AP® CALCULUS BC
2012 SCORING GUIDELINES

Question 6

The function g has derivatives of all orders, and the Maclaurin series for g is

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots.$$

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

(a) $\left|\frac{x^{2n+3}}{2n+5} \cdot \frac{2n+3}{x^{2n+1}}\right| = \left(\frac{2n+3}{2n+5}\right) \cdot x^2$

$\lim\limits_{n\to\infty}\left(\frac{2n+3}{2n+5}\right) \cdot x^2 = x^2$

$x^2 < 1 \;\Rightarrow\; -1 < x < 1$

The series converges when $-1 < x < 1$.

When $x = -1$, the series is $-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

This series converges by the Alternating Series Test.

When $x = 1$, the series is $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots$

This series converges by the Alternating Series Test.

Therefore, the interval of convergence is $-1 \le x \le 1$.

5 :
- 1 : sets up ratio
- 1 : computes limit of ratio
- 1 : identifies interior of interval of convergence
- 1 : considers both endpoints
- 1 : analysis and interval of convergence

(b) $\left|g\left(\frac{1}{2}\right) - \frac{17}{120}\right| < \frac{\left(\frac{1}{2}\right)^5}{7} = \frac{1}{224} < \frac{1}{200}$

2 :
- 1 : uses the third term as an error bound
- 1 : error bound

(c) $g'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 + \cdots + (-1)^n\left(\frac{2n+1}{2n+3}\right)x^{2n} + \cdots$

2 :
- 1 : first three terms
- 1 : general term

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6. The function g has derivatives of all orders, and the Maclaurin series for g is
$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots.$

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)+1}}{2(n+1)+3}\cdot\frac{2n+3}{(-1)^n x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2+1}}{2n+2+3}\cdot\frac{2n+3}{(-1)^n x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)^{\cancel{n+1}}x^{2n+3}}{\cancel{2n+5}}\cdot\frac{\cancel{2n+3}}{\cancel{(-1)^n} x^{2n+1}}\right|$$

$$= |x^2| < 1$$

$$-1 < x < 1$$

end point

$x=-1$ $\quad (-1)^n\cdot\frac{(-1)^{2n+1}}{2n+3} = \frac{(-1)^{3n+1}}{2n+3}$

the series is alternating, and the absolute value of each term decreases to 0

$\therefore$ converges

$x=1$ $\quad (-1)^n\cdot\frac{1^{2n+1}}{2n+3} = \frac{(-1)^n}{2n+3}$

the series is alternating, and the absolute value of each term decreases to 0 $\therefore$ converges

$\therefore$ interval of convergence is $x\in[-1,1]$.

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(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

$\because$ The Maclaurin series for g evaluated at $x=\frac{1}{2}$ is an alternating Series whose decrease in absolute value to 0

$\therefore$ Error of using the first two nozero terms is smaller than the third term of the maclaurin series

third term: $x=\frac{1}{2}$

$$\frac{(\frac{1}{2})^5}{7} = \frac{1}{32}\cdot\frac{1}{7} = \frac{1}{224} < \frac{1}{200}$$

$\therefore$ the approximation differs from $g(\frac{1}{2})$ is less than $\frac{1}{200}$

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

$$g'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 + \cdots + (-1)^n \cdot (2n+1)\frac{x^{2n}}{2n+3}$$

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6. The function g has derivatives of all orders, and the Maclaurin series for g is

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots.$$

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

$$\lim_{n\to\infty}\left|\frac{x^{2(n+1)+1}}{2(n+1)+3}\cdot\frac{2n+3}{x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{x^2\cdot 2n+3}{2n+5}\right| = |x^2| < 1$$

$$-1 < x < 1$$

when $x=1$, $\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+3} = \frac{1}{3}-\frac{1}{5}+\frac{1}{7}\cdots$ Convergent

when $x=-1$, $\sum_{n=0}^{\infty}(-1)^n\frac{(-1)}{2n+3} = -\frac{1}{3}+\frac{1}{5}-\frac{1}{7}$. Convergent

So the interval of convergence of Maclaurin series for g is $-1 \le x \le 1$.

