PROFESSIONAL DEVELOPMENT

AP® Calculus
Infinite Series

Special Focus

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Infinite Series In Calculus

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In the study of calculus, the topic of infinite series generally occurs near the end of the second semester in a typical two-semester sequence in single variable calculus. This seems to be one of the most difficult topics for students to understand and for teachers to explain clearly. It should not be surprising that these ideas are as difficult to grasp as the use of ε–δ proofs for limits, since these concepts—limits and infinite series—are related, and both took hundreds of years to formulate. Thus, even though now we have a better perspective from which to start, it is still much to ask of our students to gain a full understanding of infinite series in the two to four weeks given to their study in a beginning calculus course.

Some History of Infinite Series

Concepts surrounding infinite series were present in ancient Greek mathematics as Zeno, Archimedes, and other mathematicians worked with finite sums. Zeno posed his paradox in about 450 BCE, and Archimedes found the area of a parabolic segment in approximately 250 BCE by determining the sum of the infinite geometric series with constant ratio $1/4$ (Stillwell 1989, 170). One cannot credit Archimedes (or the Greeks) with discovering infinite series, since Archimedes worked only with finite sums and determined that certain finite sums underapproximated the area and others overapproximated the area, leaving the common limit (not known to him as a limit) as the area of the parabolic segment. However, this does point out one of the two motivations for the development of infinite series: (1) to approximate unknown areas, and (2) to approximate the value of π.

A nongeometric series appeared in *Liber calculationum* by Richard Suiseth, known as "The Calculator" (Stillwell 1989, 118), in approximately 1350. Suiseth indicated that $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{k}{2^k}+\cdots=2$. At about the same time, Nicole Oresme, a bishop in Normandy (Cajori 1919, 127), also found this infinite sum along with similar ones, and proved the divergence of the harmonic series using

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots>1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\cdots$$

However, in the same century, Madhava of Sangamagramma (c. 1340–1425) and fellow scholars of the Kerala school in southern India were making even more important discoveries about infinite series. Madhava, an astronomer and mathematician, used the now easily obtained series $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$ to estimate π, and, in addition, transformed this series into the more rapidly converging series $\sqrt{12}\left(1-\frac{1}{3\cdot 3}+\frac{1}{5\cdot 3^2}-\frac{1}{7\cdot 3^3}+\cdots\right)$ for π (Joseph, 290). More generally, he discovered what we now call the Taylor series for arctangent, sine, and cosine (Joseph 2000, 289–293; Katz 2004, 152–156). One theory is that these ideas may have been carried to Europe by Jesuit missionaries to India (Katz 2004, 156).

Moving to Europe, Portuguese mathematician Alvarus Thomas considered geometric series in 1509 (Cajori 1919, 172). Pietro Mengoli of Bologna treated particular infinite series in *Novæ quadraturae arithmeticae* in 1650, finding $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ along with proving the divergence of the harmonic series. In 1668, the theory of power series began with the publication of the series for $\ln(1+x)$ by Nicolaus Mercator, who did this by "integrating" $\frac{1}{1+x}$ (Stillwell 1989, 120).

Newton's general binomial theorem in 1665 aided the finding of series for many functions. Newton, Gregory, and Leibniz all used interpolation ideas to lead to their important results and went from finite approximations to infinite expansions. There is strong evidence that James Gregory, who was the first to publish a proof of the Fundamental Theorem of Calculus in 1668, used Taylor series in 1671, 44 years prior to Brook Taylor's results in 1715.

In 1734, Leonhard Euler gave new life to infinite series by finding that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, after attempts by Mengoli and the Bernoulli brothers had failed. In 1748, Euler used some ideas from his work in 1734 to generate what we now call the Riemann zeta function $\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}$. This function, named after Bernhard Riemann because he was the first to use complex numbers in the domain of this function, has become important because of its relationship to the distribution of prime numbers and holds the distinction today of pertaining to one of the most famous unsolved problems in mathematics, the Riemann Hypothesis.

During this time, issues of convergence of series were barely considered, which often led to confusing and conflicting statements concerning infinite series. The first important and rigorous treatment of infinite series was given by Karl Friedrich Gauss in his study of hypergeometric series in 1812 (Cajori 1919, 373). In 1816, Bernard Bolzano exhibited clear notions of convergence. Augustin-Louis Cauchy shared these ideas with the public in 1821 in his textbook *Cours d'analyse de l'École Polytechnique*. The ratio and root tests and the idea of absolute convergence were included in this text. Uniform convergence was studied in the middle of the nineteenth century, and divergent series were studied in the late nineteenth century.

Today, infinite series are taught in beginning and advanced calculus courses. They are heavily used in the study of differential equations. They are still used to approximate π as illustrated by the BBP (Bailey, Borwein, and Plouffe) formulas. One of these, discovered by Plouffe in 1995, gives the base 16-digit extraction algorithm for π using $\pi=\sum_{n=0}^{\infty}\left(\frac{4}{8n+1}-\frac{2}{8n+4}-\frac{1}{8n+5}-\frac{1}{8n+6}\right)\left(\frac{1}{16}\right)^n$.

This historical background would not be complete without mentioning Fourier series, which attempt to give values for a function using an infinite sum of trigonometric functions. These are named after Joseph Fourier, a scientist and mathematician who used them in *La Théorie Analytique de la Chaleur* in 1822 to study the conduction of heat.

You can see that the development of the concepts surrounding infinite series took a long time and involved many different mathematicians with many different ideas. It's not surprising that our students don't fully comprehend their nuances in just one semester of study.

The Heart of Infinite Series

At the heart of infinite series are three concepts:

(1) the definition of convergence of an infinite series,

(2) positive term series, and
(3) absolute convergence of series.

While absolute convergence does not appear specifically in the AP® syllabus, power series cannot be fully considered without this idea, nor can one take full advantage of the ratio and root tests.

Given students' difficulty with understanding the concept of infinite series, I believe we frequently "rush" to get to the tests for convergence and never really require students to fully understand the notion of convergence. More time should be spent with students computing partial sums and attempting to find the limits of these sequences of partial sums.

For positive term series, convergence of the sequence of partial sums is simple. Since for a positive term series the sequence of partial sums is nondecreasing, convergence of the sequence of partial sums occurs if and only if that sequence is bounded above. We probably should spend more time finding upper bounds for the sequence of partial sums of a positive term series or showing that there is no such upper bound. Nearly all of the convergence tests are founded on this one idea. The comparison test, limit comparison test, and integral test all lead directly to upper bounds for the sequence of partial sums or show that there is no such upper bound. Even the ratio and roots tests essentially are a limit comparison test with a geometric series, and show convergence if the comparison is with a geometric series whose common ratio has an absolute value of less than 1. Thus, the ratio and root tests are just formalized versions of a limit comparison test with a geometric series. Relating these tests back to upper bounds for the sequence of partial sums might help our students see the one common thread for all these tests.

For series that have both positive and negative terms, the idea of absolute convergence becomes helpful. If a series converges absolutely, then it must converge. While this idea is the one needed most frequently, our students sometimes fixate on the alternating series test, which is a very specialized test guaranteeing convergence of a particular type of infinite series. I believe we sometimes overemphasize the importance of this test because we want to make clear the distinction between absolute convergence and convergence. That is, we want to give examples of series that converge but do not converge absolutely. It is relatively difficult to do that without giving examples of series satisfying the hypotheses of the alternating series test. This is because it is difficult to show that a series not satisfying the hypotheses is convergent when it is not absolutely convergent. In fact, showing convergence of an arbitrary series can be quite difficult. No matter what rules we might develop to determine the convergence of a series, “a series can be invented for which the rule fails to give a decisive result.” (Bromwich 1965, 46). Thus, it is not surprising that infinite series is a difficult topic for our students.

Once students understand the concept of convergence and divergence of infinite series, there are two basic questions one can ask about a specific series:

(1) Does that series converge?
(2) If it converges, to what does it converge; and if it diverges, why?

We have already talked about the first question above. The second question really leads to the study of power series. Power series define functions on their intervals of convergence, and the challenge is to identify these functions. The other role of power series is to allow us to express a given function as a power series and then use that expression as a means to approximate the function with a polynomial function, which uses only simple arithmetic (addition and multiplication) to approximate functional values.

Infinite Series in AP® Calculus

The May 2008 syllabus for AP Calculus BC lists the following items:

Polynomial Approximations and Series

Concept of series

A series is defined as a sequence of partial sums, and convergence is defined in terms of the limit of the sequence of partial sums. Technology can be used to explore convergence and divergence.

Series of constants

- Motivating examples, including decimal expansion
- Geometric series with applications
- The harmonic series
- Alternating series with error bound
- Terms of series as areas of rectangles and their relationship to improper integrals, including the integral test and its use in testing the convergence of p-series
- The ratio test for convergence and divergence
- Comparing series to test for convergence or divergence

Taylor series

- Taylor polynomial approximation with graphical demonstration of convergence (for example, viewing graphs of various Taylor polynomials of the sine function approximating the sine curve)
- Maclaurin series and the general Taylor series centered at x = a
- Maclaurin series for the functions e^x, $\sin(x)$, $\cos(x)$, and $\frac{1}{1-x}$
- Formal manipulation of Taylor series and shortcuts to computing Taylor series, including substitution, differentiation, antidifferentiation, and the formation of new series from known series
- Functions defined by power series
- Radius and interval of convergence of power series
- Lagrange error bound for Taylor polynomials

These indicate both what students should know about infinite series and what they should be able to do with infinite series. It's possible to see these items being tested by examining those free-response questions that have appeared on the AP Calculus BC Exam over the years. The following table identifies questions pertaining to the particular ideas mentioned in the syllabus using the AP Calculus BC Exams from 1969 to the present. Questions can occur under multiple sections below.

Topic	**Exam Questions**
Geometric Series	1981-3a, 1981-3b, 1981-3c, 2001-6d, 2002-6c
Harmonic Series	1972-4b, 1975-4b, 1982-5b, 2002-6a, 2005-6c
Alternating Series Test	1970-6b, 1972-4b, 1975-4b, 1982-5b, 2002-6a
Alternating Series Error	1970-6c, 1971-4d, 1979-4c, 1982-5c, 1984-4c, 1990-5c, 1994-5c, 2000-3c, 2003-6b, 2006-6d, 2007-6d
Integral Comparisons	1969-7a
Integral Test	1969-7bc,1973-6c, 1992-6b
Ratio Test	1975-4a
*n*th Term Test (Divergence)	1973-6a
Comparison Test	1972-4a, 1973-6b, 1977-5a, 1980-3a, 1980-3c, 1992-6a, 1992-6c
Taylor Polynomials	1995-4c, 1997-2a, 1998-3a, 1999-4a, 2000-3a, 2004-6a, 2005-6a
Taylor Polynomial Approximation	1971-4c, 1984-4b, 1995-4a, 1995-4b, 1997-2d, 1998-3d
Lagrange Error Bound	1976-7c, 1999-4b, 2004-6c
Taylor or Maclaurin Series	1971-4a, 1982-5a, 1986-5a, 1990-5a, 1996-2a, 2003-6a, 2004-6b, 2005-6b
Radius of Convergence	1984-4a, 2000-3b

Interval of Convergence	1971-4b, 1972-4b, 1975-4b, 1976-7b, 1979-4b, 1982-5b, 1983-5b, 1987-4b, 1988-4, 1991-5c, 1994-5b, 2001-6a, 2002-6a, 2005-6c
Known Maclaurin Series	1976-7a, 1979-4a
Manipulation of Known Series	1982-5d, 1986-5b, 1987-4a, 1987-4c, 1991-5a, 1993-5a, 1993-5b, 1994-5a, 1996-2b, 1996-2c, 1998-3b, 1999-4c, 2001-6b, 2003-6c, 2006-6a, 2006-6b, 2007-6a, 2007-6b
Differentiation of Series	1983-5c, 1986-5c, 1993-5c, 1997-2b, 2002-6b, 2003-6c, 2006-6a
Antidifferentiation of Series	1990-5b, 1991-5b, 1997-2c, 1998-3c, 2001-6c, 2004-6d, 2006-6c, 2007-6c

The table above indicates that one of the most common types of questions involves the use of known series that are then modified through differentiation, integration, substitution (e.g., finding a Maclaurin series for $\sin(x^2)$ using the known series for $\sin(x)$), and/or algebraic manipulation. These are all questions that manipulate the known series specified by the course description or manipulate a series given in the stem or another part of the problem.

A second item commonly tested is that of interval of convergence. This most frequently involves using the ratio test to find the radius of convergence and then checking individually each of the endpoints of the interval generated by that radius of convergence to determine convergence or divergence. The endpoint checking has often involved the use of the alternating series test and knowledge of the convergence or divergence of well-known series, such as the harmonic series.

The third most common question involves the use of the alternating series error-bound theorem. The typical form for this question is to use the first several terms of a Taylor polynomial to approximate the value of a function at a point and then either ask the question of how much error could have been made or have the student show that the error made in the approximation is less than a specified amount.

In terms of the multiple-choice portion of the exam, the following types of questions were asked for the years when exams were released.

1997 Multiple-Choice Topics	**1998 Multiple-Choice Topics**	**2003 Multiple-Choice Topics**
Sum of a Geometric Series	Taylor Polynomial Approximation	Geometric Series
Sequence Limit Question	Convergence Tests—Several	Manipulation of Series
Taylor Polynomial	Integral Test	Manipulation of Series for e^x
Interval of Convergence	Differentiation of Series	nth Term Test and p Series
Differentiation of Series	Geometric and Alternating Series Test	Differentiation of Series and Taylor Series

	Calculator—Error in Approximation Question	Taylor Polynomials
	Interval of Convergence	
	Exponential Series	

In regard to teaching infinite series, one thing is certain: They should be taught in the context of giving a different way of expressing a function, of their use as approximating (Taylor) polynomials, and of the error made in that approximation—whether it be given by the Lagrange error bound or by the alternating series error theorem.

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Setting the Stage with Geometric Series

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One of the difficult things about teaching infinite series at the end of an AP Calculus BC course is trying to make the students see the topic as something other than a four-week detour down a side track just when the train ought to be coming into the station. It is particularly difficult to disabuse students of this notion if the teacher secretly shares their concern. That is why it is easier for both teacher and students if the emphasis is on functions from the beginning, and if calculus becomes part of the picture shortly thereafter.

There are good reasons to talk about series in an introductory calculus course, and it is helpful to keep them in mind when thinking about how to teach the topic.

Infinite series are another important application of limits. Moreover, as limits they are easier to understand than either the derivative or the integral because they do not involve those mysterious differentials.

The construction of Taylor series is not only a nice application of the derivative but also a nice review of such topics as linear approximation, slope, and concavity.

If a function can be represented by a power series, it can be differentiated or integrated as easily as a polynomial. This provides another approach to evaluating expressions like $\int e^{x^2}dx$ that would be difficult or impossible otherwise. Series figured prominently in the early history of calculus, so they ought to play some role in an introductory course.

Series figure prominently in higher-level analysis courses, so it is useful to lay the groundwork for our better students as soon as we can.

Notice that most of these good reasons for teaching series in a calculus course involve series as *functions*, not series as *infinite sums of numbers*. In fact, series of numbers are only important when considering the various tests for convergence. One school of thought is that students need to see all those tests before seeing their first power series, necessitating a detour away from functions and calculus that is difficult to motivate. A more intuitive (and historically faithful) approach is to play with power series from the beginning and see what can be done with them. The question of convergence eventually must be confronted, but in the meantime it is sufficient for students to know that convergence is an issue.

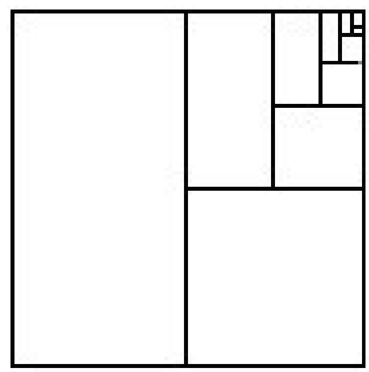
Happily, there is a way to introduce students to power series right away and simultaneously make them aware of the question of convergence, all while building on their existing knowledge of a previous topic: geometric series.

Geometric Series Basics

Even if students have not studied geometric series by name, they have encountered them in various convergent forms.

For example, a 1"-by-1" square can be cut into two halves. One half can then be cut into two quarters, one quarter can be cut into two eighths, and so on ad infinitum. This process of infinite subdivision, the basis of some of Zeno's ancient paradoxes, leads to the inevitable conclusion that

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\left(\frac{1}{2}\right)^n+\cdots=1 .$$



For another example, some well-known rational numbers have familiar decimal expansions that are actually convergent geometric series:

$$0.\overline{3}=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\cdots+3\left(\frac{1}{10}\right)^n+\cdots=\frac{1}{3}.$$

So infinite sums can converge to finite numbers, but obviously not all of them do. For example, $1 + 1 + 1 + \cdots$ is infinite, and $1-1+1-1+\cdots+(-1)^{n+1}+\cdots$ is at least ambiguous. The latter series shows the necessity of defining the *sum* of an infinite series as the limit of its sequence of partial sums.

There is a formula for the nth partial sum of a geometric series in which $r \neq 1$:

$$a+ar+ar^2+ar^3+\cdots+ar^{n-1}=\frac{a-ar^n}{1-r}.$$

(There are several ways to prove this, and most precalculus texts do.) The limit of the *n*th partial sum as n $\to\infty$ depends entirely on the fate of r^n, which goes to zero if and only if $|r|<1$. If $|r|\geq 1$, the limit diverges.

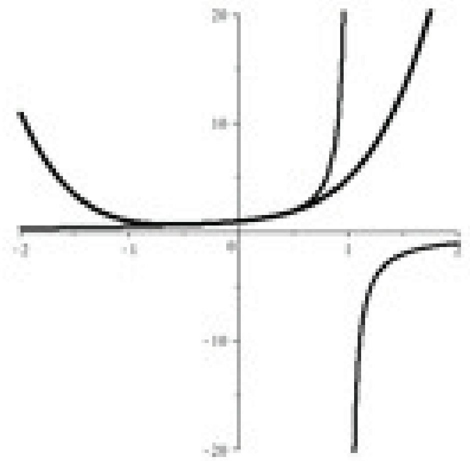
Thus, whether or not students have formally studied geometric series, they can be led very quickly to the realization that a series of the form $a+ar+ar^2+ar^3+\cdots+ar^{n-1}+\cdots$ converges if and only if $|r|<1$, in which case the sum is $\dfrac{a}{1-r}$.

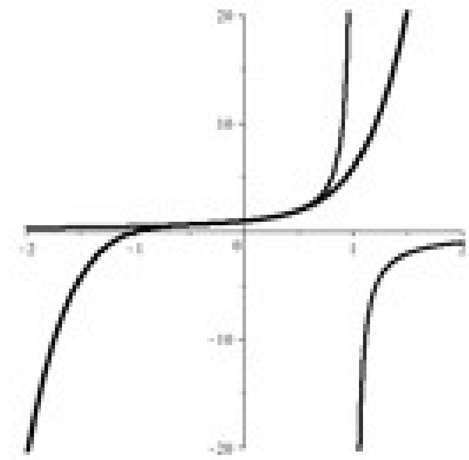
At this point, students are ready for *x*.

A Geometric Series for $f(x)=\dfrac{1}{1-x}$

If $|x|<1$, then $\dfrac{1}{1-x}$ is the sum of the geometric series $1+x+x^2+x^3+\cdots+x^{n-1}+\cdots$. The latter expression looks like a polynomial of infinite degree, but since there is no such thing we must give it a new name. We call it a *power series* (a series of powers of *x*). It is an interesting example of a function with domain (–1, 1), since we can technically plug in values of *x* outside the domain, but we get expressions like $1+2+2^2+2^3+\cdots$ that are simply meaningless. If, on the other hand, we plug in a value of *x inside* the domain, we get $\dfrac{1}{1-x}$. The interval (–1, 1) is called the *interval of convergence*.

The partial sums of a power series are polynomials. If we graph them, we get a dramatic visualization of why the interval of convergence matters, as in the graphs below of the fourth and fifth partial sums of $1+x+x^2+x^3+\cdots+x^{n-1}+\cdots$ compared to the graph of $\dfrac{1}{1-x}$:





The quartic and quintic polynomial partial sums do a very good job of approximating $\frac{1}{1-x}$ in the interval of convergence (–1, 1), but outside that interval they are not even close. The partial sums of higher degree approximate the curve progressively more closely on the interval of convergence, but with no better success outside that interval.

Exploring the Implications

- Find a power series to represent $\frac{1}{1+x}$ and give its interval of convergence.

  Solution: We use a geometric series with first term 1 and ratio $-x$.

  $$1-x+x^2-x^3+\cdots+(-1)^{n-1}x^{n-1}+\cdots.$$

  It converges for $|-x|<1$, so the interval of convergence is (–1, 1).

- Find a power series to represent $\frac{1}{1+x^2}$ and give its interval of convergence

  Solution: We use a geometric series with first term 1 and ratio $-x^2$.

  $$1-x^2+x^4-x^6+\cdots+(-1)^{n-1}x^{2n-2}+\cdots.$$

  It converges for $\left|-x^2\right|<1$, so the interval of convergence is (–1, 1).

- Find a power series to represent $\tan^{-1}(x)$ and give its interval of convergence.

  Solution: We just found a series to represent $\frac{d}{dx}\left(\tan^{-1}(x)\right)=\frac{1}{1+x^2}$:

  $$1-x^2+x^4-x^6+\cdots+(-1)^{n-1}x^{2n-2}+\cdots \quad .$$

The interval of convergence of this series is (–1, 1).

- For x in the interval of convergence, we can write the equation:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^{n-1} x^{2n-2} + \cdots$$

The antiderivatives should then differ by a constant:

$$\tan^{-1}(x) + C = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots$$

Now we can find the constant by letting $x = 0$:

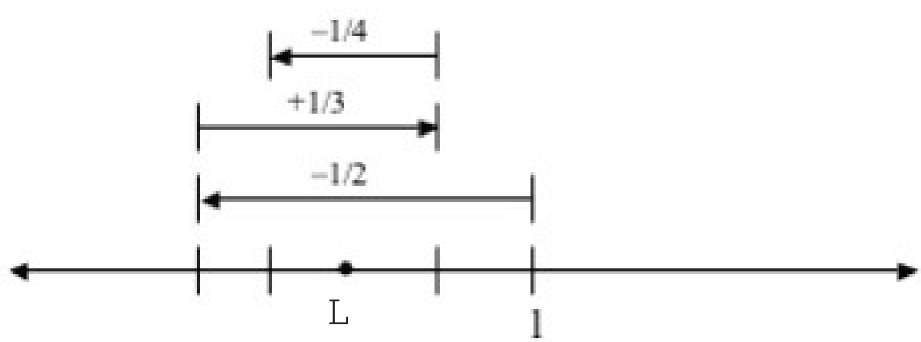
$$\begin{aligned} \tan^{-1}(0) + C &= 0 - \frac{0^3}{3} + \frac{0^5}{5} - \frac{0^7}{7} + \cdots + (-1)^{n-1}\frac{0^{2n-1}}{2n-1} + \cdots \\ 0 + C &= 0 \\ C &= 0 \end{aligned}$$

So $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots$.

We expect this equation at least to be valid on the interval (–1, 1), but in fact we have also picked up convergence at the endpoints. For example, plugging $x = 1$ into the series yields

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{n-1}\frac{1}{2n-1} + \cdots .$$

This is an example of an *alternating* series of terms of *diminishing magnitude* that *tend to a limit of 0*. Such series always converge. (This is the alternating series test.) Students can easily be convinced of this fact by tracking the partial sums of this series on a number line:



The partial sums bounce back and forth on the number line, but because each added or subtracted term is smaller than the previous one, they get closer to a limit in the process. Since the terms tend to zero, there is a limit L that becomes the sum.

It seems logical (and it can be proved using a deeper understanding of limits) that the number to which $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots+(-1)^{n-1}\frac{1}{2n-1}+\cdots$ converges ought to be $\tan^{-1}(1)$, which we know to be $\frac{\pi}{4}$. This series can, in fact, be used to compute π, but it is so close to the threshold of divergence that it converges too slowly to be of much practical value. (Students can convince themselves of this fact by trying the partial sums on their calculators. You know that some of them will.)

Beginning with geometric series, you will have thus exposed students to the idea that functions can be represented by power series, that such series have intervals of convergence for which such representations are valid, and that series can be manipulated using calculus to yield new functions. You will have shown them power series that they will eventually recognize as the Maclaurin series for three different functions (along with their intervals of convergence), and you will even have exposed them to a valuable convergence test for series of constants. In all likelihood you will have been able to accomplish this during a single class period.

Reaping the Benefits

Once students have seen that a power series can represent a function on some interval of convergence, they can discover some significant results on their own. Here are just two examples.

Challenge: Consider the function f defined by the infinite series

$$f(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots.$$

(a) Find $f(0)$.

(b) Find $f'(x)$. What is interesting about it?

(c) What can you conclude about the function f?

Students will easily guess that the function is $f(x)$ - e^x, but only the best of them will recognize that they have the information required to conclude that *$f(x)$ must be e^x.* In part (b) they discover the differential equation $f'(x) = f(x)$, and in part (a) they discover the initial condition $f(0) = 1$. The unique solution to this initial value problem is $f(x) = e^x$.

Challenge: Construct a fifth-degree polynomial P such that:

$$P(0) = 2$$
$$P'(0) = 3$$
$$P''(0) = 5$$
$$P'''(0) = 7$$
$$P^{(4)}(0) = 11$$
$$P^{(5)}(0) = 13$$

Students will usually succeed at this, and in the process they will see that the coefficient of x^n must be $\dfrac{P^{(n)}(0)}{n!}$. You are then ready to ask them to build a polynomial whose first n derivatives match the first n derivatives of a function f at 0 (the sine function, for example). The coefficient of each x^n will then be $\dfrac{f^{(n)}(0)}{n!}$. Your students will have discovered Maclaurin series!

Challenge: BC-3 from the 1981 AP Examination.

Let S be the series $S = \sum_{n=0}^{\infty} \left(\dfrac{t}{1+t}\right)^n$ where $t \neq 0$.

