term} \\ 1 : \text{remaining terms} \end{cases}$

6A1

6. A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

(a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

$$P_n = f(c) + \frac{f'(c)(x-c)}{1!} + \frac{f''(c)(x-c)^2}{2!}$$

$$P_1 = -4 + f'(0)x$$

$$-3 = -4 + f'(0)\cdot \tfrac{1}{2}$$

$$1 = \frac{f'(0)}{2}, \therefore f'(0) = 2$$

(b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

$$P_3(x) = -4 + 2x + \frac{(-2/3)x^2}{2!} + \frac{(1/3)x^3}{3!}$$

$$\boxed{= -4 + 2x - \frac{1}{3}x^2 + \frac{1}{18}x^3}$$

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6A2

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(c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

$$h(x) = \int f(2x)\,dx \approx \int P_3(2x) \qquad \frac{4x^2}{3} \qquad \frac{4}{9}x^3$$

$$P_3(2x) = -4 + 2(2x) - \frac{1}{3}(2x)^2 + \frac{1}{18}(2x)^3$$

$$h(0) + \int P_3(2x) = -4x + 2x^2 - \frac{4}{9}x^3$$

$$\boxed{P_3 \text{ for } h = 7 - 4x + 2x^2 - \frac{4}{9}x^3}$$

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6B,

NO CALCULATOR ALLOWED

6. A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

(a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

$$P_1(x) = -4 + f'(0)\left(\tfrac{1}{2}\right) = -3$$

$$f'(0)\left(\tfrac{1}{2}\right) = 1$$

$$f'(0) = 2$$

(b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

$$P_3(x) = \boxed{-4 + 2x - \frac{2/3\, x^2}{2} + \frac{1/3\, x^3}{6}}$$

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NO CALCULATOR ALLOWED

6B2

(c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

$$\cancel{T_3(x) = 7 + 2(2x) - \frac{2/3(2x)^2}{2} + \frac{1/3(2x)^2}{6}}$$

$$\cancel{\boxed{= 7 \quad 4x - \frac{4}{3}x^2 + \frac{2x^3}{9}}}$$

$$T_3(x) = 7 - 4(2x) + \frac{2(2x)^2}{2} - \frac{2/3(2x)^3}{6} \qquad \frac{8 \cdot 2}{3 \cdot 6}$$

$$\boxed{= 7 - 8x + 4x^2 - \frac{8}{9}x^3}$$

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6C1

6. A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

(a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

$-4 + x = -3$

$x = 1$

$f'(0) = 2$

$-4 + \frac{2x}{1!} = -3$

$2x = 1$

$x = \frac{1}{2}$

$P_1(\frac{1}{2}) = -3$

$\therefore f'(0) = 2$

(b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

$f(0) = -4$

$f'(0) = 2$

$f''(0) = -2/3$

$f'''(0) = 1/3$

$-4 + \frac{2x^2}{2!} - \frac{2x^3}{3 \cdot 3!} + \frac{x^4}{3 \cdot 4!}$

$\boxed{-4 + x^2 - \frac{x^3}{9} + \frac{x^4}{72}}$

$\frac{24 \cdot 3}{72}$

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6C2

(c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

$h(0) = 7$ $h'(0) = f(0)$

$h'(x) = f(2x)$ $h''(0) = f'(2\cancel{x})(2)$

$7 + \frac{-4x}{1!} + \frac{4x^2}{2!}$

$\boxed{7 - 4x + 2x^2}$

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Question 6

Overview

This problem described a function f known to have derivatives of all orders at $x = 0$. In part (a) information was provided about the value of the function at $x = 0$, as well as about the value of its first-degree Taylor polynomial about $x = 0$ at the point $x = \frac{1}{2}$. Students needed to use this information to verify that $f'(0) = 2$.
In part (b) students were given information about the second and third derivatives of f at $x = 0$. Students needed to use this additional information to find the third-degree Taylor polynomial for f about $x = 0$. In part (c) a new function h was defined in terms of f. Students needed to use information provided about $h(0)$, as well as information already provided about f, to find the third-degree Taylor polynomial for h about $x = 0$.

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 2 points in part (a), 3 points in part (b), and 1 point in part (c). In parts (a) and (b), the student's work is correct. No supporting work was required. In part (c) the student appears to have antidifferentiated $P_2(x)$ and replaced x by $2x$ in the result. This is not a legitimate method, and the student did not earn the fourth point. The student earned the third point because the answer has a constant term of 7 in a polynomial of degree three or higher.

Sample: 6C
Score: 3

The student earned 3 points: 2 points in part (a) and 1 point in part (c). In part (a) the student's actual solution begins on the third line and earned both points in part (a). The student shows that $f'(0) = 2$ together with the given values for $P_1\left(\frac{1}{2}\right)$ and $f(0)$ implies that $x = \frac{1}{2}$, thereby verifying that $f'(0) = 2$. In part (b) the student's work is incorrect. In part (c) the student earned 1 of 2 points. The student would have earned the second point with a correct expression for $h'''(0)$. Because the student attempts to use a legitimate method, the student was eligible to earn the fourth point. No additional points were earned since the answer is not a polynomial of degree three or higher, and the student does not have a value for $h'''(0)$.