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NO CALCULATOR ALLOWED

(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

Suppose $a_n = (-1)^n \frac{x^{2n+1}}{2n+3}$, $x = \frac{1}{2}$

$a_n = (-1)^n \frac{(\frac{1}{2})^{2n+1}}{2n+3}$ $g(\frac{1}{2})$ is gained using the first two terms,

So difference $< |a_3| = \frac{(\frac{1}{2})^7}{9} = \frac{1}{2^7 \times 9} < \frac{1}{200}$

So this approximation differs from $g(\frac{1}{2})$ by less than $\frac{1}{200}$.

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

Maclaurin series for $g'(x)$

$$g'(x) = \frac{1}{3} - \frac{3x^2}{5} + \frac{5x^4}{7} + \cdots\cdots \frac{(-1)^n(2n+1)x^{2n}}{2n+3} + \cdots$$

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NO CALCULATOR ALLOWED

6. The function g has derivatives of all orders, and the Maclaurin series for g is

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots.$$

$2(n+1)+1$
$2n+2+1$

$2(n+1)+3$
$2n+2+3$

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

$$\lim_{n\to\infty}\left|\frac{x^{2n+3}}{2n+5}\cdot\frac{2n+3}{x^{2n+1}}\right| < 1$$

$$\lim_{n\to\infty}\frac{x^3+3}{x+5} < 1$$

$x^3+3 < x+5$

$x^3 - x < 2$

$x(x^2-1) < 2$

$x(x-1)(x+1) < 2$

$x=0 \quad x=1 \quad x=-1$

$-1 \leq x \leq 1$

$$\frac{(-1)^3+3}{-1+5} = \frac{2}{4} = \frac{1}{2}$$

$$\frac{(1)^3+3}{1+5} = \frac{4}{6} = \frac{2}{3}$$

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(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

(Error)

$$\begin{array}{r} 32 \\ \times\ 7 \\ \hline 224 \end{array}$$

$$g\left(\tfrac{1}{2}\right) = \frac{17}{120}$$

$$\frac{x}{3} - \frac{x^3}{5} = \frac{17}{120}$$

Proving $\frac{x^5}{7} = \frac{\left(\frac{1}{2}\right)^5}{7} = \frac{\frac{1}{32}}{7} = \frac{1}{224}$

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

Maclaurin:
$$g(x) = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7}$$

take derivative

$$g'(x) = \frac{1}{3} - \frac{1}{5}(3x^2) + \frac{1}{7}(5x^4)$$

$$g'(x) = \frac{1}{3} - \frac{3x^2}{5} + \frac{5x^4}{7}$$

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Question 6

Overview

This problem presented the Maclaurin series for an infinitely differentiable function g. Part (a) asked students to use the ratio test to determine the interval of convergence for the given Maclaurin series. Students should have observed that for $x = -1$ and $x = 1$, the resulting series is alternating with terms decreasing in absolute value to 0. Therefore, the series converges for $x = -1$ and $x = 1$. Part (b) asked students to show that the approximation for $g\left(\frac{1}{2}\right)$ obtained by using the first two nonzero terms of the series differs from the actual value by less than $\frac{1}{200}$. Because this is an alternating series with terms decreasing in absolute value to 0, students should have observed that the absolute value of the third term bounds the error and is strictly less than $\frac{1}{200}$. Part (c) asked the students to find the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$. Students should have computed the symbolic derivative of the first three nonzero terms and the general term of the series for $g(x)$.

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 4 points in part (a), no points in part (b), and 2 points in part (c). In part (a) the student sets up the ratio correctly, evaluates the limit, finds the interior of the interval of convergence, and considers the endpoints. The student does not provide a reason for the convergence, so the fifth point in part (a) was not earned. In part (b) the student does not use the third term as the error bound for the first two terms, so no points were earned. In part (c) the student's work is correct.

Sample: 6C
Score: 3

The student earned 3 points: 1 point in part (a), 1 point in part (b), and 1 point in part (c). In part (a) the student sets up the ratio correctly, so the first point was earned. In part (b) the student selects the third term as the error bound for the sum of the first two terms, evaluates the third term, but never states that the error is less than $\frac{1}{200}$. In part (c) the student correctly finds the first three terms but not the general term.