(a) Find the value to which S converges when $t = 1$.
(b) Determine the values of t for which S converges. Justify your answer.
(c) Find all values of t that make the sum of the series S greater than 10.

Students are ready for this as soon as they understand geometric series. Curiously, as they learn more about series they become overqualified to solve something this simple.

Challenge BC-3 Solution

(a) When $t = 1$, $S = \sum_{n=0}^{\infty} \left(\dfrac{1}{2}\right)^n$. This is a geometric series with first term $a = 1$ and common ratio $r = \dfrac{1}{2}$. Hence $S = \dfrac{a}{1-r} = \dfrac{1}{\left(1-\dfrac{1}{2}\right)} = 2$.

(b) The series will converge if and only if $\left|\dfrac{t}{1+t}\right| < 1$ where $t \neq 0$. This will be true for $t > 0$ and for $-\dfrac{1}{2} < t < 0$.

(c) Since the series is a geometric series with first term $a = 1$ and common ratio

$r = \dfrac{t}{t+1}$, we will have $S = \dfrac{a}{1-r} = \dfrac{1}{\left(1 - \dfrac{t}{1+t}\right)} = \dfrac{1+t}{(1+t-t)} = 1+t$.

Thus $S > 10$ when $t > 9$.

Once your students have grasped the general idea of what series are and how they behave, they will be ready to tackle the rest of the topics for infinite series—eventually even those other tests for convergence!

Convergence of Taylor and Maclaurin Series

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Instructional Unit Overview

Focus: How to determine the radius and interval of convergence of a Taylor or Maclaurin series

Audience: AP Calculus BC students

In this three-day unit students are introduced to the idea of the interval and radius of convergence of a Maclaurin or Taylor series. The first lesson has students determine this interval visually and by checking suspected endpoints of the interval. Students start by considering the series for $\sin(x)$ and also the series for $\frac{1}{1-x}$ since this expression can be viewed as the sum of a geometric series. Next they investigate a series that is not geometric but appears to have a limited interval of convergence. In the process of exploring this series, the harmonic series is introduced along with the integral test. In a homework problem, students will encounter an alternating harmonic series and need to reason out its convergence. An investigation is also provided that leads students through much of this material independently or in a small-group setting rather than in a large class format.

The second lesson introduces the ratio test and formalizes the alternating series test. Time is spent practicing these tests on portions of previous AP free-response problems. An investigation is provided that asks students to explore the ratio of the terms of a series in the limit and generalize the ratio test from their results.

The third lesson addresses some interesting questions about series, such as "Does a Taylor series always converge to the function from which it was constructed?" "Can a series converge to more than one function?", and "Is every series that converges to a function the Taylor series for that function?" Reflecting on questions like these and extending beyond the normal range of material helps students gain perspective on the larger picture.

Assumed skills and knowledge: Students should be able to use a calculator or computer to graph a function and evaluate it at a point. They should understand the meaning of convergence and divergence for an infinite series. Students should know the formula for the sum of a geometric series. Students should be able to construct a Taylor or Maclaurin series for a given function. They should know the Lagrange form of the error and be able to use it to estimate the accuracy of an approximation made with a Taylor polynomial.

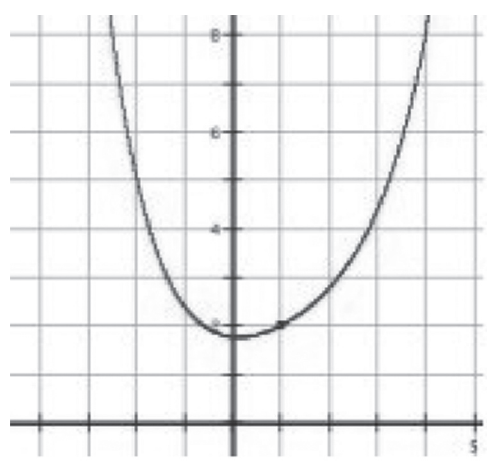
Background information on Taylor series: When finding the Taylor series for a function, it is important to determine for what values of x the series approximates the generating function. Finding this interval of convergence is sometimes a very straightforward procedure. Often the greatest challenges come in determining convergence at the endpoints of the interval.

The Taylor series for a function about $x = a$ is constructed according to the formula

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \cdots.$$

Students should recognize the first two terms of this series as the linear approximation for a function, or the equation of the line tangent to $f(x)$ at $x = a$. I like to check the students' understanding of the information presented in the first few terms of the series by using exercises like the following:

1. If the Taylor series is constructed for the function shown at right, centered at the point (1, 3), what can be determined about the coefficients of the first three terms of the series?



Solution: The first term is 2, the value of $f(1)$. The coefficient of the second term is positive because the first derivative or slope of the tangent is positive. The coefficient of the third term is also positive because the function is concave up near the point (1, 2).

2. The third-degree Taylor polynomial T about $x = 1$ for a function, $f(x)$, is $T(x) = 3 + 2(x-1)^2 - 4(x-1)^3$. What information do you know about $f(x)$?

Solution: From the first term you know that $f(1)=3$. There is no linear term so the first derivative at the point (1, 3) is 0, and this is a critical point. The coefficient of the second-degree term is positive, so the function is concave up near the point (1, 3), making this critical point a minimum. The second derivative has a value of 4 and the third derivative has a value of -24.

Other problems that test students' understanding of the construction of Taylor series give an expression for the nth derivative of the function. In student work one common error is to neglect to include the factorials in the denominators. A nice example of this sort of problem comes from the 2005 AP Calculus BC Exam, question 6.

2005 BC#6

Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the nth derivative of f at $x = 2$ is 0. When n is even and $n \geq 2$, the nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \dfrac{(n-1)!}{3^n}$.

a. Write the sixth-degree Taylor polynomial for f about $x = 2$.

Solution: $P_6(x) = 7 + \dfrac{1!}{3^2}\cdot\dfrac{1}{2!}(x-2)^2 + \dfrac{3!}{3^4}\cdot\dfrac{1}{4!}(x-2)^4 + \dfrac{5!}{3^6}\cdot\dfrac{1}{6!}(x-2)^6$

Students must pay close attention to the fact that the odd derivatives are 0. They must also realize that the sixth-degree Taylor polynomial includes terms up to and including $(x-2)^6$, and not necessarily six terms.

b. For the Taylor series for f about $x = 2$, what is the coefficient of $(x-2)^{2n}$ for $n \geq 1$?

Solution: $\dfrac{(2n-1)!}{3^{2n}}\cdot\dfrac{1}{(2n)!} = \dfrac{1}{3^{2n}(2n)}$

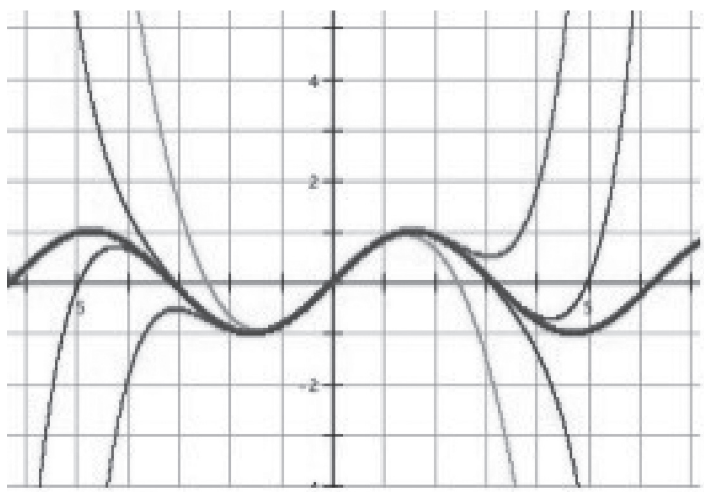
Students can either substitute $2n$ for n in the general term of the series, or observe the pattern in the terms of the series with powers of $2 \cdot 0$, $2 \cdot 1$, $2 \cdot 2$, and $2 \cdot 3$, then create a general term based on that pattern. Once again they must be careful to remember to include the factorial in the denominator and not just state the value of the derivative.

c. Find the interval of convergence of the Taylor series for f about $x = 2$. Show the work that leads to your answer.

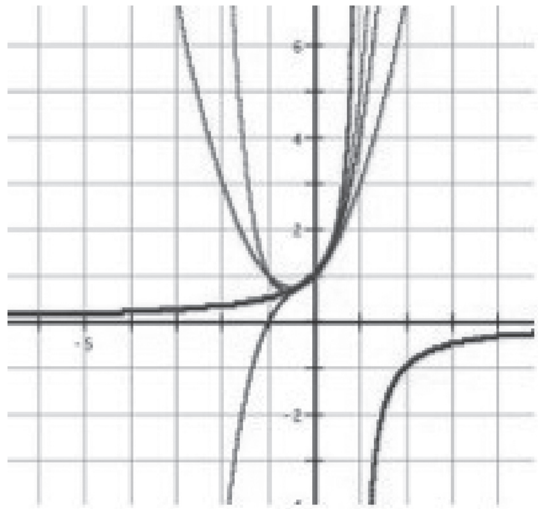
The first two parts of this question deal with constructing the Taylor polynomial and series, and can be done before studying the interval of convergence. The problem of determining the interval of convergence is the focus of the next three lessons.

Day 1: Introducing the Interval of Convergence

To introduce the idea of the interval of convergence I have students look at the Taylor series for $f(x) = \sin(x)$ and graph partial sums on their calculators. We then discuss how it appears that as one adds more terms of the series, the partial sums "fit" the function better and better. I ask them if they continued this forever, would the series fit the entire function? Or is there some x value beyond which the series just won't fit, no matter how many terms are added on? How do they know? This usually prompts a good discussion and provides a motivation for looking for these x values where the series "fits" or converges to the function.

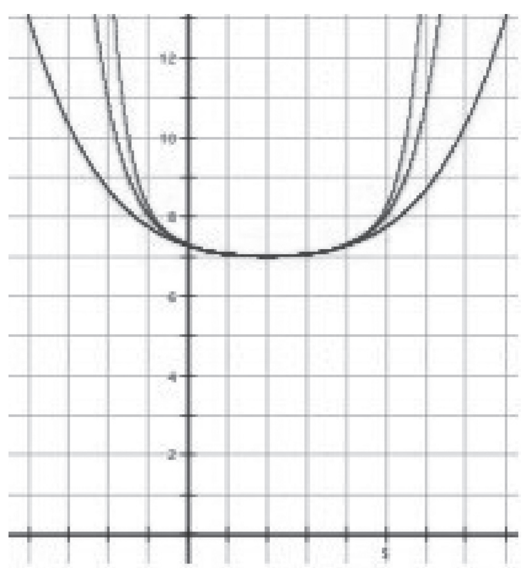


Next I like to introduce a second example that has a limited interval of convergence, for instance, the function $g(x) = \dfrac{1}{1-x}$, and look at several of its partial sums. In this case it becomes readily apparent that this series is a good fit only in a small interval. Students easily see that it is not good for any approximations when x is greater than 1, and they may also guess the lower bound of the interval to be –1. I ask them why this happens, and since we have discussed infinite geometric series previously, we can usually come to the realization that this function can be thought of as the sum of a geometric series with first term 1 and ratio x. They then make the connection that it converges only when $|x| < 1$.

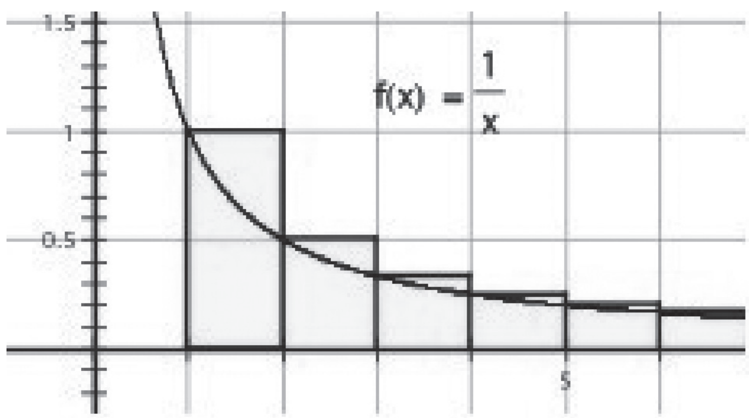


The next issue we consider is what happens if the series doesn't appear to converge for all values of x but is not geometric. How can we determine the interval of convergence?

We are now ready to return to part (c) of 2005 BC, #6. If we take our answer from part (b) and write the general term of the series we get $\frac{1}{3^{2n}(2n)} \cdot (x-2)^{2n}$. Or by rearranging a bit we can write it as $\frac{1}{2n} \cdot \left(\frac{x-2}{3}\right)^{2n}$. In this form it looks like some relative of a geometric series. We look at the graphs of some of the partial sums and try to make an estimate as to where the series will converge. Based on the graphs students usually agree that the series converges for $x = 3$ and $x = 4$, but might wonder about $x = 5$ and anything greater. So we try out one or more of these x values to see what the series looks like if we substitute them in place of x.



For instance, if $x = 3$ the general term becomes $\frac{1}{2n} \cdot \left(\frac{1}{3}\right)^{2n}$. The second factor here is the nth term of a convergent geometric series. Students can then reason that the first factor is less than 1 and thus makes the terms of the actual series smaller than those of the convergent geometric series. So the series converges when $x = 3$. I have students investigate other values of x. They quickly realize that for $x > 5$ the geometric part of this series diverges and the first factor probably doesn't compensate for this divergence. A good discussion usually surrounds the case of $x = 5$. In this case the first factor essentially becomes one-half the nth term of the harmonic series. I like to encourage this discussion as students wrestle with this important series. Eventually we are led to looking at the series graphically and thinking about the integral associated with the function $f(x) = \frac{1}{x}$.



The areas of the rectangles represent the terms of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. The sum of their areas is clearly larger than the area under the curve. We have studied this function and the improper integral $\int_1^{\infty} \frac{1}{x}\,dx$ and know that it is divergent. So it is logical that the harmonic series is also divergent. We often have to go back and redo this integral, but since this is such an important result, I think the time spent is worth it. Students are now ready to state the upper bound of the interval of convergence as 5 (with divergence at $x = 5$). By similar reasoning they realize that the series will diverge if $x < -1$. The next question is whether it converges at $x = -1$. Because of the $2n$ in the exponent, the series is identical to that when $x = 5$ so it diverges here as well.

At this point I like students to try these ideas on their own, and I ask them to investigate the Taylor series for $\cos(x)$ along with another series such as $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n \cdot 3^n}$ for their homework. They should be able to use the graphs of the partial sums and their reasoning about geometric series, along with the harmonic series to find the intervals of convergence for these series. In $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n \cdot 3^n}$, one endpoint of the interval creates an alternating series. I ask students to think about this and try to reason out on their own whether it will converge. They usually decide that if the partial sums are oscillating and those oscillations are getting smaller, the series will converge. This informal understanding of the ideas behind the alternating series test helps them when we study it formally later. They are also motivated to learn the ratio test the next day in order to make this search for the interval of convergence a bit easier.

Day 2: A Procedure for Determining the Interval of Convergence

During the next class we look at the ratio test and practice its use. I do not formally prove the test, but rather present it with the sort of informal reasoning outlined below.

The Ratio Test

Suppose the limit $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ either exists or is infinite.

Then

a. If $L < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges.

b. If $L > 1$, the series $\sum_{n=1}^{\infty} a_n$ diverges.

c. If $L = 1$, the test is inconclusive.

If $L < 1$, this means that successive terms are getting larger and it is logical that the series diverges.

If $L < 1$, this means that successive terms are getting smaller and it is possible that the series converges. Look a bit more closely at the terms of a power series $\sum_{n=0}^{\infty} c_n x^n$, where $a_n = c_n x^n$. The power on x increases regularly. This is like a geometric series. Of course, the coefficients don't always change by the same ratio, but we can still think of this as a sort of cousin to a geometric series. The ratio formed in the limit is less than some number *r* for a geometric series. Since the limit is less than 1, you can pick a number for *r* that is also less than 1 and above the ratio $\frac{a_{n+1}}{a_n}$. So our series is less than this convergent geometric series and it makes sense that if $L < 1$, the series will converge.

When $L = 1$ the result is inconclusive. In the harmonic series, $L = 1$ and the series diverges. However, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ has $L = 1$ again, but this series converges. So if $L = 1$, we can't say whether the series converges.

We go back and apply the ratio test to part (c) from 2005 AP Calculus BC, #6, and get the following:

$$\lim_{n\to\infty}\left|\frac{\frac{1}{2(n+1)}\cdot\frac{1}{3^{2(n+1)}}(x-2)^{2(n+1)}}{\frac{1}{2n}\cdot\frac{1}{3^{2n}}(x-2)^{2n}}\right| = \lim_{n\to\infty}\left|\frac{2n}{2(n+1)}\cdot\frac{3^{2n}}{3^{2n}3^2}\cdot(x-2)^2\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\cdot\frac{(x-2)^2}{9}\right| = \frac{(x-2)^2}{9}.$$

The series converges when this limit is less than 1:

$$\frac{(x-2)^2}{9} < 1 \text{ or } (x-2)^2 < 9 \text{ or } |x-2| < 3.$$

Written this way we can see the set of solutions to this inequality as an interval centered at 2 with a radius of 3. Here ideas of function transformations can help a student to understand the relationship of an inequality like this to the parent inequality $|x| < 1$. The radius of this interval is called the **radius of convergence**. In this case, $-1 < x < 5$ is a subset of the interval of convergence. In order to complete the analysis of this **interval of convergence**, the behavior at the endpoints must be investigated. After the hard work of graphing partial sums and checking values, students find using the ratio test to be much simpler and more direct. Checking the endpoints then becomes our challenge, and we review the work we did the previous day.

For $x = -1$, the series becomes $7+\sum_{n=1}^{\infty}\frac{(-3)^{2n}}{2n\cdot 3^{2n}} = 7+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$. This series is the harmonic series and it diverges.

For $x = 5$, the series becomes $7+\sum_{n=1}^{\infty}\frac{3^{2n}}{2n\cdot 3^{2n}} = 7+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$. This series diverges because of its relationship to the harmonic series. Thus the interval of convergence is $-1 < x < 5$.

Since quite often when evaluating the endpoints of an interval, the series can be compared to a harmonic or alternating harmonic series, it is now appropriate to formalize the alternating series test.

Alternating Series Test

If a series is of the form $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$, where all $a_n > 0$ and $0 < a_{n+1} < a_n$ for all n greater than some integer N, and $\lim_{n\to\infty} a_n = 0$, then the series converges.

When the hypotheses are not satisfied, other convergence tests must be used. See the article "Overview of Tests for Convergence of Infinite Series" for more details on this topic.

I next like to work through several problems where students have the opportunity to put these tests into practice.

2000 BC, #3

The Taylor series about $x = 5$ for a certain function f converges to $f(x)$ for all values of x in the interval of convergence. The nth derivative of f at $x = 5$ is given by

$f^{(n)}(5) = \frac{(-1)^n n!}{2^n(n+2)}$, and $f(5) = \frac{1}{2}$.

a. Write the third-degree Taylor polynomial for f about $x = 5$.

Solution: The coefficients of the polynomial are given by $\frac{f^{(n)}(5)}{n!} = \frac{(-1)^n}{2^n(n+2)}$.

using $n = 1, 2,$ and 3 creates $P_3(x) = \frac{1}{2} - \frac{1}{6}(x-5) + \frac{1}{16}(x-5)^2 - \frac{1}{40}(x-5)^3$.

b. Find the radius of convergence of the Taylor series for f about $x = 5$.

Solution: Using the ratio test

$$\lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}(x-5)^{n+1}}{2^{n+1}(n+3)}}{\dfrac{(-1)^{n}(x-5)^{n}}{2^{n}(n+2)}}\right| = \lim_{n\to\infty}\left|\frac{-1(n+2)(x-5)}{2(n+3)}\right| = \lim_{n\to\infty}\frac{1}{2}\cdot\frac{(n+2)}{(n+3)}|x-5| = \frac{|x-5|}{2}$$

If this limit is less than 1, the series will converge: $\frac{|x-5|}{2} < 1$ or $|x-5| < 2$.

From here we can see the interval of convergence is centered at 5 with a radius of 2. The problem did not ask for the actual interval, but it's good practice to go ahead and work this out. So far we have the interval $3 < x < 7$.

Now check the endpoints.

If $x = 3$, the series becomes $\sum_{n=0}^{\infty}\frac{(-1)^n(-2)^n}{2^n(n+2)} = \sum_{n=0}^{\infty}\frac{(2)^n}{2^n(n+2)} = \sum_{n=0}^{\infty}\frac{1}{(n+2)}$. It diverges because of its relationship to the harmonic series.

If $x = 7$, the series becomes $\sum_{n=0}^{\infty}\frac{(-1)^n(2)^n}{2^n(n+2)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+2)}$, which is an alternating harmonic series and thus converges.

So the interval of convergence is $3 < x \le 7$.

c. Show that the sixth-degree Taylor polynomial for f about $x = 5$ approximates $f(6)$ with error less than $\frac{1}{1000}$.

Solution: Since x = 6 is in the interval of convergence, we can use the series to approximate the value of $f(6)$. But how good is our approximation? If students substitute $x = 6$ in the sixth-degree polynomial, it will look like this:

$$f(6) \approx \frac{1}{2} - \frac{1}{6} + \frac{1}{16} - \frac{1}{40} + \frac{1}{96} - \frac{1}{224} + \frac{1}{512} = \frac{6787}{17920} \approx 0.3787.$$

Ask students if this is more or less than the sum of the entire series. By looking at the signs they should realize that this is more than the actual sum. Ask students how much the next term will change this total. Since the next term is subtracted, it will decrease the sum by $\frac{1}{1152}$. Then ask them to think about how close they were to the actual sum. Since decreasing the sum by $\frac{1}{1152}$ causes the total to go from too big to too small, you were less than this far from the actual sum. Since this is less than the requested accuracy, the error in this approximation is within $\frac{1}{1000}$ of the actual value. This sort of reasoning should lead students to the realization that when a series satisfies the conditions of the alternating series test, the error in any partial sum is less than the next term.

2002 BC, #6

The Maclaurin series for the function *f* is given by

$$f(x)=\sum_{n=0}^{\infty}\frac{(2x)^{n+1}}{n+1}=2x+\frac{4x^2}{2}+\frac{8x^3}{3}+\frac{16x^4}{4}+\ldots+\frac{(2x)^{n+1}}{n+1}+\ldots$$

on its interval of convergence.

a. Find the interval of convergence of the Maclaurin series for *f*. Justify your answer.

Parts (b) and (c) deal with manipulation of Taylor and Maclaurin series, which is covered in the "Manipulation of Power Series" section of this Special Focus.

Solution:

Apply the ratio test $\lim\limits_{n\to\infty}\left|\dfrac{\frac{(2x)^{n+2}}{n+2}}{\frac{(2x)^{n+1}}{n+1}}\right|=\lim\limits_{n\to\infty}\left|\dfrac{(n+1)}{(n+2)}2x\right|=|2x|$.

This indicates the series converges if $-\frac{1}{2}<x<\frac{1}{2}$. But what about the endpoints?

If $x=-\frac{1}{2}$, the series becomes $\sum\limits_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}$, an alternating harmonic series, so it converges.

If $x=\frac{1}{2}$, the series becomes $\sum\limits_{n=0}^{\infty}\frac{1}{n+1}$, the harmonic series, so it diverges.

Thus the interval of convergence is $-\frac{1}{2}\le x<\frac{1}{2}$.

After discussing these problems I present students with a problem like 2005 BC, Form B, #3. This is a nice problem that ties together understanding the meanings of the first several terms of the Taylor series, constructing the series from a definition of the derivatives and finding the radius of convergence. Doing this as a homework assignment helps students to discover just where they have questions on this process. In class we also work to find the actual interval of convergence to make the problem a complete summary of the work done so far. Evaluating convergence at the endpoints is not as easy here as in the problems presented thus far, and it will take some work together as a group to come to the right conclusions and justifications.

2005 BC, #3

The Taylor series about $x = 0$ for a certain function f converges to $f(x)$ for all values

of x in the interval of convergence. The nth derivative of f at $x = 0$ is given by

$f^{(n)}(0) = \dfrac{(-1)^{n+1}(n+1)!}{5^n (n-1)^2}$ for $n \geq 2$. The graph of f has a horizontal tangent line at $x = 0$ and $f(0) = 6$.

a. Determine whether f has a relative maximum, a relative minimum, or neither at $x = 0$. Justify your answer.

Solution: By the second derivative test, f has a relative maximum at $x = 0$ because $f'(0) = 0$ (due to the horizontal tangent) and $f''(0) < 0$.

b. Write the third-degree Taylor polynomial for f about x = 0.

Solution: Using the given information

$$f(0) = 6, f'(0) = 0, f''(0) = -\frac{3!}{5^2 1^2} = -\frac{6}{25}, f'''(0) = \frac{4!}{5^3 2^2} = \frac{6}{125}$$

$$P(x) = 6 - \frac{3}{25}x^2 + \frac{1}{125}x^3$$

c. Find the radius of convergence of the Taylor series for f about $x = 0$. Show the work that leads to your answer.

Solution: Using the ratio test:

$$\lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+2}(n+2)}{5^{n+1}n^2}x^{n+1}}{\dfrac{(-1)^{n+1}(n+1)}{5^n(n-1)^2}x^n}\right| = \lim_{n\to\infty}\left|\frac{-1(n+2)(n-1)^2 x}{5(n+1)(n)^2}\right| = \lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)\left(\frac{n-1}{n}\right)^2\frac{|x|}{5} = \frac{|x|}{5}$$

The series converges if $|x| < 5$ or $-5 < x < 5$. The radius of convergence is 5. But what about those pesky endpoints?

If $x = -5$, the series becomes

$$6+\sum_{n=2}^{\infty}\frac{(-1)^{n+1}(n+1)(-5)^n}{5^n(n-1)^2}=6+\sum_{n=2}^{\infty}\frac{(-1)(n+1)}{(n-1)^2}=6-\sum_{n=2}^{\infty}\frac{(n+1)}{(n-1)^2}.$$

Because the numerator of each term of this series is one degree less than the degree of the denominator, this series is reminiscent of the harmonic series. Note that $\frac{n+1}{(n-1)^2} > \frac{n-1}{(n-1)^2} = \frac{1}{n-1}$, the nth term of a shifted harmonic series, which is divergent. So the series $\sum_{n=2}^{\infty}\frac{(n+1)}{(n-1)^2}$ diverges by comparison, and thus so does the series $6-\sum_{n=2}^{\infty}\frac{(n+1)}{(n-1)^2}$.

