The figure above shows the graph of $f'$, the derivative of a twice-differentiable function $f$, on the closed interval $0 \leq x \leq 8$. The graph of $f'$ has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of $f'$ and the x-axis are labeled in the figure. The function $f$ is defined for all real numbers and satisfies $f(8) = 4$.

(a) Find all values of $x$ on the open interval $0 < x < 8$ for which the function $f$ has a local minimum. Justify your answer.

(b) Determine the absolute minimum value of $f$ on the closed interval $0 \leq x \leq 8$. Justify your answer.

(c) On what open intervals contained in $0 < x < 8$ is the graph of $f$ both concave down and increasing? Explain your reasoning.

(d) The function $g$ is defined by $g(x) = (f(x))^3$. If $f(3) = \frac{-5}{2}$, find the slope of the line tangent to the graph of $g$ at $x = 3$.

(a) $x = 6$ is the only critical point at which $f'$ changes sign from negative to positive. Therefore, $f$ has a local minimum at $x = 6$.

(b) From part (a), the absolute minimum occurs either at $x = 6$ or at an endpoint.

\[
\begin{align*}
  f(0) &= f(8) + \int_0^8 f'(x) \, dx \\
  &= f(8) - \int_0^8 f''(x) \, dx = 4 - 12 = -8 \\
  f(6) &= f(8) + \int_6^8 f'(x) \, dx \\
  &= f(8) - \int_6^8 f''(x) \, dx = 4 - 7 = -3 \\
  f(8) &= 4
\end{align*}
\]

The absolute minimum value of $f$ on the closed interval $[0, 8]$ is $-8$.

(c) The graph of $f$ is concave down and increasing on $0 < x < 1$ and $3 < x < 4$, because $f''$ is decreasing and positive on these intervals.

(d) $g'(x) = 3[f(x)]^2 \cdot f'(x)$

\[
g'(3) = 3[f(3)]^2 \cdot f'(3) = 3\left(-\frac{5}{2}\right)^2 \cdot 4 = 75
\]
4. The figure above shows the graph of $f'$, the derivative of a twice-differentiable function $f$, on the closed interval $0 \leq x \leq 8$. The graph of $f'$ has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of $f'$ and the $x$-axis are labeled in the figure. The function $f$ is defined for all real numbers and satisfies $f(8) = 4$.

(a) Find all values of $x$ on the open interval $0 < x < 8$ for which the function $f$ has a local minimum. Justify your answer.

\[ x = 6 \]

$f$ has a local minima at $x = 6$, because the graph of $f'$ changes from negative to positive at $x = 6$.

(b) Determine the absolute minimum value of $f$ on the closed interval $0 \leq x \leq 8$. Justify your answer.

\[
\text{local minimum } x = 6, \quad f(6) = 4
\]

\[
\int_0^6 f'(x) \, dx = 7 = f(8) - f(0) = 4 - f(6), \quad f(0) = -8
\]

The absolute minimum value of $f$ on the interval $0 \leq x \leq 8$ is $-8$ because it is the lowest value for $f$ among the endpoints and critical numbers.
(c) On what open intervals contained in $0 < x < 8$ is the graph of $f$ both concave down and increasing? Explain your reasoning.

The open intervals where the graph of $f$ is both concave down and increasing is $(0, 1) \cup (3, 4)$, or $0 < x < 1$ and $3 < x < 4$, because using the graph of $f'$, when the graph of $f''$ is positive and the slope of $f'$ is negative, that means that $f$ is increasing and $f''$ is negative, so $f$ is both concave down and increasing.

(d) The function $g$ is defined by $g(x) = (f(x))^3$. If $f(3) = -\frac{5}{2}$, find the slope of the line tangent to the graph of $g$ at $x = 3$.

$$g'(x) = 3(f(x))^2 \cdot f'(x)$$

$$g'(3) = 3(f(3))^2 \cdot f'(3) = 3\left(-\frac{5}{2}\right)^2 \cdot (4)$$

$$= 3\left(\frac{25}{4}\right) \cdot 4 = 75$$

$$g'(3) = 75$$
4. The figure above shows the graph of $f'$, the derivative of a twice-differentiable function $f$, on the closed interval $0 \leq x \leq 8$. The graph of $f'$ has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of $f'$ and the $x$-axis are labeled in the figure. The function $f$ is defined for all real numbers and satisfies $f(8) = 4$.

(a) Find all values of $x$ on the open interval $0 < x < 8$ for which the function $f$ has a local minimum. Justify your answer.

$x = 6$ because the sign of $f'$ changes from negative to positive.