If $x = 5$, the series becomes $6+\sum_{n=2}^{\infty}\frac{(-1)^{n+1}(n+1)(5)^n}{5^n(n-1)^2}=6+\sum_{n=2}^{\infty}\frac{(-1)^{n+1}(n+1)}{(n-1)^2}$.

In this case, the series alternates and satisfies the conditions of the alternating series test, so it converges. The interval of convergence is then $-5 < x \leq 5$.

Day 3: Some Interesting Questions About Convergence

Students are now familiar with constructing Taylor series and identifying the radius of convergence. The verification of endpoint convergence is something we will continue to work on as we develop the convergence tests more fully. However, it is helpful to take some time to discuss questions about convergence that may not have otherwise occurred to students. We have seen that the series for $\sin(x)$ converges for all values of x, but how do we know it converges to the function $\sin(x)$ and not something else? This question may seem silly at first to students, but it is worth examining.

It is possible for a series to converge and yet not converge to the function used to construct it. In order to show that a series does indeed converge to the original function, we must examine what happens to the error.

> If f and all its derivatives are continuous and $P_n(x)$ is the nth degree Taylor approximation to $f(x)$ about $x = a$, then the error,
>
> $\left|E_n(x)\right| = \left|f(x) - P_n(x)\right| \le \dfrac{M}{(n+1)!}\left|x-a\right|^{n+1}$, where M is the maximum value of $f^{(n+1)}$ on the interval between x and a.

Applying this formula to $f(x) = \sin(x)$ and its series expansion

$$\left|E_n(x)\right| = \left|\sin x - x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right| \le \frac{1}{(n+1)!}\left|x\right|^{n+1}, \text{ where}$$

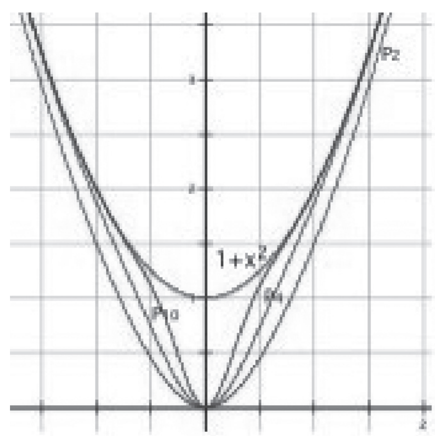
we use $M = 1$ because all derivatives of $f(x) = \sin(x)$ have a maximum value of 1, and $a = 0$ because the series is centered at 0. If this error has a limit of 0, then the series converges to the function $f(x) = \sin(x)$ for all values of x in the interval of convergence. This means we must show that $\lim\limits_{n\to\infty} \dfrac{\left|x\right|^{n+1}}{(n+1)!} = 0$. Using a calculator and experimenting with various values of x and large values of n provides evidence that this limit statement is true. To consider the limit more analytically, select a positive integer N such that $N > 2\left|x\right|$. Note that for $n \ge N$, $\dfrac{\left|x\right|}{n} < \dfrac{1}{2}$, the expression in the limit contains factors such as $\dfrac{|x|^{n+1}}{(n+1)!} \cdot \dfrac{|x|}{(n+2)} \cdot \dfrac{|x|}{(n+3)} \cdot \ \ldots$. For $n \ge N$, each factor of $\dfrac{|x|}{(n+k)}$ in the expression will be less than $\dfrac{1}{2}$. So for $n \ge N$ we can state $\left|\dfrac{x^{n+1}}{(n+1)!}\right| \le \dfrac{|x|^N}{N!} \cdot \dfrac{|x|}{N+1} \cdot \ldots \cdot \dfrac{|x|}{n} \le \dfrac{|x|^N}{N!} \cdot \dfrac{1}{2^{n-N}}$. Thus the limit will go to 0 as $n \to \infty$. Therefore the Taylor series for $\sin(x)$ converges to this function for all values of x. Students can work through similar arguments to show that the series for $\cos(x)$ converges to $\cos(x)$ for all x, and that the series for e^x converges to e^x for all values of x.

It may be instructive to look at a series that does not converge to the function that is used to construct it. For instance, the function $f(x) = \begin{cases} e^{-1/x^2} & x \ne 0 \\ 0 & x = 0 \end{cases}$ has derivatives at $x = 0$ that are all equal to 0. (We need to use the limit definition of the derivative to show this.) The Taylor series for the function is simply a series of zeroes. This series certainly converges regardless of the value of x, and it converges to 0 for all values of x. So it does not converge to $f(x)$ for any values of x but 0. Students do not generally encounter functions like this in an introductory course, but should be aware that they exist and not take for granted that every Taylor series converges to the function used to construct it.

A second interesting question is, “Can a Taylor series converge to different functions on different parts of its interval of convergence?” Again this may seem counterintuitive, but it can happen in some sense. Consider the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$. Graphing a few of the partial sums shows different behavior on each side of 0. Using the ratio test we find that the series converges for all values of x. This happens to be the series you get if you replace x in the series for $\cos(x)$ with $\sqrt{x}$. The function $\cos(\sqrt{x})$ is not defined for $x < 0$ if we stay in the real number system. So what does it converge to when $x < 0$? Students may not have learned about the hyperbolic functions, but this series actually converges to $\cos\left(\sqrt{|x|}\right)$ when $x < 0$. For some students it will be sufficient to see that this function does “match” the series when x is negative. An activity sheet is provided to help students work through the demonstration of the relationship.

A third question that can be explored is, “Is every series that converges to some function a Taylor series, for that function?” Since students have been focusing on Taylor series, it is natural for them to think that any convergent series is a Taylor series.

However, consider the series $\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$.



This is a geometric series. Using the principles for finding where a geometric series converges, you find the following: $\left|\frac{1}{1+x^2}\right| < 1$ so $1 < 1 + x^2$ or $x^2 > 0$. If we use $x = 0$, the series is convergent to 0 because all terms are 0. So this series converges for all values of x. If you apply the formula for the sum of a convergent geometric series, you get:

$\frac{x^2}{1-\frac{1}{(1+x^2)}} = \frac{x^2}{\frac{1+x^2-1}{1+x^2}} = \frac{x^2}{\frac{x^2}{1+x^2}} = 1 + x^2$ for $x \neq 0$. So here we have a series that converges everywhere, but to different functions for different values of x.

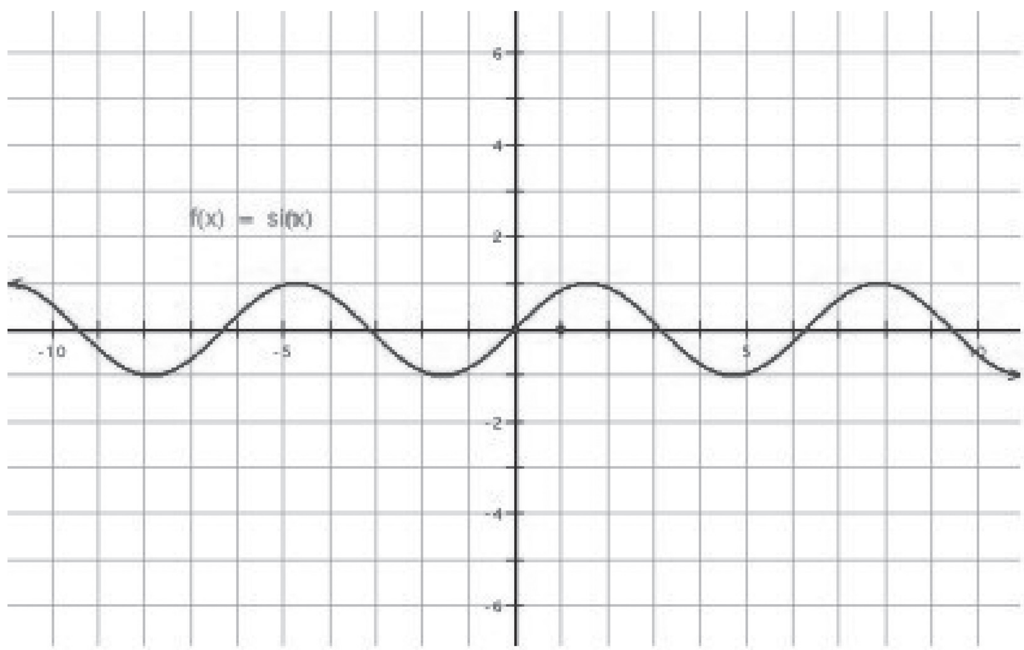
However, it is clearly not a Taylor series since the Taylor series for a polynomial is simply that polynomial. While students are not apt to encounter series like this in this course, it is nevertheless valuable for them to realize that such things can happen and that not everything is as simple as applying the ratio test and recognizing harmonic series.

The materials that follow are investigations that can be given to students.

Investigation 1: Convergence of a Taylor Series

1. Write the Taylor series for $f(x)=\sin(x)$.

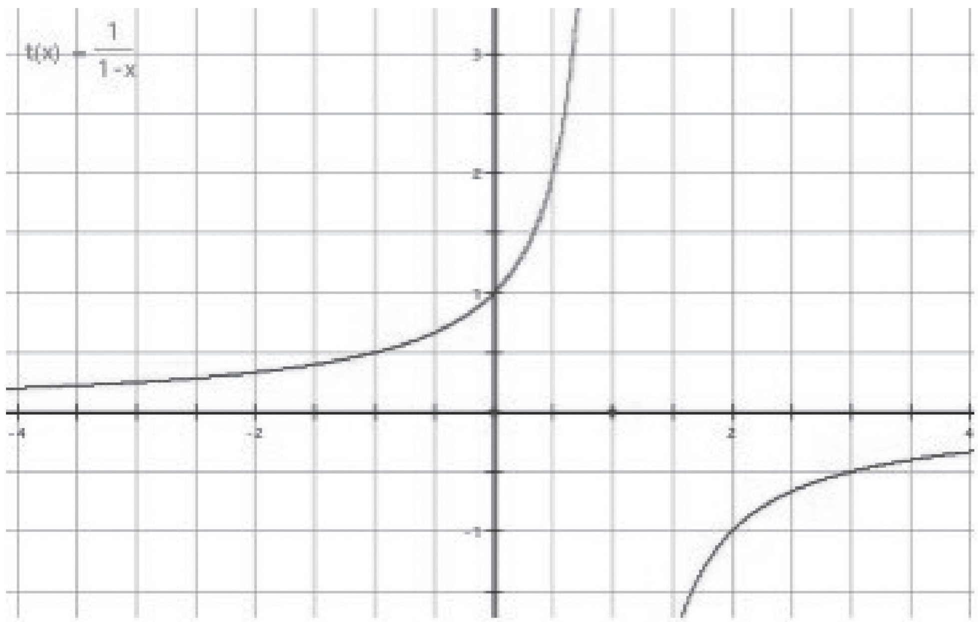
2. Graph the following Taylor polynomials for $f(x)=\sin(x)$ along with $f(x)=\sin(x)$: $T_5(x),\ T_{11}(x),\ T_{17}(x)$. Record a sketch of your graphs here.



3. Based on your graph above, for what values of x do you think you could use the series to approximate the values of $f(x)=\sin(x)$? Explain.

4. Write the Maclaurin series for $f(x)=\frac{1}{1-x}$. (Hint: Think of this function as the sum of an infinite geometric series.)

5. Graph the following Taylor polynomials for $f(x)=\frac{1}{1-x}$ along with $f(x)=\frac{1}{1-x}$: $T_4(x), T_7(x), T_{12}(x)$. Record a sketch of your graphs here.



6. Based on your graph above, for what values of *x* do you think you could use the series to approximate the values of $f(x)=\frac{1}{1-x}$? Justify your answer using ideas of convergent geometric series.

7. When you know the function that was used to create a Taylor series, you can often look at the graphs of some of the polynomials and estimate the interval of convergence from those graphs. However, when you are simply given the series

for an unknown function, it may be a bit harder to determine its interval of convergence from the graph. Consider the series $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n \cdot 3^n}$. Write the first four terms of the series.

8. Graph several of the partial sums of this series and estimate the interval of convergence based on your graphs.

9. To check your interval of convergence, you can look at the series you obtain if you substitute a value for x. For instance, if you evaluate the series for $x = 2$, you have $\sum_{n=1}^{\infty} \frac{2^{n-1}}{n \cdot 3^n}$ or $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \frac{1}{2n}$. This new series should remind you of a geometric series with its nth term multiplied by another factor. Study the following argument for the convergence of this series:

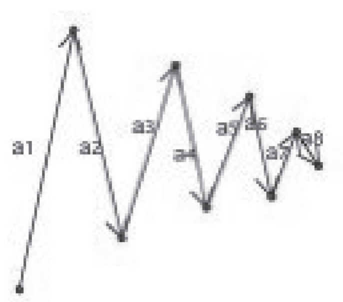
   (i) $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges because it is a geometric series with $|r| < 1$.

   (ii) $\frac{1}{2n} < 1$ for all values of $n \geq 1$.

   (iii) $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \cdot \frac{1}{2n}$ converges because its terms (all of which are positive) are less than those of the geometric series. So this sum will be less than the sum of the geometric series.

   Think about putting other values in place of x. How large can x be if the series is still to converge? In other words, what is the upper bound of the interval of convergence?

10. If you substitute a negative value for x, the terms of the series will alternate in sign. You will learn an official test for alternating series convergence later, but for now think about what happens when you add up a string of numbers that alternate in sign. If the terms you are summing are getting closer and closer to zero then the partial sums behave something like the picture below, where the upward arrows represent positive terms and the downward arrows represent negative terms. If more terms were added you can imagine that the series would converge to some point between the high and low, something like a spring bouncing and finally coming to rest at an equilibrium position.



For the series $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n \cdot 3^n}$, what negative values of x would result in the convergence of the series? In other words, what is the lower bound of the interval of convergence?

Investigation 2: The Ratio Test

When a series is geometric, it is easy to tell when it converges. However, if the series is not geometric and not closely related to a geometric series, it can be more difficult to find this interval of convergence. In this investigation you will look for relationships based on the ideas of geometric series that will be useful in determining the convergence of other series.

1. The terms of a geometric series have a constant ratio. That is, $\frac{a_{n+1}}{a_n}$ is always the same, no matter which two consecutive terms you use. The series converges if this ratio has an absolute value less than one. In nongeometric series this ratio is not constant, it can change with each pair of terms. However, since the series is infinite it is reasonable to look at the limit of this ratio to discover the behavior of the terms in the long run.

Below are two lists of series. For each one determine $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Converging Series	**Diverging Series**
$\sum_{n=0}^{\infty} \frac{2}{n^2+1}$	$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{3}{2}\right)^n$
$\sum_{n=0}^{\infty} \left(\frac{6}{7}\right)^n$	$\sum_{n=1}^{\infty} \frac{n}{n+1}$
$\sum_{n=0}^{\infty} \frac{2^n}{n!}$	$\sum_{n=1}^{\infty} \frac{n!}{n^2}$
$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$	$\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

2. If the limit of this ratio is less than 1, what seems to be true about the series? Why is this reasonable?

3. If the limit of the ratio is more than 1, what seems to be true about the series? Why is this reasonable?

4. If the limit of the ratio is 1, what seems to be true about the series? Why is this reasonable?

5. Using what you have discovered, for what interval of values do you think the series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$ will converge?

6. The results you discovered are summarized as the ratio test for convergence. It will be one of the most useful tests you learn for finding the interval of convergence of a Taylor series.

The Ratio Test

Suppose the limit $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ either exists or is infinite.

Then

a. If $L < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges.

b. If $L > 1$, the series $\sum_{n=1}^{\infty} a_n$ diverges.

c. If $L = 1$, the test is inconclusive.

Investigation 3: A Series That Converges to Two Different Functions (in Form)

1. Write the first four nonzero terms and general term of the Maclaurin series for $f(x)=\cos(x)$.

2. You know that you can create a series for $\cos(g(x))$ by replacing each x in the series with $g(x)$. You have done this with such functions as $g(x) = x^2$. Now use this idea to write a series for $f(x)=\cos(\sqrt{x})$.

3. Find the interval of convergence for your series.

4. Verify by graphing that the series seems to converge to the original function for $x \geq 0$. Sketch a graph of $f(x) = \cos(\sqrt{x})$ and two different series approximations to show this convergence.

5. What happens when $x < 0$? The interval of convergence indicates that the series will still converge, but $f(x) = \cos(\sqrt{x})$ is not defined for $x < 0$. What will the series converge to then? We can find an answer to this question by examining the function (called the hyperbolic cosine function) $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$. Write the series for e^x here.

6. Write the series for e^{-x} here.

7. Use the two series from steps 5 and 6 to get a series for $\cosh(x)$.

8. Use the series from step 7 to get a series for $\cos(\sqrt{x})$.

9. For $x < 0$, use the series from step 8 to get a series for $\cosh(\sqrt{-x})$.

10. Let $h(x)$ be the function represented by the series in step 2. Express $h(x)$ in terms of the $\cos(\sqrt{x})$ and $\cosh(\sqrt{-x})$. Verify by graphing $\cosh(\sqrt{-x})$ and two different series approximations to illustrate convergence for $x < 0$.

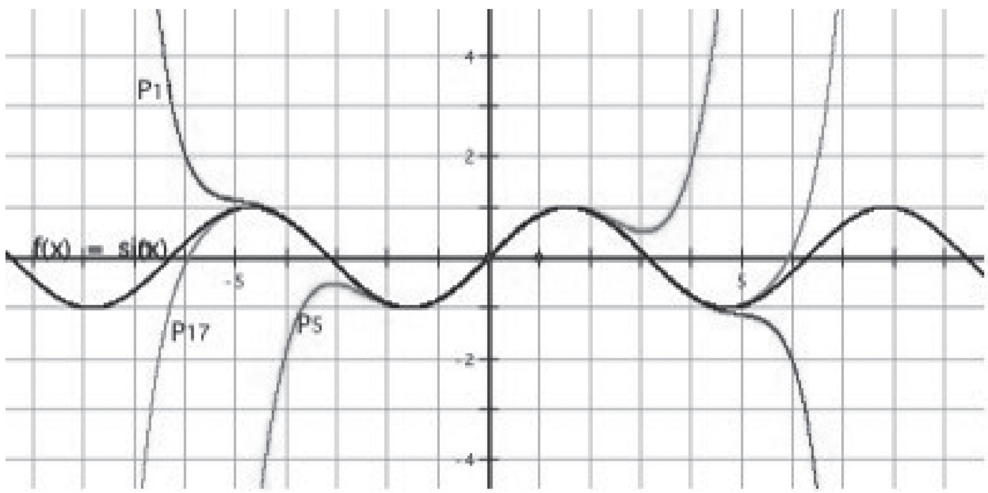
Teacher Notes on the Investigations

Investigation 1: Convergence of a Taylor Series

1. Write the Taylor series for $f(x) = \sin(x)$.

   $$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots$$

2. Graph the following Taylor polynomials for $f(x) = \sin(x)$ along with $f(x) = \sin(x)$: $T_5(x)$, $T_{11}(x)$, $T_{17}(x)$. Record a sketch of your graphs here.



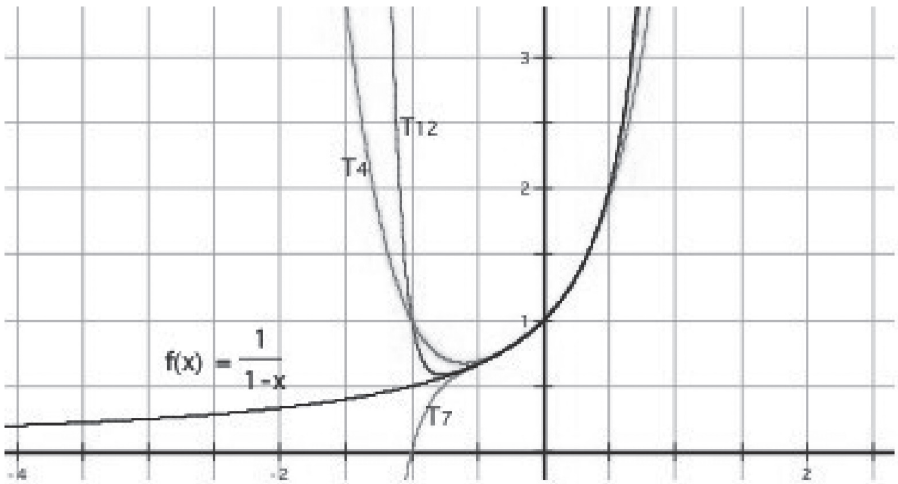
3. Based on your graph above, for what values of x do you think you could use the series to approximate the values of $f(x) = \sin(x)$? Explain.

   It appears that as you add more terms to the polynomial, it fits a larger portion of the curve. Since the series is infinitely long, it is reasonable that the series would fit the entire function. So you could use it to approximate values of $f(x) = \sin(x)$ for all values of x. However, if the x value is far from zero, it would take many terms to have a good approximation.

4. Write the Maclaurin series for $f(x) = \dfrac{1}{1-x}$. (Hint: Think of this function as the sum of an infinite geometric series.)

   $f(x) = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$

5. Graph the following Taylor polynomials for $f(x)=\dfrac{1}{1-x}$ along with $f(x)=\dfrac{1}{1-x}$: $T_4(x)$, $T_7(x)$, $T_{12}(x)$. Record a sketch of your graphs here.



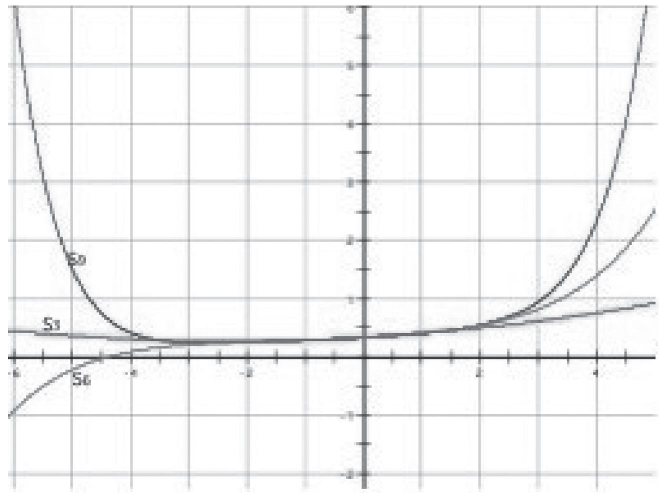
6. Based on your graph above, for what values of *x* do you think you could use the series to approximate the values of $f(x)=\dfrac{1}{1-x}$? Justify your answer using ideas of convergent geometric series.

   The series appears to "match" the function well in the interval $-1 < x < 1$. This is reasonable because it is a geometric series with ratio *x*, and thus converges when $|x| < 1$.

7. When you know the function that was used to create the Taylor series, you can often look at the graphs of some of the polynomials and estimate the interval of convergence from those graphs. However, when you are simply given the series for an unknown function, it may be a bit harder to determine its interval of convergence from the graph. Consider the series $\sum_{n=1}^{\infty} \dfrac{x^{n-1}}{n \cdot 3^n}$. Write the first four terms of the series.

   $$\frac{1}{3}+\frac{x}{2\cdot 3^2}+\frac{x^2}{3\cdot 3^3}+\frac{x^3}{4\cdot 3^4}$$

8. Graph several of the partial sums of this series and estimate the interval of convergence based on your graphs.



Based on the graphs of partial sums students may estimate the interval of convergence to be $-4 < x < 3$. This is not the correct interval, but because the series actually converges for $x = -3$, students may think the interval extends beyond this value.

9. To check your interval of convergence, you can look at the series you obtain if you substitute a value for x. For instance, if you evaluate the series for $x = 2$, you have $\sum_{n=1}^{\infty} \frac{2^{n-1}}{n \cdot 3^n}$ or $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \frac{1}{2n}$. This new series should remind you of a geometric series with its nth term multiplied by another factor. Study the following argument for the convergence of this series:

(i) $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges since it is a geometric series with $|r| < 1$.

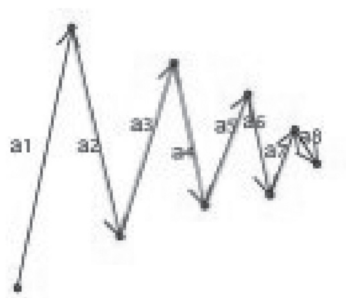
(ii) $\frac{1}{2n} < 1$ for all $n \geq 1$.

(iii) $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \cdot \frac{1}{2n}$ converges because its terms (all of which are positive) are less than those of the geometric series. So this sum will be less than the sum of the geometric series.

Think about putting other values in place of x. How large can x be if the series is still to converge? In other words, what is the upper bound of the interval of convergence?

If $x = 3$, this series becomes the harmonic series and will diverge. For any positive value less than 3, however, the series will converge.

10. If you substitute a negative value for x, the terms of the series will alternate in sign. You will learn an official test for alternating series convergence later, but for now think about what happens when you add up a string of numbers that alternate in sign. If the terms you are summing are getting closer and closer to zero, then the partial sums behave something like the picture below, where the upward arrows represent positive terms and the downward arrows represent negative terms. If more terms were added you can imagine that the series would converge to some point between the high and low, something like a spring bouncing and finally coming to rest at an equilibrium position.



For the series $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n \cdot 3^n}$, what negative values of x would result in the convergence of the series? In other words, what is the lower bound of the interval of convergence?

When trying to find the lower bound by substituting values of x, students will find that if they use a number less than –3, the "geometric" factor in the series is divergent. Using $x = –3$, they create an alternating harmonic series. Students should be able to reason that this will converge because the terms being added alternate in sign and get smaller in absolute value. When discussing this question, you may wish to formally state the alternating series test and show how it applies here.

Investigation 2: The Ratio Test

When a series is geometric, it is easy to tell when it converges. However, if the series is not geometric and not closely related to a geometric series, it can be more difficult to find this interval of convergence. In this investigation you will look for relationships based on the ideas of geometric series that will be useful in determining the convergence of other series.

1. The terms of a geometric series have a constant ratio. That is, $\frac{a_{n+1}}{a_n}$ is always the same, no matter which two consecutive terms you use. The series converges if this ratio has an absolute value less than 1. In nongeometric series this ratio is not constant; it can change with each pair of terms. However, since the series is infinite it is reasonable to look at the limit of this ratio to discover the behavior of the terms in the long run.

   Below are two lists of series. For each one determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Converging Series		**Diverging Series**	
$\sum_{n=0}^{\infty} \frac{2}{n^2+1}$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$	$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{3}{2}\right)^n$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{3}{2}$
$\sum_{n=0}^{\infty} \left(\frac{6}{7}\right)^n$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{6}{7}$	$\sum_{n=0}^{\infty} \frac{n}{n+1}$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$
$\sum_{n=1}^{\infty} \frac{2^n}{n!}$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$	$\sum_{n=1}^{\infty} \frac{n!}{n^2}$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$
$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{4}$	$\sum_{n=1}^{\infty} \frac{2^n}{n^2}$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$

2. If the limit of this ratio is less than 1, what seems to be true about the series? Why is this reasonable?
   The series converges. When the limit of the ratio is less than 1, it means the terms of the series are approaching 0 and are doing so in a way such that each pair of successive terms has a smaller ratio than the pair before. This is just like a geometric series with a ratio less than 1, and thus the series converges.

3. If the limit of the ratio is more than 1, what seems to be true about the series? Why is this reasonable?
   The series diverges. When the limit of the ratio is greater than 1, it means the terms of the series are growing in size. With each additional term the total changes by more than it did when the previous term was added. Thus the sum cannot be approaching a single value and the series diverges.

4. If the limit of the ratio is 1, what seems to be true about the series? Why is this reasonable?

The series might converge or it might diverge. Both outcomes are possible if the limit of the ratio is one. When the limit of the ratio is one this means that the terms are becoming more and more alike. If they are also becoming smaller at a fast enough rate, the series will converge. However, if they are either not approaching a limit of 0 or not doing so quickly enough, as in the harmonic series, the series will diverge.

5. Using what you have discovered, for what interval of values do you think the series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$ will converge?

   This is a geometric series with ratio $\frac{x-2}{10}$ so the series will converge when $\left|\frac{x-2}{10}\right| < 1$ or $|x-2| < 10$. This describes an interval of radius 10 centered at 2. So the interval of convergence is $-8 < x < 12$. There is no convergence at the endpoints since this is a geometric series.

6. The results you discovered are summarized as the ratio test for convergence. It will be one of the most useful tests you learn for finding the interval of convergence of a Taylor series.

The Ratio Test

Suppose the limit $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ either exists or is infinite.

Then

a. If $L < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges.

b. If $L > 1$, the series $\sum_{n=1}^{\infty} a_n$ diverges.

c. If $L = 1$, the test is inconclusive.

Investigation 3: A Series That Converges to Two Different Functions (in Form)

1. Write the first four nonzero terms and general term of the Maclaurin series for $f(x) = \cos(x)$.

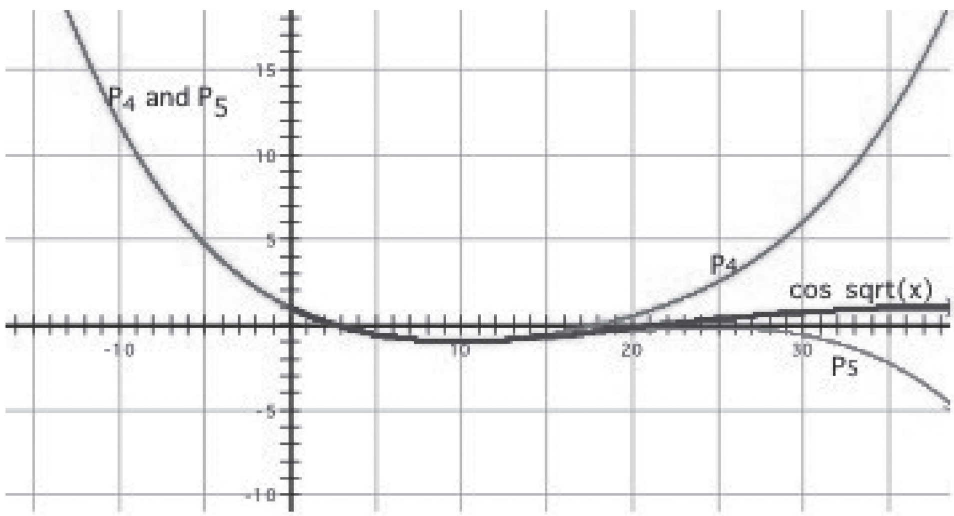
   $$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$$

2. You know that you can create a series for $\cos(g(x))$ by replacing each x in the series with $g(x)$. You have done this with such functions as $g(x) = x^2$. Now use this idea to write the series for $f(x) = \cos(\sqrt{x})$.

$$1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \ldots + (-1)^n \frac{x^n}{(2n)!} + \ldots$$

3. Find the interval of convergence for your series.
   Using the ratio test, the series converges for all values of x.

4. Verify by graphing that the series seems to converge to the original function for $x > 0$. Sketch a graph of $f(x) = \cos(\sqrt{x})$ and two different series approximations to show this convergence.



5. What happens when $x = 0$? The interval of convergence indicates that the series will still converge, but $f(x) = \cos(\sqrt{x})$ is not defined for x < 0. What will the series converge to then? We can find an answer to this question by examining the function (called the hyperbolic cosine function) $\cosh(x) = \frac{e^x + e^{-x}}{2}$. Write the series for e^x here.

$$e^x = 1 - x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

6. Write the series for e^{-x} here.

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \quad \ldots$$

7. Use the two series from steps 5 and 6 to get a series for $\cosh(x)$.

   Adding the two series from steps 5 and 6 we get

$$e^x + e^{-x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = 2 + 2\frac{x^2}{2!} + 2\frac{x^4}{4!} + \ldots.$$

   Thus

$$\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

8. Use the series from step 7 to get a series for $\cosh\left(\sqrt{x}\right)$.

$$\cosh\left(\sqrt{x}\right) = 1 + \frac{\left(\sqrt{x}\right)^2}{2!} + \frac{\left(\sqrt{x}\right)^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{\left(\sqrt{x}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$$

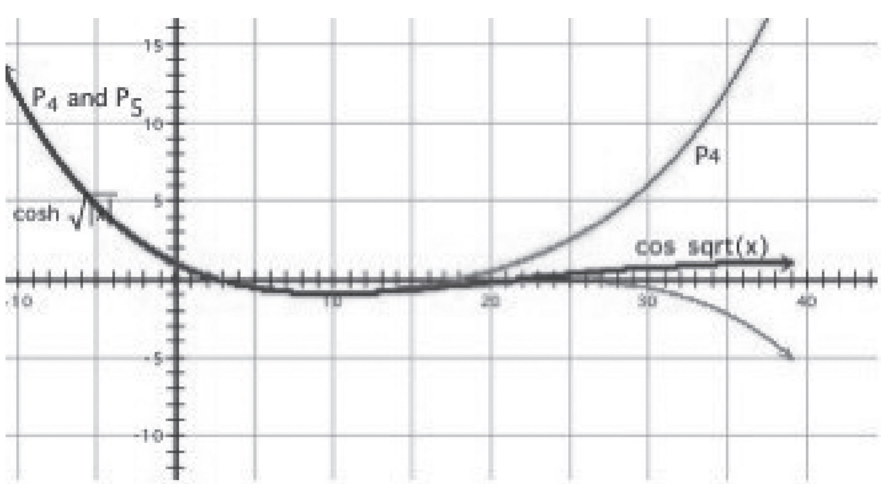
9. For $x < 0$, use the series from step 8 to get a series for $\cosh\left(\sqrt{-x}\right)$.

$$\cosh\left(\sqrt{-x}\right) = \sum_{n=0}^{\infty} \frac{\left(\sqrt{-x}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-x)^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$$

10. Let $h(x)$ be the function represented by the series in step 2. Express $h(x)$ in terms of the $\cos\left(\sqrt{x}\right)$ and $\cosh\left(\sqrt{-x}\right)$. Verify by graphing $\cosh\left(\sqrt{-x}\right)$ and two different series approximations to illustrate convergence for $x < 0$.

$$h(x) = \left\{\begin{matrix} \cos\left(\sqrt{x}\right) & \text{if } x \geq 0 \\ \cosh\left(\sqrt{-x}\right) & \text{if } x < 0 \end{matrix}\right\} = \left\{\begin{matrix} \cos\left(\sqrt{x}\right) & \text{if } x \geq 0 \\ \cosh\left(\sqrt{|x|}\right) & \text{if } x < 0 \end{matrix}\right\}$$

A graph is given by:



Students are likely to be unfamiliar with the hyperbolic functions. The connections between the hyperbolic functions and the trigonometric functions are interesting and somewhat unexpected. One can also use complex numbers to explore the relationships between these two types of functions.

For example, since i^{2n} equals 1 when n is even and –1 when n is odd, it is easy to use the Maclaurin series for $\cos(x)$ to check that $\cos(ix) = \cosh(x)$. Alternatively, this identity follows from the identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, which plays a very important role in complex analysis.

Many students find complex numbers fascinating, and when they realize that there is a meaning to the sine or cosine of an imaginary number they are intrigued. Students interested in this topic could investigate what happens with the series for $\sin(x)$ when x is replaced by $\sqrt{x}$. This will lead to the identity $\sin(ix) = i\sinh(x)$.

Overview of Tests for Convergence of Infinite Series

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To the Teacher

By the time students have seen all of the tests for convergence of infinite series they are usually overwhelmed by the sheer number of tests. Moreover, the number of ways of asking a question that requires the use of a test for convergence can be daunting as well. Before applying a test for convergence or divergence of a series, students need first to recognize that they need to know whether a given series converges (or, in the case of series where the nth term is a function of x, the radius or interval of convergence). Sometimes, this recognition is trivial: A question may simply ask whether a series converges. Other times, a student may need to think a bit to see that they need to know whether a series converges. Once that's done, he or she then needs to select an appropriate test and apply it. This is not a simple task by any means.

This unit attempts to summarize the process for students. First, we'll look at the contexts where the convergence tests are applied. Then we'll examine each test, giving a rationale for why it works, and consider some questions that address the conceptual sides of the tests. There are ample examples for straightforward testing of convergence of series in textbooks; we won't provide many more of these. Consequently, this unit should be seen as a supplement rather than a replacement for textbook coverage of convergence tests.

By examining the AP questions identified in the "Introduction to Infinite Series" at the beginning of these materials, you can see some of the ways students have been asked questions that require a test for convergence on past AP Exams.

Note that there are two broad classes of series: those with constant terms, and those whose terms depend on a variable. Tests for convergence can be applied to both.

I. When do I need to test for convergence of an infinite series?

- Which of the following series converge?
- Which of the following series diverge?
- For what values of k does a series whose nth term is a function of k converge?
- Is it possible to evaluate $f(a)$ with arbitrary accuracy using its Taylor Series expansion at $x = b$?
- What is the radius or interval of convergence for a particular series?

All of these are contexts that require you to choose and apply tests for convergence. In some cases, the test for convergence isn't really much of a test at all. You may be expected to simply *know* that a certain series converges or diverges. Typical examples that rarely if ever require justification on the free-response section of the AP Exam include geometric series, harmonic and alternating harmonic series, and p-series. Students could simply assert: "This is the harmonic series. It diverges," or "This is the alternating harmonic series. It converges." The geometric series and p-series tests are almost as easy to apply. To show that $\sum_{n=1}^{\infty}\left(\frac{e}{\pi}\right)^n$ converges, a student can simply state, "This is a geometric series with a common ratio whose absolute value is less than 1. It converges." To show that $\sum_{n=1}^{\infty}\frac{2}{\sqrt[3]{n^2}}$ diverges, it's sufficient to say "This is a p-series with $p < 1$. It diverges." It would be clearer if the student said, "$\sum_{n=1}^{\infty}\left(\frac{e}{\pi}\right)^n$ converges to $\frac{e}{\pi - e}$ because it is a geometric series with common ratio $0 < \frac{e}{\pi} < 1$" and "$\sum_{n=1}^{\infty}\frac{2}{\sqrt[3]{n^2}}$ diverges because it is a p-series with $p = \frac{2}{3} < 1$." Note that these sorts of applications are really based on your ability to recognize that a series matches a particular form, and your ability to recall a fact about the convergence of series that have that form. There is no real *procedure* to follow to apply these tests.

All of the other tests share a more complicated pattern of use. First, you need to verify the hypotheses of the test. Then you need to assert the proper conclusion based on the hypotheses. How this is done varies from test to test. Here is a list of convergence tests needed for the AP Calculus BC Exam:

Summary of Tests for Convergence of Infinite Series

***n*th Term Test or Divergence Test**	If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.
Geometric Series	$\sum_{i=1}^{\infty} ar^{i-1}$ converges if and only if $\lvert r\rvert < 1$. If the series converges, its sum is $\frac{a}{1-r}$.
Integral Test	If $f(x)$ is continuous, positive, and decreasing for all $x \geq M > 0$, then $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_M^{\infty} f(x)\,dx$ converges.
***p*-Series**	$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.
Comparison Test	Suppose $0 \leq a_n \leq b_n$ for all $n \geq N$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.
Ratio Test	If $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges. If $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = 1$, then no conclusion can be made about $\sum_{n=1}^{\infty} a_n$.

Alternating Series Test	If $a_n > 0$, decreasing, and $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty}(-1)^{n-1} a_n$ and $\sum_{n=1}^{\infty}(-1)^{n} a_n$ converge.
Absolute Convergence Test	If $\sum_{n=1}^{\infty}\lvert a_n\rvert$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Additional tests for convergence include the limit comparison test and the root test, but these are not tested on the AP Exam. Of course, students can use these tests on the exam, but no questions will require their application.

What Each Test Really Says

*n*th Term Test or Divergence Test

Indirect reasoning may help you understand why the nth term test works. Suppose the limit of the nth term were a number $L \neq 0$. If that were the case, then eventually the series would behave like a series where each term was a nonzero constant, L. Essentially, the tail end of the series acts like $L + L + L + \cdots$, and so the sum could be made arbitrarily large.

Note that having an nth term that approaches 0 is a necessary *but not a sufficient* condition for convergence. A frequent mistake is assuming that the *converse* of the statement of the nth term test is also true. For example, a student might look at the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$ and reason incorrectly that since $\frac{1}{n}\to 0$ as $n\to\infty$, the series $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$ must converge. You should simply know that the harmonic series diverges, although this fact is easily verified (see the discussion in the section on the integral test). Having an nth term that approaches 0 tells us nothing about the convergence or divergence of a series. Having an nth term that does not approach 0 tells us that a series definitely diverges. That's the nth term test.

Geometric Series

If a geometric series $\sum_{i=1}^{\infty} ar^{i-1}$ has first term a and common ratio r then the sum of the first n terms is given by $\frac{a(1-r^n)}{1-r}$. Looking at the behavior of this sum as $n\to\infty$ informs us about the convergence of that geometric series. The only way $\lim_{n\to\infty}\frac{a(1-r^n)}{1-r}$ can exist is if $|r| < 1$. When $|r| < 1$, the term r^n goes to 0 as $n\to\infty$, and the infinite geometric series converges to $\frac{a}{1-r}$. Note that this is the only test that tells us not only that a series converges but *what* the series converges *to*.

Applying this test requires two steps:

- Recognizing that you have a geometric series.
- Finding the common ratio.

Once you've done that, it's a simple matter to compare the absolute value of the common ratio to 1, and then make a conclusion about the series based on that comparison.

Integral Test

The integral test is based on left and right Riemann sums. The pictures below give ample justification for the test.

FIGURE 1

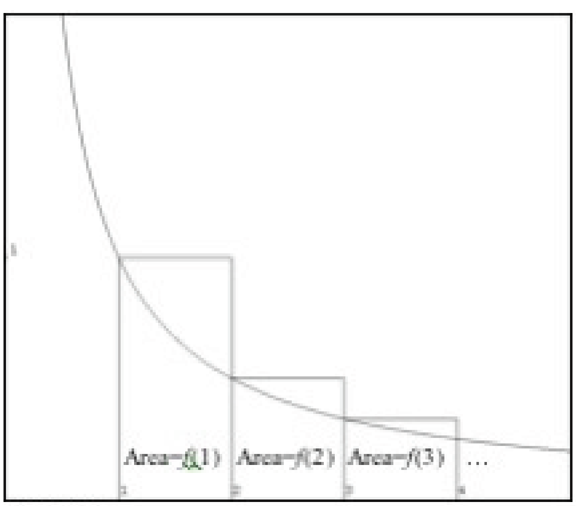
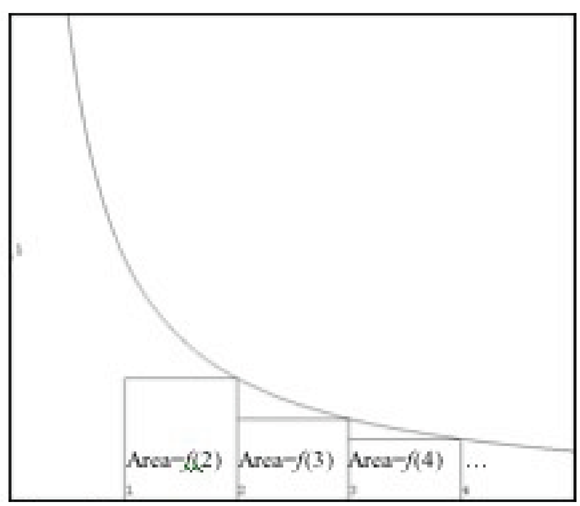


FIGURE 2



First, you have to imagine extending the domain of the sequence $f(k)$ that generates the series $\sum_{k=1}^{\infty} f(k)$ to include the positive real numbers. The graph of the resulting function is shown in Figures 1 and 2. Notice in Figure 1 above that each left endpoint rectangle has base 1 and height $f(k)$ for $k = 1,2,3...$. The sum of the areas of these rectangles is the same as the series and greater than the area under the graph of $y = f(x)$ for x from 1 to ∞. So if the integral $\int_1^{\infty} f(x)\,dx$ diverges, the series, which is greater, must diverge as well.

Similarly in Figure 2, a right Riemann sum is illustrated. Again, the area of each rectangle is a term in the series. The series sum is less than the area under the curve, again from 1 to ∞, plus $f(1)$, so if the integral $\int_1^{\infty} f(x)\,dx$ converges, the series converges. Adding the first term, $f(1)$, doesn't affect the convergence.

This is the essence of the integral test. If f is *eventually* continuous, decreasing and positive (for $x \geq M$), then $\int_M^{\infty} f(x)\,dx$ and $\sum_{n=1}^{\infty} f(n)$ either both converge or both diverge. You can begin the summation of the series at $n = 1$ instead of at M because you can remove any finite number of beginning terms of a series without affecting its convergence or divergence.

The integral test is commonly used to show that the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$ diverges. Notice that the function $f(x)=\frac{1}{x}$ is indeed continuous, positive, and decreasing for $x > 0$. The integral test tells us that since $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{N\to\infty}\left(\int_1^{N} \frac{1}{x}\,dx\right) = \lim_{N\to\infty}\left(\ln(N)\right) = \infty$, the harmonic series also diverges. You'll see the integral test applied in the section covering p-series. As with all tests for convergence, it's important to verify the hypotheses of the integral test before blindly applying the test.

Interesting Observation 'On the Side'

You might guess that since the divergence of the harmonic series was determined by the divergence of the natural logarithm function as the inputs get infinitely large, there might be a connection between the sum of the first n terms of the harmonic series and $\ln(n)$. In fact, it has been shown that $\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)-\ln(n)$ approaches a constant as $n \to \infty$. That is, $\lim_{n\to\infty}\left(\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)-\ln(n)\right)$ exists. The value of that limit is called Euler's constant, denoted by γ (the Greek letter gamma), and has the approximate value of 0.5772. It's an open question whether γ is rational or not.

p-Series

A series of the form $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is, by definition, a *p*-series. The *p*-series test says that such a series converges as long as the exponent, *p*, is greater than 1. This test results directly from applying the integral test to the series. That is, since $\int_1^{\infty}\frac{1}{x^p}dx$ converges if and only if $p > 1$, $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if and only if $p > 1$. Note that for $p > 0$ and $x > 0$, $f(x)=\frac{1}{x^p}$ is indeed a continuous, positive, and decreasing function of *x* so the hypotheses of the integral test are satisfied. If $p \le 0$, the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges by the *n*th term test. For example, if $p = -2$, we have $\sum_{n=1}^{\infty}\frac{1}{n^{-2}}=\sum_{n=1}^{\infty}n^2$. There's no way that one converges! In fact, it fails the *n*th term test.

> Applying the *p*-series test involves the same sort of knowledge as applying the test for a geometric series:
>
> - Recognizing that you have a *p*-series.
> - Finding the value of *p*.

Once you've done that, it's a simple matter to compare the value of *p* to 1, and then make a conclusion about the series based on that comparison.

One remarkable fact that becomes apparent when you think about the *p*-series test is how special the value of $p = 1$ is. It establishes a magical boundary between two wholly different classes of series. That is, since $p = 0.99$ in the series $\sum_{n=1}^{\infty}\frac{1}{n^{0.99}}$, it diverges. So you can make the partial sums of $\frac{1}{1^{0.99}}+\frac{1}{2^{0.99}}+\frac{1}{3^{0.99}}+\frac{1}{4^{0.99}}+\cdots$ as large as you want. Increase *p* just a little, and the behavior is fantastically different. You can't make partial sums of $\frac{1}{1^{1.01}}+\frac{1}{2^{1.01}}+\frac{1}{3^{1.01}}+\frac{1}{4^{1.01}}+\ldots$ as large as you want, or even any larger than 101, as the following discussion demonstrates. Amazing!

Interesting Observation "On the Side"

If you look just a little closer, the ideas behind the integral test let you establish an upper bound for $\frac{1}{1^{1.01}}+\frac{1}{2^{1.01}}+\frac{1}{3^{1.01}}+\frac{1}{4^{1.01}}+\ldots$. (The following is not a required topic for the AP Exam.) Adding the first 10 terms of the series gives a sum of 2.902261. If you look at Figure 2 from the discussion of the integral test, it should be clear that the sum of our series from the eleventh term out to infinity is less than $\int_{10}^{\infty}\frac{1}{x^{1.01}}dx$. It's straightforward to evaluate this improper integral:

$$\int_{10}^{\infty}\frac{1}{x^{1.01}}dx = \lim_{N\to\infty}\int_{10}^{N}\frac{1}{x^{1.01}}dx = \lim_{N\to\infty}\left(\frac{x^{-0.01}}{-0.01}\Bigg|_{10}^{N}\right) = \frac{10^{-0.01}}{0.01} < 97.724.$$

So the entire sum must be less than 2.903 + 97.724 = 100.627. So, while you can make partial sums of $\frac{1}{1^{.99}}+\frac{1}{2^{.99}}+\frac{1}{3^{.99}}+\frac{1}{4^{.99}}+\ldots$ as large as you want, the sum $\frac{1}{1^{1.01}}+\frac{1}{2^{1.01}}+\frac{1}{3^{1.01}}+\frac{1}{4^{1.01}}+\ldots$ is not larger than 101. Incredible.

Comparison Test

Conceptually, the comparison test is straightforward. First, it's important to note that the test applies only to series with positive (or nonnegative) terms. A positive term series whose terms are less than those of a convergent series must also converge. And a positive term series whose terms are greater than the terms of a divergent series must also diverge.

Note that the comparison test tells us nothing about a series whose terms are *larger* than the terms of a *convergent* series, and nothing about a series whose terms are *smaller* than the terms of a *divergent* series. In these cases, you must try another test, or pick another series with which to compare.

Usually, the hardest part when applying the comparison test is picking the series with which to compare. Often, this is some well-known series. Here's a simple example:

$\sum_{n=1}^{\infty}\frac{\sin^2(n)}{n^2}$ converges because, for all n, $\frac{\sin^2(n)}{n^2}\le\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges because it is a p-series with $p = 2$. Similarly, $\sum_{n=1}^{\infty}\frac{\ln(n)}{n}$ diverges because, for all $n < 2$, $\frac{\ln(n)}{n}>\frac{1}{n}$. Notice that all we need is for the terms to *eventually* get larger than the terms of the known series, since we can remove any finite number of terms from the front of a series without affecting its convergence or divergence.

Ratio Test

The ratio test is frequently used to find the radius of convergence for a power series. Essentially, it says that if the ratio of adjacent terms in the series *eventually* approaches a number whose absolute value is less than 1, then the series converges. A geometric series has a constant common ratio. Applying the ratio test checks whether a nonconstant ratio has a limit as $n \to \infty$. The conditions of the two tests are almost the same.

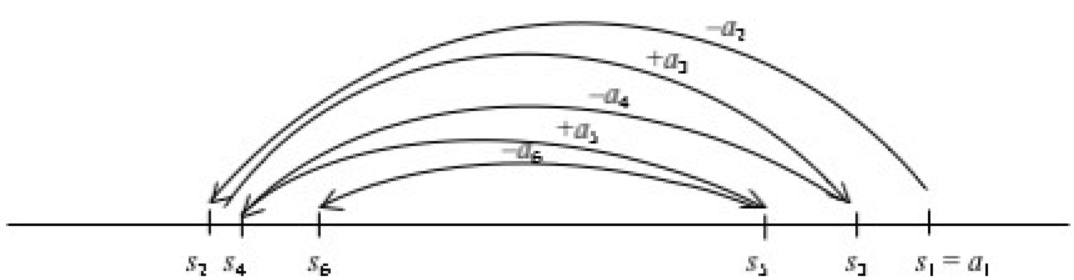
- A geometric series converges when $|r| < 1$.
- A series converges absolutely by the ratio test when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.
- A geometric series diverges when $|r| > 1$.
- A series diverges by the ratio test when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$.