(b) Determine the absolute minimum value of $f$ on the closed interval $0 \leq x \leq 8$. Justify your answer.

$x = 6$ because that is the smallest $x$ where $f(x)$ is a relative minimum.
(c) On what open intervals contained in \(0 < x < 8\) is the graph of \(f\) both concave down and increasing? Explain your reasoning.

\[ f(x) \text{ is concave down and increasing when } f'' \text{ is negative and } f' \text{ is positive.} \]

This occurs in \((0, 1)\) and \((3, 4)\).

(d) The function \(g\) is defined by \(g(x) = (f(x))^3\). If \(f(3) = -\frac{5}{2}\), find the slope of the line tangent to the graph of \(g\) at \(x = 3\).

\[ g'(x) = 3(f(x))^2 \cdot f'(x) \]

\[ g'(3) = 3\left(-\frac{5}{2}\right)^2 \cdot 4 \]

\[ g'(3) = -\frac{75}{4} \]

\[ g'(3) = -\frac{300}{4} \]

\[ g'(3) = -75 \]
4. The figure above shows the graph of $f'$, the derivative of a twice-differentiable function $f$, on the closed interval $0 \leq x \leq 8$. The graph of $f'$ has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of $f'$ and the x-axis are labeled in the figure. The function $f$ is defined for all real numbers and satisfies $f(8) = 4$.

(a) Find all values of $x$ on the open interval $0 < x < 8$ for which the function $f$ has a local minimum. Justify your answer.

$\text{Min of } f \text{ is when } f' \text{ changes from } (-) \text{ to } (+)$

$\text{Min at } x = 6$

(b) Determine the absolute minimum value of $f$ on the closed interval $0 \leq x \leq 8$. Justify your answer.
(c) On what open intervals contained in $0 < x < 8$ is the graph of $f$ both concave down and increasing? Explain your reasoning.

\[
\text{F is concave down when } F'' \text{ is } (-)
\]
\[
\text{F is increasing when } F' \text{ is } (+)
\]
\[
\text{Both from } (0,1), \text{ and } (3,4)
\]

(d) The function $g$ is defined by $g(x) = (f(x))^3$. If $f(3) = -\frac{5}{2}$, find the slope of the line tangent to the graph of $g$ at $x = 3$.

\[
g(3) = (f(3))^3
\]
\[
-\frac{5}{2}^3 = -\frac{125}{8}
\]
\[
x = 3
\]
\[
\frac{125}{8} = y
\]
\[
y = \frac{125}{8} = 48(x-3)
\]
Question 4

Overview

This problem described a function $f$ that is defined and twice differentiable for all real numbers, and for which $f(8) = 4$. The graph of $y = f'(x)$ on $[0, 8]$ is given, along with information about locations of horizontal tangent lines for the graph of $f'$ and the areas of the regions between the graph of $f'$ and the x-axis over this interval. Part (a) asked for all values of $x$ in the interval $(0, 8)$ at which $f$ has a local minimum. Students needed to recognize that this occurs where $f'$ changes sign from negative to positive. Part (b) asked for the absolute minimum value of $f$ on the interval $[0, 8]$. Students needed to use the information about the areas provided with the graph, as well as $f(8)$, to evaluate $f(x)$ at $0$ and at the local minimum found in part (a). Part (c) asked for the open intervals on which the graph of $f$ is both concave down and increasing. Students needed to recognize that this is given by intervals where the graph of $f'$ is both decreasing and positive. Students were to determine these intervals from the graph. Part (d) introduced a new function $g$ defined by $g(x) = (f(x))^3$, and included that $f(3) = -\frac{5}{2}$. Students were asked to find the slope of the line tangent to the graph of $g$ at $x = 3$. Students needed to recognize that this slope is given by $g'(3)$. In order to determine this value, students needed to apply the chain rule correctly and read the value of $f'(3)$ from the graph.

Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: 1 point in part (a), 1 point in part (b), 2 points in part (c), and 2 points in part (d). In part (a) the student’s work is correct. In part (b) the student earned the point for considering $x = 0$ and $x = 6$. The student does not report a correct answer and was not eligible for the justification point. In part (c) the student’s work is correct. In part (d) the student earned the 2 points for $g'(x)$ but did not earn the answer point.

Sample: 4C
Score: 3

The student earned 3 points: 1 point in part (a) and 2 points in part (c). In part (a) the student’s work is correct. In part (b) the student does not consider $x = 0$ and $x = 6$, does not find the answer, and is not eligible for the justification point. In part (c) the student’s work is correct. In part (d) the student makes a chain rule error in the derivative and did not earn the answer point.