The only difference between the two is the case where $r = 1$. A geometric series with $r = 1$ diverges. But the ratio test is inconclusive when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$. Notice that if the nth term has an expression raised to the nth power (as it will in any power series), when you calculate $\frac{a_{n+1}}{a_n}$ the powers vanish. Moreover, when you divide a term with $(n+1)!$ by a term with $n!$, the factorials vanish. So, it should come as no surprise that the ratio test is invariably used to find the radius of convergence of power series, including Taylor series.

One other important fact about the ratio test is that when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, we know more than just that $\sum_{n=1}^{\infty} a_n$ converges. In particular, we know that $\sum_{n=1}^{\infty} |a_n|$ converges.

Alternating Series Test

The alternating series test has three parts to its hypothesis: The terms in the series must alternate in sign, they must decrease in absolute value, and the nth term must approach 0 as n approaches infinity. If all three conditions are met, the series converges. The justification for the alternating series test typically involves a look at how the sequence of partial sums, $s_1 = a_1$, $s_2 = a_1 - a_2$, $s_3 = a_1 - a_2 + a_3$, ..., behaves. (Here, we're assuming each of the terms a_k is positive.) Most texts have a graph like the following:



Adding a_3 to the second partial sum s_2 results in a third partial sum that is less than the first, s_1. This must be true because we subtracted more (a_2) from s_1 than was added back (a_3). In this context, if eventually the amount a partial sum changes by goes to 0, the partial sums must converge. This analysis shows why the alternating series test requires that terms decrease to 0 in absolute value. If the terms didn't decrease, then one term could bounce us past an earlier partial sum. And of course if the terms don't approach 0, then the sequence of partial sums can't converge by the nth term test.

Similar to the way the integral test gives rise to an upper bound for the error in stopping an infinite summation at the nth term (sometimes called the truncation error), so too does the alternating series test allow us to determine an error bound. This time, though, the error bound is a required topic for AP Calculus BC. If an alternating series $\left(\sum_{n=1}^{\infty} b_n\right)$ converges with $\sum_{n=1}^{\infty} b_n = S$ [note that here we're using $b_n = (-1)^{n+1} a_n$ where the sequence (a_n) is positive and *decreasing*] and the nth partial sum, $\sum_{n=1}^{N} b_n = S_N$, then the alternating series error bound says simply that the error $|S - S_N|$ satisfies $|S - S_N| < |b_{N+1}|$. That is, the truncation error is no greater in magnitude than the magnitude of the first omitted term. If $b_{N+1} > 0$, then $S_N < S$ and if $b_{N+1} < 0$, then $S_N > S$. Here's a simple application of the test and error bound. The alternating harmonic series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots$, has terms that alternate and decrease to 0 in absolute value. The alternating series test guarantees that the series converges. In fact, it converges to $\ln(2) \approx 0.693$. Adding the first five terms gives $S_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = \frac{47}{60} = 0.78\overline{3}$. The error bound says that the truncation error is no greater in magnitude than the first omitted term, $\left|-\frac{1}{6}\right| = 0.1\overline{6}$. Indeed, S_5 is within $0.1\overline{6}$ of $\ln(2)$.

Absolute Convergence Test

Though rarely a topic on recent AP Exams, the absolute convergence test is perhaps the simplest test of all. It says that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. In other words, a series that converges absolutely must converge. You can also be sure that $-\sum_{n=1}^{\infty} |a_n| \le \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} |a_n|$.

This test can be more useful than first meets the eye. Both the integral test and the comparison test require terms that are nonnegative. If you're presented with a series $\sum_{n=1}^{\infty} a_n$ that does *not* satisfy this requirement, you could try testing whether $\sum_{n=1}^{\infty} |a_n|$ converges. If it does, this test guarantees that $\sum_{n=1}^{\infty} a_n$ converges as well. You might also try the alternating series test.

Interesting Observation "On the Side"

When a series converges absolutely, the order in which the terms are added makes no difference. The sum will be the same. The same is *not* true for series that converge but not absolutely. Once again, consider the alternating harmonic series. Earlier, we noted that the sum $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots = \ln(2)$. But if we rearrange the terms (two positive terms and then a negative term—repeated) $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$ we'll get a convergent series. Curiously, though, this series does *not* converge to ln(2). You can show that this series converges to $\frac{3\ln(2)}{2}$ by using the series for ln(2) in $\ln(2) + \frac{1}{2}\ln(2)$. For a series like $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots$ that does converge absolutely (the series of absolute values is a p-series with $p = 2$), it doesn't matter how you rearrange the terms; the series will still converge to the same number.

Questions

1. If $\sum_{n=1}^{\infty} a_n$ diverges, which of the following must be true?

   I. $\lim_{n\to\infty} a_n \neq 0$ II. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \neq \frac{1}{2}$ III. If $f(n) = a_n$, then $\int_1^{\infty} f(x)\,dx$ diverges.

2. If $\sum_{n=1}^{\infty} a_n$ converges, which of the following is a valid conclusion?

   I. $\lim_{n\to\infty} a_n \neq \frac{1}{2}$ II. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \neq 2$ III. $\sum_{n=1}^{\infty} a_n = 0$

3. If $\lim_{n\to\infty} a_n = 0$, which of these must be true?

   I. $\sum_{n=1}^{\infty} a_n$ converges II. $\sum_{n=1}^{\infty} a_n$ diverges III. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

4. If $a_1 + a_2 + a_3 + \ldots$ converges and $a_n > 0$ for all n, then which of these must be true?

   I. $a_1 + a_2 - a_3 - a_4 + a_5 + a_6 - a_7 - a_8 + \ldots$ also converges.

II. $\lim_{n\to\infty} a_n = 0$

III. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$

5. If $a_1 + a_2 + a_3 + \ldots$ converges, which of these must be true?

   I. $a_1 + a_2 - a_3 - a_4 + a_5 + a_6 - a_7 - a_8 + \ldots$ also converges.

   II. $\lim_{n\to\infty} a_n = 0$

   III. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$

6. How many terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots$ must be added before the alternating series error bound would guarantee that the sum is within 0.001 of ln(2)?

Answers

1. Option I need not be true because a series can diverge even when its nth term goes to 0. For example, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges but $\lim_{n\to\infty} \frac{1}{n} = 0$.

   Option II must be true, because if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{2}$, then $\sum_{n=1}^{\infty} a_n$ would converge by the ratio test.

   Option III need not be true because we don't know if $f(x)$ is continuous, positive, and decreasing. If $f(x) = a_n$ for $n \le x < n + 1$ and $n = 1, 2, 3, \ldots$ then $\int_1^{\infty} f(x)dx = \sum_{n=1}^{\infty} a_n$ diverges. However, if $f(x) = 0$ when x is not a positive integer, then $\int_1^{\infty} f(x)dx = 0$ converges.

2. Option I must be true because for a convergent series, $\lim_{n\to\infty} a_n = 0$ by the nth term test.

   Option II must be true because if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$, then $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 2$, and the series must diverge by the ratio test.

   Option III need not be true, as $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1 \ne 0$ illustrates. The series could converge to any number.

Notice the difference between Options I and III.

3. None must be true. Having an nth term that goes to zero tells us nothing about the convergence of a series. Counterexamples to I, II, and III, respectively, are $\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$, and $\sum_{n=1}^{\infty} \frac{1}{n}$.

4. Option I must be true. Since the series in I converges absolutely, it converges.

   Option II must be true. When a series converges, its nth term must go to 0 by the nth term test.

   Option 3 need not be true. All we know is that if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists, then $0 \le \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \le 1$.

5. Notice the difference between this question and the previous one. Here we are not told that $a_n > 0$ for all n, so we don't know if the given series converges absolutely. The terms could really alternate in sign. The only option that must be true is Option II.

6. By the alternating series error bound, adding 999 terms guarantees the error is less than the absolute value of the thousandth term, or 0.001. In fact, $\sum_{n=1}^{999} \frac{(-1)^{n+1}}{n} = 0.6936$ is within 0.001 of $\ln(2) \approx 0.6931$.

Java Applets for Series

http://www.slu.edu/classes/maymk/SeriesGraphs/SeriesGraphs.html
http://www.slu.edu/classes/maymk/SeqSeries/SeqSeries.html
http://www.scottsarra.org/applets/calculus/SeriesGrapherApplet.html
(This one offers a nice way to increase the degree and graph Taylor polynomial approximations.)

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Instructional Unit: Manipulation of Power Series

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Overview

Focus: The different ways a power series can be manipulated.

Audience: AP Calculus BC students.

Background Information Required: Prior to this unit students should have been exposed to Maclaurin series for e^x, $\cos(x)$, $\sin(x)$, and $\frac{1}{1-x}$. In addition, they should know about the Taylor series, $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, for a function f about $x = a$, along with knowledge about radius and interval of convergence.

Unit Summary: This two- to four-day unit will summarize and review the ways of manipulating known power series. It will look at the four different ways existing series can be manipulated:

(i) using a substitution
(ii) using algebra
(iii) using differentiation
(iv) using integration

The first section will examine the use of substitution and algebra to manipulate series and will have a worksheet that can be given directly to students. The second section will concentrate on differentiation and integration, and will also have a worksheet. At the end of the second section, students will be expected to use a combination of these four different ways of manipulating a series. Students should not only be able to give a specified number of terms in the manipulated series, but should also be able to give the general term. In all of the work, we will pay attention to the domain on which our statements are true. Before the beginning of the unit, students should be reminded of the Maclaurin series they have already seen:

(a) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ for all values of x

(b) $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$ for all values of x

(c) $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ for all values of x

(d) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ for $-1 < x < 1$

Section 1: Series Manipulation Using Substitution and Algebra

The easiest way to show these manipulations is to give examples. I recommend starting with the following examples for substitution.

Example 1

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n = 1 + (-x) + (-x)^2 + (-x)^3 + \ldots = 1 - x + x^2 - x^3 + \cdots$$

for $-1 < -x < 1$ because we are replacing x in the geometric series with $-x$. These inequalities are equivalent to $-1 < x < 1$. Note that the first four terms for the new series are given by the last expression, whereas the general term can be described using either of the two summands in the two expressions using summation notation.

Example 2

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \ldots = 1 - x^2 + x^4 - x^6 + \cdots$$

is valid for $-1 < -x^2 < 1$ because we are replacing x in the geometric series with $-x^2$. The interval on which these equations hold is equivalent to $-1 < x < 1$ because we must have $x^2 < 1$. Again, the first four terms and general term are given in the same way as in Example 1.

Example 3

$$\frac{1}{x}=\frac{1}{1-(1-x)}=\sum_{n=0}^{\infty}(1-x)^n=\sum_{n=0}^{\infty}(-1)^n(x-1)^n$$
$$=1+(1-x)+(1-x)^2+(1-x)^3+\cdots=1-(x-1)+(x-1)^2-(x-1)^3+\cdots$$

holds for $-1<1-x<1$ or, equivalently, for $0<x<2$. Notice that this gives the Taylor series for the function given by $f(x)=\frac{1}{x}$ centered at $a=1$ (i.e., the point (1, 1)).

Example 4

$$\cos\left(\frac{\pi}{2}-x\right)=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\pi}{2}-x\right)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}\frac{(-1)^n\left(x-\frac{\pi}{2}\right)^{2n}}{(2n)!}=1-\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}+\frac{\left(x-\frac{\pi}{2}\right)^4}{4!}-\frac{\left(x-\frac{\pi}{2}\right)^6}{6!}+\ldots$$

is true for all values of $\left(x-\frac{\pi}{2}\right)$ and thus for all values of x. We note that this gives us the Taylor series for $\sin(x)$ centered at $a=\frac{\pi}{2}$ or at the point $\left(\frac{\pi}{2},1\right)$.

Example 5

$$e^{-x^2}=\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}=1+(-x^2)+\frac{(-x^2)^2}{2!}+\frac{(-x^2)^3}{3!}+\ldots=1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\ldots$$

for all values of x.

We now switch to some examples of manipulation of a series using algebra.

Example 1

$\frac{x}{1-x}=x\cdot\frac{1}{1-x}=\sum_{n=0}^{\infty}x\cdot x^n=\sum_{n=0}^{\infty}x^{n+1}=x+x^2+x^3+x^4+\ldots$ holds for $-1<x<1$ because we are multiplying the geometric series by x, and the geometric series converges for $-1<x<1$.

Example 2

$$\frac{1+x}{1-x^2}=(1+x)\cdot\frac{1}{1-x^2}=(1+x)\cdot\sum_{n=0}^{\infty}\left(x^2\right)^n=\sum_{n=0}^{\infty}(1+x)\cdot x^{2n}=\sum_{n=0}^{\infty}\left(x^{2n}+x^{2n+1}\right) \quad \text{again}$$
$$=(1+x)+\left(x^2+x^3\right)+\left(x^4+x^5\right)+\left(x^6+x^7\right)+\ldots=1+x+x^2+x^3+x^4+\ldots$$

is valid for $-1 < x^2 < 1$ or equivalently $-1 < x < 1$ because we are replacing x in the geometric series with x^2. Note that this was a rather silly way to proceed since we could have simplified the algebraic expression first to get $\frac{1+x}{1-x^2}=\frac{1+x}{(1-x)(1+x)}=\frac{1}{1-x}$ for $x \neq -1$.

Example 3

$$\frac{x}{3+x}=\left(\frac{x}{3}\right)\cdot\frac{1}{1-\left(\frac{-x}{3}\right)}=\left(\frac{x}{3}\right)\cdot\sum_{n=0}^{\infty}\left(\frac{-x}{3}\right)^n=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{3^{n+1}}=\frac{x}{3}-\frac{x^2}{3^2}+\frac{x^3}{3^3}-\frac{x^4}{3^4}+\cdots \text{ is true}$$

for $-1<\frac{-x}{3}<1$, or equivalently for $-3 < x < 3$, since we replaced x with $\frac{-x}{3}$ in the geometric series.

Example 4

Using the Maclaurin series for $\sin(x)$ find $\lim_{x\to 0}\left(\frac{\sin(x)-x}{x^3}\right)$.

$$\sin(x)-x=\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\right)-x=-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots \text{ so}$$

$$\frac{\sin(x)-x}{x^3}=\left(\frac{1}{x^3}\right)\cdot\left(-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\right)=-\frac{1}{3!}+\frac{x^2}{5!}-\frac{x^4}{7!}+\cdots. \text{ Thus,}$$

$\lim_{x\to 0}\left(\frac{\sin(x)-x}{x^3}\right)=\lim_{x\to 0}\left(-\frac{1}{3!}+\frac{x^2}{5!}-\frac{x^4}{7!}+\ldots\right)=-\frac{1}{6}$. This last equality comes from the fact that a power series is continuous on the interior of its interval of convergence. Since the series is convergent for all values of x, we can simply use $x = 0$ in the series after we've finished with the algebraic manipulation to compute the limit.

Example 5

Find the Maclaurin series for $\frac{1}{1+x}+\frac{1}{1-x}$.

$$\begin{aligned}\frac{1}{1+x}+\frac{1}{1-x} &= \sum_{n=0}^{\infty}(-1)^n x^n + \sum_{n=0}^{\infty} x^n \\ &= \left(1-x+x^2-x^3+\cdots\right)+\left(1+x+x^2+x^3+\cdots\right) \\ &= 2+2x^2+2x^4+2x^6+\cdots \\ &\left(=\frac{2}{1-x^2}\right)\end{aligned}$$

Note that the last parenthetical line is derived from the sum of a geometric series and is included because if we had initially simplified the sum of the two fractions, the result would be the last expression. Note that these equations are valid for $-1 < x < 1$ because the Maclaurin series for both of the summands have this interval as their interval of convergence. In general, this type of algebraic work will be valid on the intersection of the two intervals of convergence.

Exercise Set A Questions

1. (a) Find the first four nonzero terms and the general term of the Maclaurin series for $\sin(x^2)$ and give its interval of convergence.

   (b) Use the series in part (a) to find the first four nonzero terms and general term of the Maclaurin series for $x \sin(x^2)$.

2. Find the first four terms and general term of the Maclaurin series for $\dfrac{x^2}{2-x^2}$ and give its interval of convergence.

3. (2007 BC, 6b) Use the Maclaurin series for e^{-x^2} to find $\lim_{x\to 0}\left(\frac{1-x^2-e^{-x^2}}{x^4}\right)$.

4. Define the function f by $f(x)=\begin{cases}\frac{\sin(x)}{x} & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$.

   (a) Find the first four nonzero terms and the general term of the Maclaurin series for f.

   (b) Use part (a) to determine $f^{(n)}(0)$.

   (c) Indicate why f has a local maximum at $x = 0$.

5. Use the fact that $\cos(x)=\sin\left(\frac{\pi}{2}-x\right)$ to find the first four nonzero terms and general term of the Taylor series for $\cos(x)$ at $a=\frac{\pi}{2}$.

6. (2001 BC, 6b) A function f is defined by $f(x)=\frac{1}{3}+\frac{2x}{3^2}+\frac{3x^2}{3^3}+\cdots+\frac{(n+1)x^n}{3^{n+1}}+\cdots$.

   Find $\lim_{x\to 0}\left(\frac{f(x)-\frac{1}{3}}{x}\right)$.

7. (1998 BC, 3b) The function f has derivatives of all orders for all real numbers with $f(0) = 5$,

$f'(0) = -3$, $f''(0) = 1$, and $f'''(0) = 4$. If $g(x) = f(x^2)$, find the fourth-degree Taylor polynomial for g about $a = 0$.

8. (a) (1996 BC, 2c) The Maclaurin series for the function $f(x)$ is $1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\cdots+\frac{x^n}{(n+1)!}+\cdots$. Find the first three nonzero terms and the general term of the Maclaurin series for $g(x) = xf(x)$.

   (b) Give a formula for $g(x)$ that does not involve an infinite series.

9. (1993 BC, 5b) Let f be the function given by $f(x)=e^{\frac{x}{2}}$. Write the first three nonzero terms and the general terms for the series about $x = 0$ for the function given by $g(0)=\frac{1}{2}$ and $g(x)=\frac{e^{\frac{x}{2}}-1}{x}$ for $x \neq 0$.

10. Find the first four terms and general term of the Taylor series for $\frac{x}{2-x}$ centered at $a = 1$ and give its interval of convergence. Note that $2 - x = 1 - (x - 1)$ and x = $(x - 1) + 1$.

Exercise Set A Solutions

1. (a) Find the first four nonzero terms and the general term of the Maclaurin series for $\sin\left(x^2\right)$ and give its interval of convergence.

   (b) Use the series in part (a) to find the first four nonzero terms and general term of the Maclaurin series for $x\sin\left(x^2\right)$.

   (a) $\sin(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots + \dfrac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$ so

$$\sin\left(x^2\right) = x^2 - \frac{\left(x^2\right)^3}{3!} + \frac{\left(x^2\right)^5}{5!} - \frac{\left(x^2\right)^7}{7!} + \cdots + \frac{(-1)^n\left(x^2\right)^{2n+1}}{(2n+1)!} + \cdots$$

$$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots + \frac{(-1)^n x^{4n+2}}{(2n+1)!} + \cdots$$

   The interval of convergence is all real numbers.

   (b) $x\sin\left(x^2\right) = x^3 - \dfrac{x^7}{3!} + \dfrac{x^{11}}{5!} - \dfrac{x^{15}}{7!} + \cdots + \dfrac{(-1)^n x^{4n+3}}{(2n+1)!} + \cdots$

2. Find the first four terms and general term of the Maclaurin series for $\dfrac{x^2}{2-x^2}$ and give its interval of convergence.

$$\frac{x^2}{2-x^2} = \frac{x^2}{2}\cdot\frac{1}{1-\frac{x^2}{2}}$$

$$= \frac{x^2}{2}\cdot\left(1 + \frac{x^2}{2} + \left(\frac{x^2}{2}\right)^2 + \left(\frac{x^2}{2}\right)^3 + \cdots + \left(\frac{x^2}{2}\right)^n + \cdots\right)$$

$$= \frac{x^2}{2} + \frac{x^4}{2^2} + \frac{x^6}{2^3} + \frac{x^8}{2^4} + \cdots + \frac{x^{2n+2}}{2^{n+1}} + \cdots$$

   The series converges when $-1 < \dfrac{x^2}{2} < 1$ or equivalently when $x^2 < 2$ and thus when $-\sqrt{2} < x < \sqrt{2}$.

3. (2007 BC, 6b) Use the Maclaurin series for e^{-x^2} to find $\lim\limits_{x\to 0}\left(\dfrac{1-x^2-e^{-x^2}}{x^4}\right)$.

   $e^{-x^2} = 1 + \left(-x^2\right) + \dfrac{\left(-x^2\right)^2}{2!} + \dfrac{\left(-x^2\right)^3}{3!} + \cdots = 1 - x^2 + \dfrac{x^4}{2} - \dfrac{x^6}{6} + \cdots$ so

   $1 - x^2 - e^{-x^2} = -\dfrac{x^4}{2} + \dfrac{x^6}{6} - \cdots$ giving $\dfrac{1-x^2-e^{-x^2}}{x^4} = -\dfrac{1}{2} + \dfrac{x^2}{6} - \cdots$. Therefore

   $\lim\limits_{x\to 0}\left(\dfrac{1-x^2-e^{-x^2}}{x^4}\right) = -\dfrac{1}{2}$.

4. Define the function f by $f(x)=\begin{cases}\dfrac{\sin(x)}{x} & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$. Find the first four nonzero terms and the general term of the Maclaurin series for f and use them to determine $f^{(n)}(0)$ along with indicating why f has a local maximum at $x = 0$.

(a) $\sin(x)=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\cdots+\dfrac{(-1)^n x^{2n+1}}{(2n+1)!}+\cdots$ so

$$f(x)=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots+\frac{(-1)^n x^{2n}}{(2n+1)!}+\cdots\ .$$

(b) Thus $f^{(n)}(0)=\begin{cases}0 & \text{if } n \text{ is odd}\\ \dfrac{(-1)^{n/2}}{n+1} & \text{if } n \text{ is even}\end{cases}$ for $n>0$.

(c) Since $f'(0)=0$ and $f''(0)=-\dfrac{1}{3}$, there is a local maximum at $x=0$ by the second derivative test for local extrema.

5. Use the fact that $\cos(x) = \sin\left(\dfrac{\pi}{2}-x\right)$ to find the first four nonzero terms and general term of the Taylor series for $\cos(x)$ at $a=\dfrac{\pi}{2}$.

$$\cos(x)=\sin\left(\frac{\pi}{2}-x\right)=-\sin\left(x-\frac{\pi}{2}\right)$$

$$=-\left(\left(x-\frac{\pi}{2}\right)-\frac{\left(x-\frac{\pi}{2}\right)^3}{3!}+\frac{\left(x-\frac{\pi}{2}\right)^5}{5!}-\frac{\left(x-\frac{\pi}{2}\right)^7}{7!}+\cdots+\frac{(-1)^n\left(x-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}+\cdots\right)$$

$$=-\left(x-\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^3}{3!}-\frac{\left(x-\frac{\pi}{2}\right)^5}{5!}+\frac{\left(x-\frac{\pi}{2}\right)^7}{7!}+\cdots+\frac{(-1)^{n+1}\left(x-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}+\cdots$$

6. (2001 BC, 6b) A function f is defined by $f(x)=\dfrac{1}{3}+\dfrac{2x}{3^2}+\dfrac{3x^2}{3^3}+\cdots+\dfrac{(n+1)x^n}{3^{n+1}}+\cdots$.

Find $\lim\limits_{x\to 0}\left(\dfrac{f(x)-\frac{1}{3}}{x}\right)$.

$\dfrac{f(x)-\frac{1}{3}}{x}=\dfrac{2}{3^2}+\dfrac{3x}{3^3}+\cdots+\dfrac{(n+1)x^{n-1}}{3^{n+1}}+\cdots$ so $\lim\limits_{x\to 0}\left(\dfrac{f(x)-\frac{1}{3}}{x}\right)=\dfrac{2}{9}$.

7. (1998 BC, 3b) The function f has derivatives of all orders for all real numbers with $f(0)=5$, $f'(0)=-3$, $f''(0)=1$, and $f'''(0)=4$. If $g(x)=f\left(x^2\right)$ find the fourth-degree Taylor polynomial for g about $a=0$.

   The second-degree Taylor polynomial for f about $a=0$ is given by

   $T(x)=f(0)+f'(0)x+\dfrac{f''(0)}{2}x^2=5-3x+\dfrac{1}{2}x^2$. Thus the fourth-degree Taylor polynomial for $g(x)=f\left(x^2\right)$ about $a=0$ is given by $S(x)=T\left(x^2\right)=5-3x^2+\dfrac{1}{2}x^4$.

8. (a) (1996 BC, 2c) The Maclaurin series for the function $f(x)$ is $1+\dfrac{x}{2!}+\dfrac{x^2}{3!}+\dfrac{x^3}{4!}+\cdots+\dfrac{x^n}{(n+1)!}+\cdots$. Find the first three nonzero terms and the general term of the Maclaurin series for $g(x)=xf(x)$.

   (b) Give a formula for $g(x)$ that does not involve an infinite series.

   Since $g(x)=xf(x)=x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dfrac{x^4}{4!}+\cdots+\dfrac{x^{n+1}}{(n+1)!}+\cdots$ we have that $g(x)=e^x-1$.

9. (1993 BC, 5b) Let f be the function given by $f(x)=e^{\frac{x}{2}}$. Write the first three nonzero terms and the general terms for the series about $x=0$ for the function given by $g(0)=\dfrac{1}{2}$ and $g(x)=\dfrac{e^{\frac{x}{2}}-1}{x}$ for $x\neq 0$.

   $e^x=1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\cdots+\dfrac{x^n}{n!}+\cdots$ so

   $e^{\frac{x}{2}}=1+\dfrac{x}{2}+\dfrac{\left(\frac{x}{2}\right)^2}{2!}+\dfrac{\left(\frac{x}{2}\right)^3}{3!}+\cdots+\dfrac{\left(\frac{x}{2}\right)^n}{n!}+\cdots=1+\dfrac{x}{2}+\dfrac{x^2}{2^2\cdot 2!}+\dfrac{x^3}{2^3\cdot 3!}+\cdots+\dfrac{x^n}{2^n\cdot n!}+\cdots,$

   which leads to

   $\dfrac{e^{\frac{x}{2}}-1}{x}=\dfrac{1}{2}+\dfrac{x}{2^2\cdot 2!}+\dfrac{x^2}{2^3\cdot 3!}+\cdots+\dfrac{x^{n-1}}{2^n\cdot n!}+\cdots.$

10. Find the first four terms and general term of the Taylor series for $\frac{x}{2-x}$ centered at $a=1$ and give its interval of convergence. Note that $2-x=1-(x-1)$ and $x=(x-1)+1$.

$$\begin{aligned}\frac{x}{2-x}&=((x-1)+1)\cdot\frac{1}{1-(x-1)}\\&=((x-1)+1)\cdot\left(1+(x-1)+(x-1)^2+(x-1)^3+\cdots+(x-1)^n+\cdots\right)\\&=\left((x-1)+(x-1)^2+(x-1)^3+(x-1)^4+\cdots+(x-1)^{n+1}+\cdots\right)+\\&\quad\left(1+(x-1)+(x-1)^2+(x-1)^3+\cdots+(x-1)^n+\cdots\right)\\&=1+2(x-1)+2(x-1)^2+2(x-1)^3+2(x-1)^4+\cdots+2(x-1)^n+\cdots\end{aligned}$$

The geometric series here converges for $-1<x-1<1$ which is equivalent to $0<x<2$.

Section 2: Series Manipulation Using Differentiation and Integration

Theoretical Background

Let $f(x)=\sum_{n=0}^{\infty}a_n(x-a)^n$ be a power series with radius of convergence R. We then know that the function f is defined on the interval $(a-R,a+R)$ and may (or may not) be defined when $x=a-R$ or $x=a+R$.

Under the conditions above we have the following two theorems:

(1) f is differentiable on $(a-R,a+R)$ and $f'(x)=\sum_{n=0}^{\infty}na_n(x-a)^{n-1}$

(2) For c and d in $(a-R,a+R)$ we have $\int_c^d f(x)\,dx=\sum_{n=0}^{\infty}a_n\int_c^d(x-a)^n\,dx$.

Even if the series for $f(x)$ converges at $x=a-R$ or $x=a+R$, the series for f' may not. We don't prove these theorems in AP Calculus BC, but we allow students to use them. Likewise, considering $\int_a^x f(t)\,dt$, where x is in $(a-R$ or $x=a+R)$, implies that

$g(x)=\sum_{n=0}^{\infty}\frac{a_n}{n+1}(x-a)^{n+1}=\sum_{n=1}^{\infty}\frac{a_{n-1}}{n}(x-a)^n$ is the antiderivative of *f* on $(a-R,a+R)$ with *g*(*a*) +0. The simplest way to talk about these theorems is to indicate to the students that they know that the derivative of a sum is the sum of the derivatives, and that this now extends to infinite sums, at least on the interior of their intervals of convergence. The same kind of thing can be said about integrals.

Again, the easiest way to show these manipulations is to give examples. I recommend starting with the following examples.

Example 1

$$\ln(1+x)=\int_0^x \frac{1}{1-(-t)}dt=\sum_{n=0}^{\infty}\int_0^x(-t)^n\,dt=\sum_{n=0}^{\infty}(-1)^n\int_0^x t^n dt=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+(-1)^{n-1}\frac{x^n}{n}+\cdots$$

for $-1<-x<1$, , or equivalently for $-1<x<1$. The theorem says nothing

about convergence at the endpoints of this interval. The alternating series test shows that the series above converges at $x = 1$, but when $x = -1$ we get the negative of the harmonic series, which diverges. Even then it's not clear what the series converges to when $x = 1$. By using the Lagrange error formula with $x = 1$, it can be shown that $\ln(2)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{(-1)^{n-1}}{n}+\cdots$. Therefore, $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+(-1)^{n-1}\frac{x^n}{n}+\cdots$ for $-1<x\le 1$.

Example 2

Using indefinite integration:

$$\arctan(x)=\int\frac{1}{1-(-x^2)}dx=\sum_{n=0}^{\infty}(-1)^n\int x^{2n}dx=c+x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots+(-1)^n\frac{x^{2n+1}}{2n+1}+\cdots$$

for $-1<x<1$. If $x = 0$, we get $c = 0$ so that

$\arctan(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots+(-1)^n\frac{x^{2n+1}}{2n+1}+\cdots$. Again, the theorem says nothing about convergence at the endpoints, but it can also be shown that this holds for $-1\le x\le 1$. I don't know that I would try to show this here, but in case a student asks, consider:

$\dfrac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \dfrac{(-1)^{n+1} t^{2n+2}}{1+t^2}$, which comes from the partial sum of a geometric series. Therefore,

$$\begin{aligned}
\arctan(1) &= \int_0^1 \frac{1}{1+t^2}\,dt \\
&= \int_0^1 1dt - \int_0^1 t^2 dt + \int_0^1 t^4 dt - \int_0^1 t^6 dt + \cdots + (-1)^n \int_0^1 t^{2n} dt + (-1)^{n+1} \int_0^1 \frac{t^{2n+2}}{1+t^2}\,dt \\
&= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^n \frac{1}{2n+1} + (-1)^{n+1} \int_0^1 \frac{t^{2n+2}}{1+t^2}\,dt
\end{aligned}$$

Considering $\left| (-1)^{n+1} \displaystyle\int_0^1 \frac{t^{2n+2}}{1+t^2}\,dt \right| \le \displaystyle\int_0^1 t^{2n+2} dt = \frac{1}{2n+3}$ we get

$$\arctan(1) = \lim_{n\to\infty}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^n \frac{1}{2n+1}\right) + \lim_{n\to\infty}\left((-1)^{n+1} \int_0^1 \frac{t^{2n+2}}{1+t^2}\,dt\right) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

Example 3

$\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ for $-1 < x < 1$ so $\left(\dfrac{1}{1-x}\right)^2 = \dfrac{d}{dx}\left(\dfrac{1}{1-x}\right) = 1 + 2x + 3x^2 + 4x^3 + \cdots$ for $-1 < x < 1$. If we continue this process, we get

$2\left(\dfrac{1}{1-x}\right)^3 = \dfrac{d}{dx}\left(\left(\dfrac{1}{1-x}\right)^2\right) = 2 + 6x + 12x^2 + 20x^3 + \cdots + (n+2)(n+1)x^n + \cdots$ or

$\left(\dfrac{1}{1-x}\right)^3 = 1 + 3x + 6x^2 + 10x^3 + \cdots + \dfrac{(n+2)(n+1)}{2} x^n + \cdots$ for $-1 < x < 1$. One could continue this to get

$3\left(\dfrac{1}{1-x}\right)^4 = 3 + 12x + 30x^2 + 60x^3 + \cdots + \dfrac{(n+3)(n+2)(n+1)}{2} x^n + \cdots$ and so

$\left(\dfrac{1}{1-x}\right)^4 = 1 + 4x + 10x^2 + 20x^3 + \cdots + \dfrac{(n+3)(n+2)(n+1)}{3\cdot 2} x^n + \cdots$.

Continuing this further yields

$$\left(\frac{1}{1-x}\right)^k = 1 + kx + \binom{k+1}{k-1}x^2 + \binom{k+2}{k-1}x^3 + \cdots + \binom{k+n-1}{k-1}x^n + \cdots$$
$$= 1 + kx + \binom{k+1}{2}x^2 + \binom{k+2}{3}x^3 + \cdots + \binom{n+k-1}{n}x^n + \cdots$$

which yields the binomial series when x is replaced with $-x$.

Example 4

$$\frac{x}{x^2+3x+2} = \frac{-1}{x+1} + \frac{2}{x+2}$$
$$= (-1)\frac{1}{1-(-x)} + \frac{1}{1-\left(-\frac{x}{2}\right)}$$
$$= (-1)\left(1 - x + x^2 - x^3 + \cdots (-1)^n x^n + \cdots\right) + \left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots + (-1)^n \frac{x^n}{2^n} + \cdots\right)$$
$$= \frac{1}{2}x - \frac{3}{4}x^2 + \frac{7}{8}x^3 + \cdots + (-1)^{n+1}\left(1 - \frac{1}{2^n}\right)x^n + \cdots$$

These equations are valid for $-1 < x < 1$ because the first infinite series is valid for those values of x and the second infinite series is valid for $-2 < x < 2$.

Example 5

Show that $y = f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$ is a solution to the differential equation $xy' + y = \cos(x)$.

$$y' = f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2nx^{2n-1}}{(2n+1)!} = -2\frac{x^1}{3!} + 4\frac{x^3}{5!} - 6\frac{x^5}{7!} + \cdots \text{ so}$$
$$xy' + y = \sum_{n=0}^{\infty} \frac{(-1)^n 2nx^{2n}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos(x).$$

Example 6

If $f(x) = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots + \frac{x^n}{(n+1)!} + \cdots$ give the Maclaurin series for $g(x) = xf(x)$ and express $g(x)$ in terms of a known function rather than an infinite series.

$xf(x) = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{n+1}}{(n+1)!} + \cdots = e^x - 1$. Thus we note that

$$f(x) = \begin{cases} \frac{e^x - 1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}.$$

Example 7

If $f(x) = -\frac{x}{2} + \frac{2}{3}x^2 - \frac{3}{4}x^3 + \cdots + (-1)^n \frac{n}{n+1}x^n + \cdots$ then

$\frac{f(x)}{x} = -\frac{1}{2} + \frac{2}{3}x^1 - \frac{3}{4}x^2 + \cdots + (-1)^n \frac{n}{n+1}x^{n-1} + \cdots$ implying $\frac{f(x)}{x} = g'(x)$ where

$g(x) = -\frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \cdots + (-1)^n \frac{1}{n+1}x^n + \cdots$.

Now $x \cdot g(x) = -\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots = \ln(1+x) - x$.

Thus $g(x) = \frac{\ln(1+x) - x}{x} = \frac{\ln(1+x)}{x} - 1$ for $x \neq 0$. From this we get

$$\frac{f(x)}{x} = g'(x) = \frac{x \cdot \frac{1}{1+x} - \ln(1+x) \cdot 1}{x^2} = \frac{1}{x(1+x)} - \frac{\ln(1+x)}{x^2}.$$

Finally, $f(x) = \frac{1}{(1+x)} - \frac{\ln(1+x)}{x}$ for $x \neq 0$ and $f(0) = 0$ from the original series. Note that this implies $0 = \lim_{x \to 0} f(x) = \lim_{x \to 0}\left(\frac{1}{(1+x)} - \frac{\ln(1+x)}{x}\right) = 1 - \lim_{x \to 0}\frac{\ln(1+x)}{x}$ or equivalently $\lim_{x \to 0}\frac{\ln(1+x)}{x} = 1$.

Example 8

Use the series for $f(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ to approximate $\int_0^2 \frac{\sin(x)}{x}dx$.

We note that the integral given is actually an improper integral since the integrand is undefined at $x = 0$. However, the integrand has a removable discontinuity there and removing the discontinuity yields $f(x)$. So

$$\int_0^2 f(x)dx = \int_0^2 1dx - \int_0^2 \frac{x^2}{3!}dx + \int_0^2 \frac{x^4}{5!}dx - \int_0^2 \frac{x^6}{7!}dx + \ldots + \int_0^2 \frac{(-1)^n x^{2n}}{(2n+1)!}dx + \ldots$$

$$= 2 - \frac{2^3}{3 \cdot 3!} + \frac{2^5}{5 \cdot 5!} - \frac{2^7}{7 \cdot 7!} + \ldots + \frac{(-1)^n 2^{2n+1}}{(2n+1) \cdot (2n+1)!} + \ldots$$

We note that this is an alternating series satisfying the conditions of the alternating series remainder theorem since

$$\frac{2^{2n+1}}{(2n+1)\cdot(2n+1)!} > \frac{2}{2n+3}\cdot\frac{2}{2n+3}\cdot\frac{2^{2n+1}}{(2n+2)\cdot(2n+1)!} = \frac{2^{2n+3}}{(2n+3)\cdot(2n+3)!} \text{ for all } n \geq 0.$$

Therefore, $\int_0^2 f(x)\,dx \approx 2 - \frac{2^3}{3\cdot 3!} + \frac{2^5}{5\cdot 5!} - \frac{2^7}{7\cdot 7!} = \frac{17698}{11025} \approx 1.60562$ with error less than $\frac{2^9}{9\cdot 9!} < 0.000157$.

Exercise Set B Questions

1. (a) Find the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int \sin(x^2)\,dx$ with $g(0)=1$ and give its interval of convergence.

   (b) Use the series in part (a) to find the first four nonzero terms and general term of the Maclaurin series for $h(x) = \int xg(x)\,dx$ that has $h(0)=0$.

2. Find the first four terms and general term of the Maclaurin series for $\frac{d}{dx}\left(\frac{x^2}{2-x^2}\right)$ and give its interval of convergence.

3. Use the Maclaurin series for e^{-x^2} to approximate $\int_0^1 e^{-x^2}\,dx$. Be sure to provide an estimate of the amount of error there could be in your approximation.

4. (2001 BC, 6c) A function f is defined by $f(x) = \frac{1}{3} + \frac{2x}{3^2} + \frac{3x^2}{3^3} + \cdots + \frac{(n+1)x^n}{3^{n+1}} + \cdots$. Write the first three nonzero terms and general term for an infinite series representing $\int_0^1 f(x)\,dx$. Use your series to find $\int_0^1 f(x)\,dx$.

5. (1998 BC, 3c) The function f has derivatives of all orders for all real numbers with $f(0) = 5$, $f'(0) = -3$, $f''(0) = 1$, and $f'''(0) = 4$. Write the third-degree Taylor polynomial about $a = 0$ for $h(x) = \int_0^x f(t)\,dt$.

Exercise Set B Solutions

1. (a) Find the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int \sin(x^2)\,dx$ with $g(0)=1$ and give its interval of convergence.

   (b) Use the series in part (a) to find the first four nonzero terms and general term of the Maclaurin series for $h(x) = \int xg(x)\,dx$ that has $h(0)=0$.

(a) From the series for $\sin(x)$ we get

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots + \frac{(-1)^n x^{4n+2}}{(2n+1)!} + \cdots .$$

$$g(x) = \int \sin(x^2)dx$$

Thus

$$= c + \int x^2 dx - \int \frac{x^6}{3!}dx + \int \frac{x^{10}}{5!}dx - \int \frac{x^{14}}{7!}dx + \cdots + \int \frac{(-1)^n x^{4n+2}}{(2n+1)!}dx + \cdots$$

$$= c + \frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \cdots + \frac{(-1)^n x^{4n+3}}{(4n+3)\cdot(2n+1)!} + \cdots$$

Since $g(0)=1$ we get $c=1$ and thus

$$g(x) = 1 + \frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \cdots + \frac{(-1)^n x^{4n+3}}{(4n+3)\cdot(2n+1)!} + \cdots .$$

The interval of convergence is all real numbers since that was the interval of convergence for the series for $\sin(x)$.

(b) From part (a) we get

$$xg(x) = x + \frac{x^4}{3} - \frac{x^8}{7\cdot 3!} + \frac{x^{12}}{11\cdot 5!} - \frac{x^{16}}{15\cdot 7!} + \cdots + \frac{(-1)^n x^{4n+4}}{(4n+3)\cdot(2n+1)!} + \cdots$$ and since $h(0)=0$, we get

$$h(x) = \frac{x^2}{2} + \frac{x^5}{15} - \frac{x^9}{9\cdot 7\cdot 3!} + \frac{x^{13}}{13\cdot 11\cdot 5!} - \frac{x^{17}}{17\cdot 15\cdot 7!} + \cdots + \frac{(-1)^n x^{4n+5}}{(4n+5)\cdot(4n+3)\cdot(2n+1)!} + \cdots .$$

2. Find the first four terms and general term of the Maclaurin series for $\frac{d}{dx}\left(\frac{x^2}{2-x^2}\right)$ and give its interval of convergence.

From Problem 2 in Exercise Set A, we have

$$\frac{x^2}{2-x^2} = \frac{x^2}{2} + \frac{x^4}{2^2} + \frac{x^6}{2^3} + \frac{x^8}{2^4} + \cdots + \frac{x^{2n+2}}{2^{n+1}} + \cdots$$ for

$-1 < x < 1$. Thus

$$\frac{d}{dx}\left(\frac{x^2}{2-x^2}\right) = \frac{d}{dx}\left(\frac{x^2}{2}\right) + \frac{d}{dx}\left(\frac{x^4}{2^2}\right) + \frac{d}{dx}\left(\frac{x^6}{2^3}\right) + \frac{d}{dx}\left(\frac{x^8}{2^4}\right) + \cdots + \frac{d}{dx}\left(\frac{x^{2n+2}}{2^{n+1}}\right) + \cdots$$

$$= x + \frac{4}{2^2}x^3 + \frac{6}{2^3}x^5 + \frac{8}{2^4}x^7 + \cdots + \frac{(2n+2)x^{2n+1}}{2^{n+1}} + \cdots,$$

which has interval of convergence $-\sqrt{2} < x < \sqrt{2}$. The terms, including the general term, could be simplified, but this is not necessary on the exam.

3. Use the Maclaurin series for e^{-x^2} to approximate $\int_0^1 e^{-x^2}\,dx$. Be sure to provide an estimate of the amount of error there could be in your approximation.

The series for e^{-x^2} is given by $e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots$.

Thus $\int_0^1 e^{-x^2}\,dx = \int_0^1 1dx - \int_0^1 x^2 dx + \int_0^1 \frac{x^4}{2}dx - \int_0^1 \frac{x^6}{6}dx + \ldots + \int_0^1 \frac{(-1)^n x^{2n}}{n!}dx + \ldots$.

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \ldots + \frac{(-1)^n}{(2n+1)\cdot n!} + \ldots.$$

Since the denominators of the fractions above are increasing, the series for $\int_0^1 e^{-x^2}\,dx$ satisfies the conditions of the alternating series approximation theorem. Thus students can use the sum of any number of terms in the series above and then have error less than the absolute value of the first omitted term. For example, $\int_0^1 e^{-x^2}\,dx \approx \frac{2}{3}$ with error less than $\frac{1}{10}$ or $\int_0^1 e^{-x^2}\,dx \approx \frac{23}{30}$ with error less than $\frac{1}{42}$.

To have the integral accurate to three decimal places, i.e., error less than $0.0005 = \frac{1}{2000}$, we must use as the last term in the sum the one with $n = 5$. This will guarantee an error of less than $\frac{1}{9360} < 0.000107$.

4. (2001 BC, 6c) A function f is defined by $f(x) = \frac{1}{3} + \frac{2x}{3^2} + \frac{3x^2}{3^3} + \cdots + \frac{(n+1)x^n}{3^{n+1}} + \cdots$.

Write the first three nonzero terms and general term for an infinite series representing $\int_0^1 f(x)\,dx$. Use your series to find $\int_0^1 f(x)\,dx$.

$$\int_0^1 f(x)\,dx = \int_0^1 \frac{1}{3}dx + \int_0^1 \frac{2x}{3^2}dx + \int_0^1 \frac{3x^2}{3^3}dx + \ldots + \int_0^1 \frac{(n+1)x^n}{3^{n+1}}dx + \ldots$$

$$= \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^2} + \ldots + \frac{1}{3^{n+1}} + \ldots = \frac{\left(\frac{1}{3}\right)}{\left(1 - \frac{1}{3}\right)} = \frac{1}{2}$$

5. (1998 BC, 3c) The function f has derivatives of all orders for all real numbers with $f(0) = 5$, $f'(0) = -3$, $f''(0) = 1$, and $f'''(0) = 4$. Write the third-degree Taylor

polynomial about $a = 0$ for $h(x) = \int_0^x f(t)\,dt$.

The second-degree Taylor polynomial for *f* about $a = 0$ is given by

$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = 5 - 3x + \frac{1}{2}x^2$. Thus the third-degree Taylor polynomial about $a = 0$ for *h* is given by $\int_0^x p(t)\,dt = 5x - \frac{3}{2}x^2 + \frac{1}{6}x^3$.

Applications of Series to Probability

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We will begin by considering a specific situation involving coin tossing that leads, in a natural way, to an infinite series. Along the way, we will introduce some of the terminology that appears in books on probability.

Suppose you had a fair coin and decided to toss it until you obtained a head (H). If you counted the number of tosses required, you might let X denote the number of tosses. Thus, with T representing tails,

TTH would give X = 3, H would give $X = 1$, TTTTH would give $X = 5$.

In probabilistic language, the variable X is called a *random variable*, i.e., a variable whose value is determined by chance.

Now, since the coin is fair, the probabilities of the three sequences above are 1/8, 1/2 and $1/2^5 = 1/32$, respectively. If we use P to denote probability, we could say

$$P(X=3)=1/8,\ P(X=1)=1/2,\ P(X=5)=1/32\,.$$

More generally then, $P(X=x)=1/2^x$ for $x = 1, 2, K$. Note carefully the standard convention of using the lower case equivalent of the name of the random variable to represent a value of the random variable.

Now consider the infinite series $\sum_{x=1}^{\infty} P(X=x)=\sum_{x=1}^{\infty}(1/2)^x$. If we believe that sooner or later, the coin must land head up, then we should believe that the value of this sum is 1. In fact, it is, since the series is a geometric series with first term and ratio 1/2, and the sum of an infinite geometric series is given by

$$a+ar+ar^2+\ldots=\sum_{k=0}^{\infty} ar^k=\frac{a}{1-r}\ if\,|r|<1.$$

Each random variable has an associated *distribution function*, F. Here we write

$$F(x) + P(X \le x).$$

Even though $F(x)$ is defined for all real values of x, we will only consider values of x that are positive integers. With this understanding, we see that

$F(x) = P(X \le x) = \sum_{k=1}^{x} P(X = k) = \sum_{k=1}^{x} (1/2)^k = \frac{1}{2} \cdot \left(\frac{1-(1/2)^x}{1-1/2} \right) = 1-(1/2)^x$, using the standard formula for the sum of a finite geometric series, i.e.,

$$a + ar + ar^2 + \ldots + ar^n = \sum_{k=0}^{n} ar^k = \frac{a\left(1-r^{n+1}\right)}{1-r} \text{ if } r \ne 1.$$

Note that the distribution function for the random variable is really just the xth partial sum for the infinite series $\sum_{k=1}^{\infty} (1/2)^k$, and this shows that partial sums have a useful probabilistic interpretation. In fact, the distribution function is even more useful than it might seem at first since it is clear that if a and b are integers with $1 \le a \le b$, then $P(a < X \le b) = F(b) - F(a)$. Thus, for example, the probability that more than three but no more than seven tosses are needed to obtain a head is given by

$$P(3 < X \le 7) = F(7) - F(3) + (1 - 1/2^7) - (1 - 1/2^3) = 1/2^3 - 1/2^7 = 15/128.$$

We are about to introduce another probabilistic concept but need to make some preliminary remarks first. Suppose that we obtained N values of the random variable X. We let V be the set of these values, say, $V = \{x_1, x_2, \cdots, x_N\}$. If N were large, we would believe that the average of the values in V provided a representative "average value" for the random variable X. To find the average of the values in V we would compute $\frac{x_1 + x_2 + \ldots + x_N}{N}$.

There is a more convenient way to compute this value, however. If we let $n(x)$ denote the number of x_1 in V that equal x, then $\frac{x_1 + x_2 + \ldots + x_N}{N} = \frac{1}{N} \sum_{x=1}^{\infty} x \cdot n(x)$. (1)

To see why (1) is true, let's look at an example. Suppose $V = \{2 ,4, 2, 3, 1, 6, 1, 2, 2, 3\}$. Then

$$\frac{2+4+2+3+1+6+1+2+2+3}{10} = \frac{1 \cdot 2 + 2 \cdot 4 + 3 \cdot 2 + 4 \cdot 1 + 5 \cdot 0 + 6 \cdot 1}{10}$$

$$= \frac{1 \cdot n(1) + 2 \cdot n(2) + 3 \cdot n(3) + 4 \cdot n(4) + 5 \cdot n(5) + 6 \cdot (n6)}{10}$$

To see why (1) is useful, note that $\frac{1}{N}\sum_{x=1}^{\infty} x \cdot n(x) = \sum_{x=1}^{\infty} x \cdot \frac{n(x)}{N}$. Now if N is large, then $\frac{n(x)}{N}$ will be very close to $P(X = x)$ so we will get a very good approximation to the average value of X if we use $\sum_{x=1}^{\infty} x \cdot P(X = x)$, and we take this sum to be the definition of the *expected value* or *mean* of the random variable X. We denote the expected value of X by $E(X)$ and think of it as the average of a large number of values of X, a "long-run" average, if you like.

In the current case, $E(X) = \sum_{x=1}^{\infty} x(1/2)^x$, another infinite series, and we would like to find the value of this series. We will use a geometric method first and then use a method that exploits some important properties of power series.

Consider the following arrangement in which the bottom row contains the terms in the infinite geometric series $\sum_{x=1}^{\infty} (1/2)^x$, the next row up contains all of these terms except 1/2, etc.

$$\begin{bmatrix} & & & & \cdots \\ & & & 1/16 & \cdots \\ & & 1/8 & 1/16 & \cdots \\ & 1/4 & 1/8 & 1/16 & \cdots \\ 1/2 & 1/4 & 1/8 & 1/16 & \cdots \end{bmatrix}$$

If we sum the entries in the bottom row, we get 1; if we sum the entries in the next row up, we get 1/2; in the next row up, 1/4; etc. Thus, the sum of all the entries is $1 + \frac{1}{2} + \frac{1}{4} + \ldots$, which is 2. On the other hand, we note that exactly one of the entries is 1/2, exactly two are 1/4, exactly three are 1/8, etc. Thus, the sum of the entries can also be expressed as $\sum_{x=1}^{\infty} x(1/2)^x$, and therefore we have

$$E(X) = \sum_{x=1}^{\infty} x(1/2)^x = 2\,.$$

CAUTIONARY NOTE: In general, evaluating infinite series by rearranging the terms in the series can lead to invalid results. However, in this case, the series in question is absolutely convergent, and rearrangement is allowed.

As an alternative to the argument above, consider $f(x) = \dfrac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{k=0}^{\infty} x^k$, which is valid for $|x| < 1$, using the formula for the sum of an infinite geometric sequence. Now, if we differentiate both sides, we obtain

$$\frac{1}{(1-x)^2} = f'(x) = \sum_{k=1}^{\infty} kx^{k-1} \text{ so that } \frac{x}{(1-x)^2} = xf'(x) = \sum_{k=1}^{\infty} kx^k .$$

If we let $x = 1/2$ here, we get $\sum_{k=1}^{\infty} k(1/2)^k = \dfrac{1/2}{(1-1/2)^2} = 2$, confirming the previous computation.

It might seem strange that the average value of the random variable X is 2, since X can take on arbitrarily large values. However, it does so with small probability. Remember that X is 1 or 2 with probability 3/4 = 1/2 +1/4, and

$$P(X > 4) = \sum_{k=5}^{\infty} \frac{1}{2^k} = \frac{\left(\frac{1}{2^5}\right)}{\left(1 - \frac{1}{2}\right)} = \frac{1}{16} .$$

The following exercise generalizes everything we have done so far.

Exercise I

Suppose a coin lands head up with probability p where $0 < p < 1$. Suppose that this coin is tossed repeatedly until a head appears and that X_p denotes the number of tosses required. Let $q = 1 - p$. The random variable X_p is said to be geometric with parameter p. (Note that in the example we just considered, $p = 1/2$ and the random variable is $X_{1/2}$.)

(a) Convince yourself that $P(X_p = x) = q^{x-1}p$ for $x = 1, 2, 3, \ldots$.

(b) Show that $\sum_{x=1}^{\infty} P(X_p = x) = \sum_{x=1}^{\infty} q^{x-1} p = 1$. (Hint: This is an easy application of the formula for the sum of an infinite geometric series.)

(c) Show that the distribution function of X_p is $1 - q^x$ for $x = 1, 2, 3, \ldots$.

(d) Use both of the methods given above to show that $E(X_p) = 1/p$, and then convince yourself that it is reasonable that if p is close to 0, the average number of tosses needed to get a head should be large (because it is very hard to get a head), and that if p is close to 1, the average number of tosses should be very close to 1 (because we will typically get a head on the first toss).

Next we extend the ideas in Exercise I. Using the coin from the exercise, toss the coin until *it lands* head up twice and let B denote the number of tosses that are required. If $B=b$, then since exactly two of the tosses are H's, the remaining tosses are T's and the first H can appear on any one of the first $b-1$ tosses, we have $P(B = b) = (b-1)q^{b-2}p^2$. Note that, much as in part (b) of Exercise I, we have

$$\sum_{b=2}^{\infty}(b-1)q^{b-2}p^2 = p^2\sum_{b=2}^{\infty}(b-1)q^{b-2} = p^2\sum_{b=1}^{\infty}bq^{b-1} = p^2\frac{d}{dq}\left(\sum_{b=0}^{\infty}q^b\right) = p^2\frac{d}{dq}\left(\frac{1}{1-q}\right) = \frac{p^2}{(1-q)^2}.$$

However, since $q = 1-p$, $\frac{p^2}{(1-q)^2} = 1$ and therefore $\sum_{b=2}^{\infty}P(B=b) = 1$, so our claim that $P(B=b) = (b-1)q^{b-2}p^2$ is consistent. Now we will find the expected value of B. We have

$$E(B) = \sum_{b=2}^{\infty}bP(B=b) = \sum_{b=2}^{\infty}b(b-1)q^{b-2}p^2 = p^2\frac{d^2}{dq^2}\left(\sum_{b=0}^{\infty}q^b\right) = p^2\frac{d^2}{dq^2}\left(\frac{1}{1-q}\right) = p^2\frac{2}{(1-q)^3}.$$

But this last expression equals $2/p$, and this should really not come as much of a surprise. Note that on average we need $1/p$ tosses to get the first head and then another $1/p$ tosses to get the second head, or $2/p$ tosses in all. The random variable B has a name as well; it is called a *negative binomial random variable* with parameters $n = 2$ and p.

Before moving on to another "famous" random variable, we need to extend the definition of expected value to a more general setting. Suppose that X is a random variable that takes on the values $\{x_1, x_2, \ldots\}$. Then the expected value of X, i.e., $E(X)$, is defined as $\sum_{i=1}^{\infty}x_iP(X=x_i)$, or more simply $\sum xP(X=x)$, where the second sum is understood to be taken over all of the values assumed by X. An example should help clarify this idea.

Example 1

Suppose X takes on the values 1, 3, 5,... with probabilities 1/2, 1/4, 1/8, ... respectively, i.e., $P(X=2k-1) = 1/2^k$ for k = 1, 2,.... Then the expected value of X is given by

$$E(X) = 1\cdot\frac{1}{2} + 3\cdot\frac{1}{4} + 5\cdot\frac{1}{8} + \ldots = \sum_{k=1}^{\infty}(2k-1)\cdot\frac{1}{2^k}.$$

We can find the sum of this series using ideas like those used above, but there is an easier way. Note that if Y is an arbitrary random variable and $Z = a + bY$ where a and b are constants with b, not zero, then since $Z = a + bY$ if and only if $Y = y$, we have

$$E(Z) = \sum(a+by)P(Z=a+by) = \sum(a+by)P(Y=y) = \left(\sum aP(Y=y)\right) + \left(\left(\sum byP(Y=y)\right)\right).$$

The final expression above can be written as $\left(a\sum P(Y=y)\right)+\left(b\left(\sum yP(Y=y)\right)\right)$, and we see that the value of this expression is $a+bE(Y)$ since $\sum P(Y=y)=1$ as the probability that Y takes on some value is 1 and, by definition of expected value, $\sum yP(Y=y)=E(Y)$. Thus, we have shown that $E(a+bY)=a+bE(Y)$. This is a very useful result. In this case, using the notation of Exercise I, we see that the random variable X in this example can be written as $2X_{1/2}-1$, and then using the formula we have just established $E(X) = 2E(X_{1/2})-1 = 2 \cdot 2 - 1 = 3$.

It also needs to be noted that not every random variable has an expected value. Here is a simple example that illustrates this fact.

Example 2

Let $X = x$ with probability $\frac{1}{x(x+1)}$ for x = 1, 2, 3,.... We need to check that the sum of the $P(X = x)$ equals 1 to be sure that we have a well-defined random variable. To this end, consider $\sum_{x=1}^{n}\frac{1}{x(x+1)}$, the nth partial sum of the infinite series $\sum_{x=1}^{\infty}\frac{1}{x(x+1)}$. Since $\frac{1}{x(x+1)}=\frac{1}{x}-\frac{1}{x+1}$, the partial sum "telescopes" as follows:

$$\sum_{x=1}^{n}\frac{1}{x(x+1)}=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{n+1}.$$

Since the limit of this last expression is 1 as n goes to infinity, the infinite series $\sum_{x=1}^{\infty}\frac{1}{x(x+1)}$ does have a value of 1 as required. Now what about the expected value of X? Consider $E(X)=\sum_{x=1}^{\infty}xP(X=x)=\sum_{x=1}^{\infty}\frac{x}{x(x+1)}=\sum_{x=1}^{\infty}\frac{1}{x+1}$.

But we recognize this last series as the harmonic series with its first term missing, and we know that this series diverges, or diverges to positive infinity. We would say then either that X has an infinite expected value or that the expected value of X does not exist.

As a final example before going on to our next (and last) famous random variable, we note that random variables can assume negative values. When they do, there may be some ambiguity about the existence of their expected values. The following example illustrates this.

Example 3

Suppose that $X = 1, 2, 3, -4, \ldots$ with $P\left(X = (-1)^{k+1}k\right) = \frac{1}{k(k+1)}$ for $k = 1, 2, \ldots$. Then, as in Example 2, $\sum P(X = x) = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$ so the random variable is well defined. Moreover, $E(X) = \sum xP(X = x) = \sum_{k=1}^{\infty} (-1)^{k+1} k \frac{1}{k(k+1)} = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots$, and this series obviously satisfies the hypotheses of the alternating series test and hence converges. Thus, X does have a finite expected value, at least for the moment. In fact, since $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots = 1 - \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right)$, and the expression in parentheses is the "alternating harmonic series," we see that $E(X) = 1 - \ln(2)$.

However, many mathematicians would claim that $E(X)$ does not exist, since for them, the existence of the expected value requires that $\sum |x| P(X = x)$ converge, i.e., that the series that defines the expected value *converges absolutely*. And, of course, we have already seen that this does not happen in this case. Thus, for these mathematicians, the expected value of X does not exist.

We introduce another justly famous random variable. If λ is a positive real number, define the random variable, X, by $P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$ for $x = 0, 1, 2, 3, \ldots$. This is a *Poisson* random variable, named after the French mathematician Siméon-Denis Poisson (1781–1840). We say that it has *parameter* λ. Before exploring some of the properties of X, let us check that it is well defined, i.e., that $\sum_{x=0}^{\infty} P(X = x) = \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} = 1$.

In fact, this is an immediate consequence of the well-known identity $\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$, which is valid for all real λ. This familiar fact is usually established by considering the Maclaurin Series of e^{λ}, showing first that the series converges for all real λ (using the ratio test, perhaps) and then showing that the series converges to e^{λ} for all real λ (using one of the forms of Taylor's theorem, perhaps). An alternative derivation involving separable differential equations uses the following ideas: the derivative with respect to t of the series $\sum_{x=0}^{\infty} \frac{t^x}{x!}$ is $\sum_{x=0}^{\infty} \frac{xt^{x-1}}{x!} = \sum_{x=1}^{\infty} \frac{xt^{x-1}}{x!} = \sum_{x=1}^{\infty} \frac{t^{x-1}}{(x-1)!} = \sum_{x=0}^{\infty} \frac{t^x}{x!}$ so the function $y = y(t)$ satisfies $y' = y$, and since $y(0) = 1$. $y(t) = e^t$. A key step is this derivation is the observation that $x/x! = x/(x(x-1)!) = 1/x(x-1)!$ for $x > 0$.

Let's find the expected value of X. Using, once again, $x/x! = 1/(x-1)!$, we have

$$E(X)=\sum_{x=0}^{\infty} x\frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^x}{(x-1)!} = \lambda e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!} = \lambda e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = \lambda e^{-\lambda}e^{\lambda} = \lambda .$$

Thus, the parameter λ is actually the expected value of the random variable.

Poisson Random Variables

Poisson random variables are useful in a number of settings. They generally model the number of occurrences of a "rare" event in time intervals of fixed length, for example, the number of extinctions of marine invertebrate families in a time interval of fixed length. As a second example, the so-called binomial probabilities can be approximated by Poisson probabilities. If p is a probability and n is positive integer, then $C(n,k)p^k(1-p)^{n-k}$ is a binomial probability where $C(n,k)=\dfrac{n!}{k!(n-k)!}$ is a so-called *binomial coefficient*. Computing one of these binomial probabilities is quite a chore if n is large. However, it turns out that

$$C(n,k)p^k(1-p)^{n-k} \approx \frac{\lambda^k e^{-\lambda}}{k!} \text{ if } \lambda = np \text{ and } n \text{ is large and } p \text{ is small}.$$

This is not too hard to prove in general, but we will prove only one case, i.e., $k = 1$, in which case we need to show that

$$np^1(1-p)^{n-1} \approx \frac{\lambda^1 e^{-\lambda}}{1!} = \lambda e^{-\lambda} \text{ if } \lambda = np \text{ and } n \text{ is large and } p \text{ is small}.$$

We will establish this by showing that the limit of the left-hand side, as n goes to infinity, is the right-hand side. We assume that $\lambda = np$ so that $p = \lambda/n$.

$$np^1(1-p)^{n-1} = \lambda\left(1-\frac{\lambda}{n}\right)^{n-1} = \frac{\lambda\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^1}.$$

As n goes to infinity, the numerator of the fraction goes to $\lambda e^{-\lambda}$, and the denominator goes to 1. Thus, we have the result we need. (To find the limit of the numerator we used the well-known result $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$ with r = $-\lambda$. You can use L'Hopital's Rule to prove this limit theorem.) To see how good an approximation we have, consider $n = 25$ and $p = 0.05$, and note that $np^1(1-p)^{n-1} = 0.364986$ while $\lambda e^{-\lambda} = 0.358131$.

We finish this unit with an interesting result. Suppose that for $i = 1, 2$, X_i is a Poisson random variable with parameter λ_i We say that X_1 and X_2 are *independent* *random variables* if $P(X_1 = x_1 \text{ and } X_2 = x_2) = P(X_1 = x_1)P(X_2 = x_2)$. (Independence is a crucial property in probability theory and is a much more natural idea than it might seem from this definition.) Consider the "new" random variable $S = X_1 + X_2$. Let's see if we can compute $P(S = s)$ where s is a nonnegative integer. We see that

$$P(S=s)=\sum_{x=0}^{s} P(X_1 = x \text{ and } X_2 = s-x)=\sum_{x=0}^{s} P(X_1 = x)P(X_2 = s-x)=\sum_{x=0}^{s}\frac{\lambda_1{}^x e^{-\lambda_1}}{x!}\frac{\lambda_2{}^{s-x}e^{-\lambda_2}}{(s-x)!}.$$

But $\displaystyle\sum_{x=0}^{s}\frac{\lambda_1{}^x e^{-\lambda_1}}{x!}\frac{\lambda_2{}^{s-x}e^{-\lambda_2}}{(s-x)!}=\frac{e^{-\lambda_1}e^{-\lambda_2}}{s!}\sum_{x=0}^{s}\frac{s!}{x!(s-x)}\lambda_1{}^x\lambda_2{}^{s-x}=\frac{e^{-(\lambda_1+\lambda_2)}}{s!}\sum_{x=0}^{s}C(s,x)\lambda_1{}^x\lambda_2{}^{s-x}$,

and we (hopefully) recognize this last sum as the result of applying the binomial theorem to $(\lambda_1 + \lambda_2)^S$. Then we have proved that

$P(S=s)=\dfrac{e^{-(\lambda_1+\lambda_2)}}{s!}(\lambda_1+\lambda_2)^s=\dfrac{(\lambda_1+\lambda_2)^s e^{-(\lambda_1+\lambda_2)}}{s!}$. But this implies that S is a Poisson random variable with parameter $\lambda_1 + \lambda_2$. This is certainly an interesting result. The following exercise, which concludes this unit, gives another interesting result of the same type.

Exercise II

Suppose that X_1 and X_2 are independent geometric random variables with the same parameter p. So, $P(X_i = x) = pq^{x-1}$ where $q = 1 - p$. Let $S = X_1 + X_2$.

(a) Show that $P(S = s) = (s - 1)p^2q^{s-2}$ for $s = 2, 3, \ldots$.

(b) Compare the probabilities given in part (a) with those associated with the negative binomial distribution that was introduced right after Exercise I and conclude that S is a negative binomial random variable with parameters $n = 2$ and p.

(c) Convince yourself that the sum of two independent geometric random variables, each with parameter p, should be a negative binomial random variable with parameters $n = 2$ and p. (Hint: If we toss a coin until we get two heads (H's), we could let X_1 denote the number of tosses required to get the first head and X_2 denote the number of tosses required to get the second head. Then $S = X_1 + X_2$ is the total number of tosses required to get two H's.)

Approximating the Sum of a Convergent Series

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The AP Calculus BC Course Description mentions how technology can be used to explore convergence and divergence of series, and lists various tests for convergence and divergence as topics to be covered. But no specific mention is made of actually estimating the sum of a series, and the only discussion of error bounds is for alternating series and the Lagrange error bound for Taylor polynomials. With just a little additional effort, however, students can easily approximate the sums of many common convergent series and determine how precise those approximations will be.

Approximating the Sum of a Positive Series

Here are two methods for estimating the sum of a positive series whose convergence has been established by the integral test or the ratio test. Some fairly weak additional requirements are made on the terms of the series. Proofs are given in the appendix.

Let $S = \sum_{n=1}^{\infty} a_n$ and let the nth partial sum be $S_n = \sum_{k=1}^{n} a_k$.

1. Suppose $a_n = f(n)$ where the graph of f is positive, decreasing, and concave up, and the improper integral $\int_1^{\infty} f(x)\,dx$ converges. Then

$$S_n + \int_{n+1}^{\infty} f(x)\,dx + \frac{a_{n+1}}{2} < S < S_n + \int_{n}^{\infty} f(x)\,dx - \frac{a_{n+1}}{2}. \quad (1)$$

In Example 2 you will see how this holds for $n \geq N$ if we only know f is positive, decreasing, and concave up on the interval $[N, \infty)$.

2. Suppose (a_n) is a positive decreasing sequence and $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=L<1$.

If $\frac{a_{n+1}}{a_n}$ decreases to the limit L, then $S_n+a_n\left(\frac{L}{1-L}\right)<S<S_n+\frac{a_{n+1}}{\left(1-\frac{a_{n+1}}{a_n}\right)}$, (2)

If $\frac{a_{n+1}}{a_n}$ increases to the limit L, then $S_n+\frac{a_{n+1}}{\left(1-\frac{a_{n+1}}{a_n}\right)}<S<S_n+a_n\left(\frac{L}{1-L}\right)$. (3)

Example 1: $S=\sum_{n=1}^{\infty}\frac{1}{n^2}$

The function $f(x)=\frac{1}{x^2}$ is positive with a graph that is decreasing and concave up for $x\geq 1$, and $a_n=f(n)$ for all n. In addition, $\int_1^{\infty}f(x)dx$ converges. This series converges by the integral test. By inequality (1),

$$S_n+\frac{1}{n+1}+\frac{1}{2(n+1)^2}<S<S_n+\frac{1}{n}-\frac{1}{2(n+1)^2}.$$

This inequality implies that S is contained in an interval of width

$$\frac{1}{n}-\frac{2}{2(n+1)^2}-\frac{1}{n+1}=\frac{1}{n(n+1)^2}.$$

If we wanted to estimate S with error less than 0.0001, we could use a value of n with $\frac{1}{n(n+1)^2}<0.0002$ and then take the average of the two endpoints in inequality (4) as an approximation for S. The table feature on a graphing calculator shows that n = 17 is the first value of n that works. Inequality (4) then implies that 1.6449055 < S < 1.6450871[1] and a reasonable approximation would be

$$S\approx\frac{1.6449055+1.6450870}{2}\approx 1.645$$

to three decimal places. With n = 100, inequality (4) actually shows that 1.6449339 < S < 1.6449349, and hence we know for sure that S = 1.64493.... Of course, in this case we actually know that $S=\frac{\pi^2}{6}$ = 1.644934066.... Notice also that $S_{100}\approx 1.6349839$, so the partial sum with 100 terms is a poor approximation by itself.

1. We will use the convention for positive endpoints of truncating the left endpoint of the interval and rounding up the right endpoint. This will make the interval slightly larger than that given by the actual symbolic inequality.

Example 2: $S=\sum_{n=1}^{\infty}\frac{n}{n^4+1}$

Let $f(x)=\frac{x}{x^4+1}$. The graph of f is decreasing and concave up for $x \geq 2$. Also $\int_n^{\infty}\frac{x}{x^4+1}=\frac{\pi}{4}-\frac{1}{2}\arctan(n^2)$ and so the improper integral converges. Let $T=\sum_{n=2}^{\infty}\frac{n}{n^4+1}$ with partial sums T_n for $n \geq 2$. By inequality (1),

$$T_n+\frac{\pi}{4}-\frac{1}{2}\arctan((n+1)^2)+\frac{n+1}{2((n+1)^4+1)}<T<T_n+\frac{\pi}{4}-\frac{1}{2}\arctan(n^2)-\frac{n+1}{2((n+1)^4+1)}$$

for $n \geq 2$. But now add a_1, the first term of the series for S, to each term of this inequality to see that

$$S_n+\frac{\pi}{4}-\frac{1}{2}\arctan((n+1)^2)+\frac{n+1}{2((n+1)^4+1)}<S<S_n+\frac{\pi}{4}-\frac{1}{2}\arctan(n^2)-\frac{n+1}{2((n+1)^4+1)}$$

for $n \geq 2$. Using $n = 10$ in this inequality yields $0.6941559<S<0.6942724$. We can conclude that $S \approx 0.694$ to three decimal places.

Example 3: $S=\sum_{n=0}^{\infty}\frac{1}{n!}$

The terms of this series are decreasing. In addition,

$$\frac{a_{n+1}}{a_n}=\frac{1}{(n+1)!}\cdot\frac{n!}{1}=\frac{1}{n+1}$$

which decreases to the limit $l = 0$. By inequality (2)

$$S_n<S<S_n+\frac{\frac{1}{(n+1)!}}{1-\frac{1}{n+1}}=S_n+\frac{1}{n!n}$$

for all n. Using $n = 10$ in this inequality yields $2.7182818 < S < 2.7182819$ and hence $S \approx 2.7182818$. These, of course, are the first seven decimal places of $e = 2.718281828\ldots$.

Example 4: $S=\sum_{n=1}^{\infty}\frac{1}{n^2 5^n}$

We have $\frac{a_{n+1}}{a_n}=\frac{1}{(n+1)^2 5^{n+1}}\cdot\frac{n^2 5^n}{1}=\left(\frac{n}{n+1}\right)^2\cdot\frac{1}{5}$, which increases to the limit $L=\frac{1}{5}$.

According to inequality (3)

$$S_n + \frac{\frac{1}{(n+1)^2 5^{n+1}}}{1-\frac{1}{5}\left(\frac{n}{n+1}\right)^2} < S < S_n + \frac{1}{n^2 5^n}\cdot\frac{\frac{1}{5}}{\frac{4}{5}},$$

which simplifies to

$$S_n + \frac{1}{\left(4n^2+10n+5\right)5^n} < S < S_n + \frac{1}{4n^2 5^n}.$$

With $n = 5$, this inequality shows that $0.2110037 < S < 0.2110049$.

Example 5: $S = \sum_{n=1}^{\infty} \frac{n!}{n^n}$

We have $\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n}$, which is less than 1 for all n and which decreases to the limit $L = \frac{1}{e}$. From inequality (2) we get (after some simplification)

$$S_n + \frac{n!}{n^n}\frac{1}{e-1} < S < S_n + \frac{n!}{(n+1)^n - n^n}.$$

Using $n = 10$ gives $1.8798382 < S < 1.8792548$.

Approximating the Sum of an Alternating Series

Let $S = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$ and let the nth partial sum be $S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$. We assume that (a_n) is a positive decreasing sequence that converges to 0.

1. The standard error bound is given by $S_n - a_{n+1} < S < S_n + a_{n+1}$. (5)
2. Suppose the sequence defined by $b_n = a_n - a_{n+1}$ decreases monotonically to 0. (One way to achieve this is if $a_n = f(n)$ where f is positive with a graph that is decreasing asymptotically to 0 and concave up.) Then

   if $S_n < S$, then $S_n + \frac{a_{n+1}}{2} < S < S_n + \frac{a_n}{2}$; (6)

   if $S < S_n$, then $S_n - \frac{a_n}{2} < S < S_n - \frac{a_{n+1}}{2}$. (7)

Both of these can be summarized by the inequality $\frac{a_{n+1}}{2} < |S - S_n| < \frac{a_n}{2}$.

Inequality (5) is credited to Leibniz and is the error bound described in the AP Calculus BC Course Description. Inequalities (6) and (7) are consequences of a proof published in 1962 by Philip Calabrese, then an undergraduate student at the Illinois Institute of Technology (Calabrese 1962). Calabrese proved that $|S - S_n| < \varepsilon$ if $a_n \le 2\varepsilon$, and that furthermore, if $a_n = 2\varepsilon$ for some n, then S_n is the first partial sum within ε of the sum S. See the appendix for the derivation of inequalities (6) and (7).

Example 6: $S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{2n-1}$

This is an alternating series that converges by the alternating series test. If $f(x) = \frac{4}{2x-1}$, then the graph of f is positive, decreasing to 0, and concave up for $x \ge 1$. For odd n, inequality (7) implies that

$$S_n - \frac{2}{2n-1} < S < S_n - \frac{2}{2n+1}. \tag{8}$$

If we wanted to estimate the value of S with error less than 0.0001, the typical method using the error bound from inequality (5) would use a value of n for which $a_{n+1} = \frac{4}{2n+1} < 0.0001$. This would require using 20,000 terms. On the basis of inequality (8), however, we can take as an estimate for S the midpoint of that interval, that is, for odd n,

$$S \approx S_n - \frac{1}{2}\left(\frac{2}{2n+1} + \frac{2}{2n-1}\right) = S_n - \frac{4n}{4n^2-1}, \tag{9}$$

with an error less than half the width of the interval. So for an error less than 0.0001, we only need $\frac{1}{2}\left(\frac{2}{2n-1} - \frac{2}{2n+1}\right) = \frac{2}{4n^2-1} < 0.0001.$

The first odd solution is $n = 71$, just a bit less than 20,000! The estimate from (9) using $n = 71$ is $S \approx 3.1415912$, with error less than 0.0001. Since $S = \pi$, this estimate is actually within 1.4×10^{-6} of the true value. By the way, the partial sum S_{71} is approximately 3.1556764.

Example 7: $S = \sum_{n=0}^{\infty} (-1)^n \frac{18^n}{(2n)!}$

This is an alternating series that converges by the alternating series test. Let $b_n = a_n - a_{n+1}$. It is not obvious that the sequence b_n decreases monotonically to 0. An investigation with the table feature of a graphing calculator, however, suggests that this is true for $n \geq 3$. We can therefore use inequality (6) when n is an odd integer greater than 3 (note that inequality (6) holds for odd n's because this series starts with $n = 0$.) Hence $S_n + \frac{1}{2}\frac{18^{n+1}}{(2n+2)!} < S < S_n + \frac{1}{2}\frac{18^n}{(2n)!}$ for odd $n \geq 3$.

With $n = 9$ we can estimate that S lies in the interval (–0.4526626, –0.4526477)[2], an interval of length 1.49×10^{-5}. But wait, we can actually do better than this! Since the terms of this series decrease so quickly because of the factorial in the denominator, we actually have $a_{n+1} < \frac{1}{2}a_n$ for n ≥ 3. So if we combine inequalities (5) and (6), we can deduce that for this series,

$$S_n + \frac{1}{2}\frac{18^{n+1}}{(2n+2)!} < S < S_n + \frac{18^{n+1}}{(2n+2)!} \text{ for odd } n \geq 3.$$

Now $n = 9$ gives the interval (–0.4526626, –0.4526618) containing the value of S, an interval of length 8×10^{-7}. (Note: What is the exact sum of this series?)

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2. For negative endpoints, round down the left endpoint and truncate the right endpoint.

Appendix

Proof of Inequality (1)

Let $S = \sum_{n=1}^{\infty} a_n$ and let $S_n = \sum_{k=1}^{n} a_k$. Suppose $a_n = f(n)$ where the graph of f is positive, decreasing to 0, and concave up, and the improper integral $\int_1^{\infty} f(x)\,dx$ converges. The series converges by the integral test. Because the graph is concave up, the area of the shaded trapezoid of width 1 shown in Figure 1 is greater than the area under the curve. Therefore $\int_{n+1}^{n+2} f(x)\,dx < \frac{1}{2}\left(a_{n+1} + a_{n+2}\right)$.

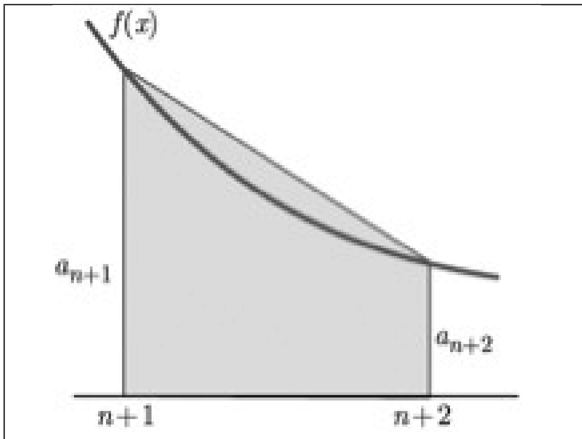


Figure 1

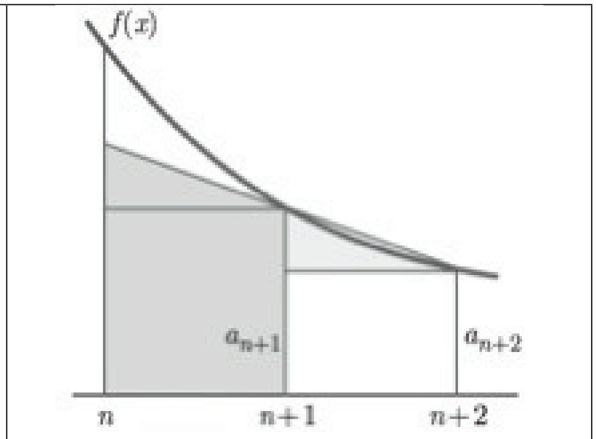


Figure 2

$$\text{Hence } \int_{n+1}^{\infty} f(x)\,dx < \frac{1}{2}\left(a_{n+1} + a_{n+2}\right) + \frac{1}{2}\left(a_{n+2} + a_{n+3}\right) + \frac{1}{2}\left(a_{n+3} + a_{n+4}\right) + \cdots$$
$$= \frac{1}{2} a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$
$$= S - S_n - \frac{1}{2} a_{n+1}$$

In Figure 2, the graph of f lies above the tangent line at $x = n+1$ (because of the positive concavity) and therefore also lies above the continuation of the secant line between $x = n+1$ and $x = n+2$. This implies that the area of the shaded trapezoid in

Figure 2 of width 1 between $x = n$ and $x = n+1$ is less than the area under the curve, and so $\int_n^{n+1} f(x)\,dx > a_{n+1} + \frac{1}{2}(a_{n+1} - a_{n+2})$.

Hence

$$\int_n^{\infty} f(x)\,dx > a_{n+1} + \frac{1}{2}(a_{n+1} - a_{n+2}) + a_{n+2} + \frac{1}{2}(a_{n+2} - a_{n+3}) + a_{n+3} + \frac{1}{2}(a_{n+3} - a_{n+4}) + \cdots$$

$$= \frac{1}{2}a_{n+1} + a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

$$= \frac{1}{2}a_{n+1} + S - S_n$$

Proof of Inequalities (2) and (3)

Let $S = \sum_{n=1}^{\infty} a_n$ and let $S_n = \sum_{k=1}^{n} a_k$. Suppose (a_n) is a positive decreasing sequence and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$, where the ratios decrease to L. The series converges by the ratio test.

Let $r = \frac{a_{n+1}}{a_n} < 1$. Then $\frac{a_{k+1}}{a_k} < r$ for all $k \geq n$. Hence

$$a_{n+1} < a_n r$$

$$a_{n+2} < a_{n+1} r < a_n r^2$$

$$a_{n+3} < a_{n+2} r < a_n r^3$$

We therefore conclude that $S - S_n = \sum_{k=n+1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{n+k} < \sum_{k=1}^{\infty} a_n r^k = \frac{a_n r}{1-r} = \frac{a_{n+1}}{1 - \frac{a_{n+1}}{a_n}}$.

But we also have $L < \frac{a_{k+1}}{a_k}$ for all $k \geq n$. By a similar argument as above,

$$S - S_n = \sum_{k=n+1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{n+k} > \sum_{k=1}^{\infty} a_n L^k = a_n \frac{L}{1-L}.$$

Combining these two results gives inequality (2). A similar argument for the inequalities with r and L reversed proves inequality (3).

Proof of Inequalities (6) and (7)

Let $S = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$ and let $S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$, where (a_n) is positive decreasing sequence that converges to 0. Let $b_n = a_n - a_{n+1}$, where we assume that the sequence (b_n) also decreases monotonically to 0. Then $S = S_n + (-1)^n (b_{n+1} + b_{n+3} + b_{n+5} + \cdots)$ and

$S = S_{n-1} + (-1)^{n+1}\left(b_n + b_{n+2} + b_{n+4} + \cdots\right).$

Because the sequence (b_n) decreases,

$\left|S - S_n\right| = b_{n+1} + b_{n+3} + b_{n+5} + \cdots < b_n + b_{n+2} + b_{n+4} + \cdots = \left|S - S_{n-1}\right|.$

Therefore $\left|S - S_n\right| < \left|S - S_{n-1}\right|$. Similarly, $\left|S - S_{n+1}\right| < \left|S - S_n\right|$. But S lies between the successive partial sums, so it follows that

$$a_n = \left|S_n - S_{n-1}\right| = \left|S - S_n\right| + \left|S - S_{n-1}\right| > 2\left|S - S_n\right|$$

and

$$a_{n+1} = \left|S_{n+1} - S_n\right| = \left|S - S_{n+1}\right| + \left|S - S_n\right| < 2\left|S - S_n\right|.$$

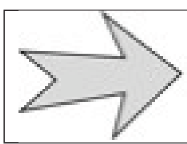
Combining these two results shows that $\dfrac{a_{n+1}}{2} < \left|S - S_n\right| < \dfrac{a_n}{2}$, from which inequalities (6) and (7) can be obtained.

‘Positively Mister Gallagher. Absolutely Mister Shean.’

Steve Greenfield
Rutgers University
Piscataway, New Jersey

The Title

The comedy team of Gallagher and Shean was very popular in U.S. vaudeville during the early 1900s. Their theme song was a huge hit, and its refrain began with the line quoted above, “Positively Mister Gallagher. Absolutely Mister Shean!” This seems appropriate for an exposition of infinite series that shows how absolutely convergent infinite series are almost as nice as convergent series with positive terms.



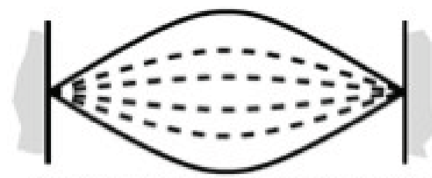
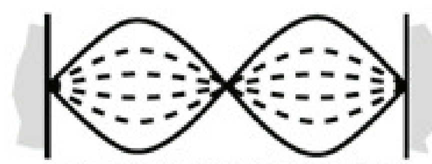
I’ll use this “advice arrow” for comments about classroom implementation of some of the ideas discussed. I’ll try to record frankly whether it is likely to succeed or not, and make some other suggestions.

Other Kinds of Series

Power series are not the only series that are widely used. In December 2007, Google had about 1,120,000 responses to the phrase “power series” and 573,000 responses to “Fourier series.” Fourier series are infinite sums of sine and cosine functions. They were developed to understand vibrational problems in the late 1700s, and then were used by Fourier to analyze heat transfer about 50 years later. The nicest Fourier series to consider are those that are absolutely convergent. Any convergent series that can be rearranged and regrouped without changing the sum must be an absolutely convergent series. Such series are easier to discuss theoretically, and computations with partial sums are also more likely to have results that are more reliable numerically than computations with conditionally convergent series.

Sounds and Fourier Series

I'll consider Fourier series in the context of sounds and hearing. Sound is vibration in a transmitting medium. This could be solid (you can hear through metal or wood) or liquid (through water), but usually is gaseous. The standard transmission medium is the atmosphere.

A vibration is produced by, say, a string moving back and forth. The string produces a dominant tone that usually has the most energy. A picture of a string producing this dominant tone is shown to the right. What doesn't appear, of course, is the kinetic aspect of the vibration: how fast the string oscillates. The profile of the string when it is most distorted from the neutral, rest position is $k \sin(x)$ on the interval $[0, \pi]$ (if the units are chosen correctly). The constant k measures the amount of energy (loudness).	 The dominant tone or frequency
Natural sound production is complex, and pure tones are rarely produced by natural processes. Vibrating objects also produce overtones or harmonics, which have frequencies with integer multiples of the original frequency. The fingering of string or wind instruments can strengthen the production of some of the harmonics. Drawn to the right is an attempt to show the vibrating string producing the first overtone, which would be modeled by $\sin(2x)$ on $[0, \pi]$.	 The first harmonic or overtone

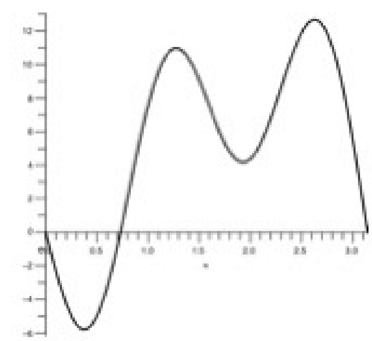
A real vibrating string (a guitar) or membrane (a drum) or column of air (a flute) produces a complicated collection of tones and overtones. A Web page of the Baylor College of Medicine states:

> *The sensory organs of the eye, ear, tongue and skin are each sensitive to specific forms of energy. The nose and tongue detect chemical energy, the eye detects light energy, the skin detects heat and mechanical energy. Sound is a form of mechanical energy. Mechanical forces can be steady, like … weight … or they can vibrate, like your car when it goes over speed bumps. Sound is generated by mechanical vibrations (such as a vibrating piano string). This sets up small oscillations of air molecules that in turn cause adjacent molecules to oscillate as the sound propagates away from its source. Sound is called a pressure wave because when the molecules of air come closer together the pressure increases (compression) as they move further apart the pressure decreases (rarefaction). … The velocity of sound in air is around 1,100 ft/sec [about 1/3 km/sec] … Sound waves travel fastest in solids, slower in liquids and slowest in air. Sound vibrations extend from a few cycles per second to millions of cycles per second. Human hearing is limited to a range of between 20 to 20,000 cycles per second.*[3]

3. From "How the Ear Works--Nature's Solutions for Listening" by William E. Brownell, Ph.D., Baylor College of Medicine (http://www.bcm.edu/oto/research/cochlea/Volta/index.html). Used by permission of William E. Brownell, Ph.D.

One model of sound for a vibrating string with the ends held fixed would use a *finite Fourier sine series*, $\sum_{j=1}^{N} a_j \sin(jx)$. The dominant tone would occur with $j = 1$, and higher j's would be used for the overtones. The constants a_j would indicate the relative strengths or amplitudes of the sound. A model of a more general situation where the ends of the string would move as well would include cosine terms, but we'll just consider sine series in this discussion. Please note that mathematical modeling of what happens to sound and how it is translated to nerve impulses in the human ear is not completely satisfactory, and is an object of current research.

To the right is a picture of a specific finite Fourier sine series, $f(x) = 8\sin(x) - 4\sin(2x) - 6\sin(4x)$. The signal, f, doesn't seem to display in any obvious way the specific amplitudes 8, −4, and −6, along with the corresponding frequencies, 1, 2, and 4. But in fact, these can be "read off" the graph using some clever natural phenomena. Inside the ear, in a structure called the *cochlea*, there is a collection of cells (called colloquially *hair cells*, but very different from and much more sophisticated than the external hair covering mammals) that vibrate sympathetically to varying frequencies of the sounds transmitted. What they do is respond to the amplitude of sound waves of appropriate frequency. The response principally depends on the length of the hair cell, but there are variations depending on their structure and location. Mathematically, the amount of response is modeled by $\int_0^{\pi} f(x)\sin(mx)\,dx$ where m is a positive integer.



Computing Some Integrals

Let's compute $\frac{2}{\pi}\int_0^{\pi} f(x)\sin(mx)dx$ for $f(x) = 8\sin(x) - 4\sin(2x) - 6\sin(4x)$.

$$\frac{2}{\pi}\int_0^{\pi} f(x)\sin(mx)dx = 8\cdot\frac{2}{\pi}\int_0^{\pi}\sin(x)\sin(mx)dx - 4\cdot\frac{2}{\pi}\int_0^{\pi}\sin(2x)\sin(mx)dx - 6\cdot\frac{2}{\pi}\int_0^{\pi}\sin 4(x)\sin(mx)dx$$

A standard exercise using integration by parts and half-angle formulas will show:

$$\frac{2}{\pi}\int_0^{\pi}\sin(nx)\sin(mx)dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$ when m and n are integers. The factor of $\frac{2}{\pi}$ is chosen to give an integral of 1 when $m = n$.

How a Physicist Would Compute These Integrals

Since the integrals just mentioned are so important, another way of computing them should be mentioned. *Of course* (!) $e^{ix} = \cos(x) + i\sin(x)$(Euler's formula: compare the Taylor series on both sides of the equation). Then $e^{-ix} = \cos(-x) + i\sin(-x) =$

$\cos(x) - i\sin(x)$ and we can subtract the equations to get $e^{ix} - e^{-ix} = 2i\sin(x)$ so that $\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$. Now let's compute the antiderivative. I'll do the $n \neq m$ case:

$$
\begin{aligned}
\int \sin(nx)\sin(mx)dx &= \int\left(\frac{1}{2i}\left(e^{inx} - e^{-inx}\right)\right)\left(\frac{1}{2i}\left(e^{imx} - e^{-imx}\right)\right)dx \\
&= \left(\frac{1}{2i}\right)^2 \int \left(e^{i(n+m)x} + e^{-i(n+m)x} - e^{i(n-m)x} - e^{-i(n+m)x}\right)dx \\
&= -\left(\frac{1}{2}\right)^2\left(\frac{e^{i(n+m)} - e^{-i(n+m)}}{i(n+m)} - \frac{e^{i(n-m)} - e^{-i(n-m)}}{i(n-m)}\right) + C \\
&= -\frac{1}{2(n+m)}\left(\frac{e^{i(n+m)} - e^{-i(n+m)}}{2i}\right) + \frac{1}{2(n-m)}\left(\frac{e^{i(n-m)} - e^{-i(n-m)}}{2i}\right) + C \\
&= -\frac{\sin((n+m)x)}{2(n+m)} + \frac{\sin((n-m)x)}{2(n-m)} + C
\end{aligned}
$$

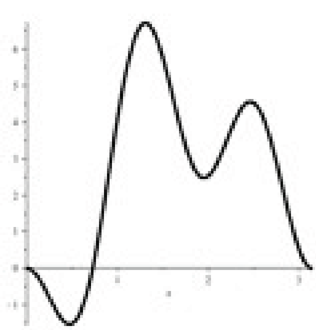
Back to Fourier Series

The "hair" in the cochlea that is attuned to the mth frequency responds in proportion to $\frac{2}{\pi}\int_0^{\pi} f(x)\sin(mx)dx$, which is (for $f(x) = 8\sin(x) - 4\sin(2x) - 6\sin(4x)$),

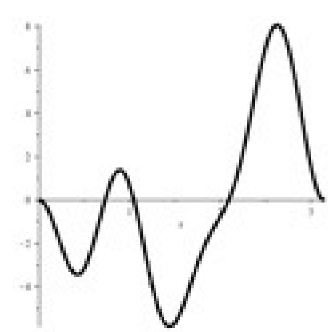
$$
\frac{2}{\pi}\int_0^{\pi} f(x)\sin(mx)dx = 8\cdot\overset{0 \text{ if } m \neq 1;\ 1 \text{ if } m = 1}{\frac{2}{\pi}\int_0^{\pi}\sin(x)\sin(mx)dx} - 4\cdot\overset{0 \text{ if } m \neq 2;\ 1 \text{ if } m = 2}{\frac{2}{\pi}\int_0^{\pi}\sin(2x)\sin(mx)dx} - 6\cdot\overset{0 \text{ if } m \neq 4;\ 1 \text{ if } m = 4}{\frac{2}{\pi}\int_0^{\pi}\sin 4(x)\sin(mx)dx}
$$

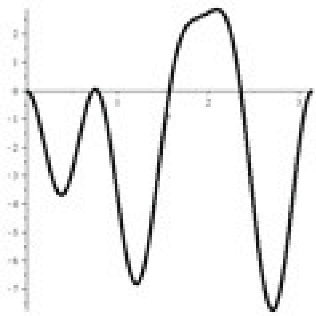
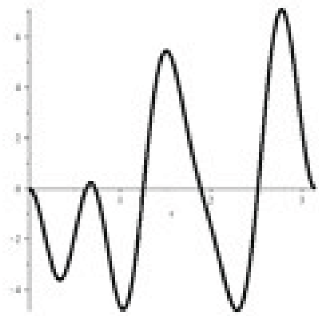
$$
\text{Therefore } \frac{2}{\pi}\int_0^{\pi} f(x)\sin(mx)dx = \begin{cases} 8 & \text{if } m = 1 \\ -4 & \text{if } m = 2 \\ -6 & \text{if } m = 4 \\ 0 & \text{otherwise} \end{cases}.
$$

Here's the product $\frac{2}{\pi} f(x)\sin(x)$ on the interval $[0, \pi]$. The integral of this product over the whole interval is 8.



Here's the product $\frac{2}{\pi} f(x)\sin(3x)$ on the interval $[0, \pi]$. The integral of this product over the whole interval is 0.

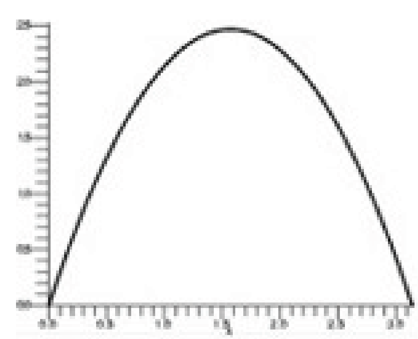


Here’s the product $\frac{2}{\pi} f(x)\sin(4x)$ on the interval [0, π] The integral of this product over the whole interval is -6. 	Here’s the product $\frac{2}{\pi} f(x)\sin(5x)$ on the interval [0, π]. The integral of this product over the whole interval is 0. 

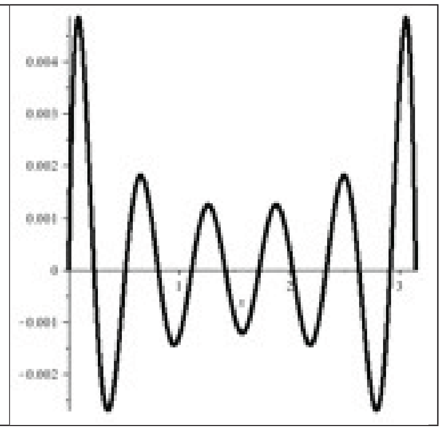
The pictures help to confirm that the computations are both mysterious and wonderful. You can actually see the integrals turning out to be the values already found.

Fourier Coefficients

Let me try to describe what happens in general. First, let’s look at those functions that can be written as finite sums of constant multiples of sine functions. If $f(x) = \sum_{m=1}^{N} a_m \sin(mx)$, then the results above are still valid: $a_m = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(mx)\,dx$. These numbers are called *Fourier coefficients*. They are named after Jean Baptiste Joseph Fourier (1768–1830), who had an extremely eventful life that included a great deal of political turmoil. Acquainted with both Napoleon and Robespierre, certainly he was not the stereotypical remote scholar and teacher. Fourier asserted that any function could be written as a (possibly infinite) sum of sine and cosine functions. Such an infinite series is now called a Fourier *series*.

Let’s consider $f(x) = x(\pi - x)$ on the interval [0, π]. If we believe that $f(x) = \sum_{m=1}^{\infty} a_m \sin(mx)$, then previous manipulations (integrations) suggest that the value of each a_m is given by the formula $a_m = \left(\frac{2}{\pi}\right)\int_0^{\pi} f(x)\sin(mx)\,dx$. For this $f(x)$ we can use integration by parts to get $a_m = \frac{8}{m^3\pi}$ if m is odd and 0 if m is even. (I *think* I could do these integrations by hand, integrating by parts twice, but I used Maple instead because I could.)	

The tenth partial sum of the Fourier sine series for this $f(x)$ is
$\frac{8}{\pi}\sin(x)+\frac{8}{27\pi}\sin(3x)+\frac{8}{125\pi}\sin(5x)+\frac{8}{343\pi}\sin(7x)+\frac{8}{729\pi}\sin(9x)$.
I would show the graph of this function, only most human beings couldn't really see any distinctions between the parabola above and this graph. Here is a graph of the ***difference*** between the original function and the partial sum. Please note the vertical scale. The difference between the partial sum and the function is between –.005 and .003, not visible to me in the original scale.



Almost 50 years passed before Fourier's claims about convergence of what's now called the Fourier series were proved for suitable functions. It is now known that if a function is twice differentiable then it will be equal to the sum of its Fourier series, but that fact is well beyond the scope of this paper.

Instructional Advice!

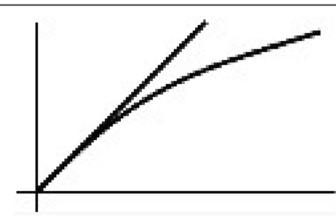
I presented the material in the preceding few pages as general background. I would be very cautious about showing it to a general audience of students (this means I wouldn't). What follows is material I have used in class to successfully stimulate questions and help students learn. I'll begin with an investigation that combines several aspects of calculus and can be used both in class and to construct thoughtful homework questions. I'll return to Fourier series later.

Mean Square Error

Here's a homework question I have assigned, accompanied by *hoped-for* versions of answers.

Part (a) For x near 0, sin(x) is well approximated by its tangent line at x = 0. What is this tangent line?

Answer This part should be easy for the well-educated calculus student. Since sin′ = cos, the slope of the tangent line to y = sin(x) at x = 0 is cos(0) = 1, and an equation for this tangent line is $y = x$. Although a picture is not requested, students may think of an image resembling what's shown.



Part (b) Approximation over an interval is preferred over approximation near a point for many purposes. One criterion for assessing the accuracy of such an approximation is *mean square error*. The mean square error between a straight line $y = Ax$ going through the origin and the function sin(x) over the interval [0, 1] is given by the definite integral $\int_0^1 (\sin(x) - Ax)^2\,dx$. Find the value of A that minimizes this integral.

Hint: Expand the integrand, compute the integral, and find the A minimizing the result.

Answer: This is decidedly *not* a routine question for most of my students. Let's call $f(A)$ the value of $\int_0^1 (\sin(x) - Ax)^2\,dx$.. We can expand $(\sin(x) - Ax)^2$ to get $(\sin(x))^2 - 2Ax\sin(x) + A^2x^2$. Now integrate (I am omitting the use of integration by parts; I usually assign this problem soon after integration by parts is introduced).

$$\int_0^1 \left((\sin(x))^2 - 2Ax\sin(x) + A^2x^2\right)dx = \left(-\frac{1}{2}\cos(x)\sin(x) + \frac{x}{2} - 2A(-x\cos(x) + \sin(x)) + \frac{A^2x^3}{3}\right)\Bigg|_0^1$$

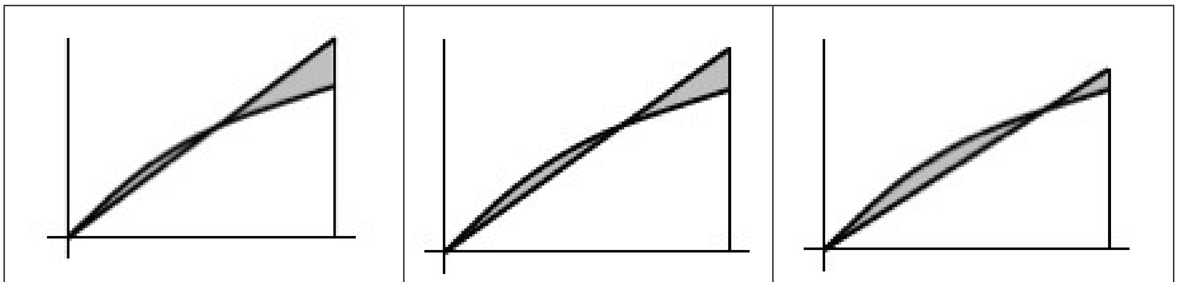
$$= -\frac{1}{2}\cos(1)\sin(1) + \frac{1}{2} - 2A\sin(1) + 2A\cos(1) + \frac{A^2}{3}$$

$$\approx 0.27268 - 0.60234A + 0.33333A^2$$

So we know that $f(A) = 0.27268 - 0.60234A + 0.33333A^2$, a quadratic polynomial with a positive second-degree coefficient, has a unique minimum when $f'(A) = -0.60234 + 0.666666A = 0$, and that happens when $A \approx 0.90351$.

Comment: I hope when I assign this problem, and when it is subsequently discussed in class, that students may have made some connection with how they treat numerical data obtained in a science lab. Very frequently a "best" linear approximation idea is mentioned there, and sometimes the least squares method of fitting data may even be suggested (the formulas needed are frequently available on calculators). I used 1 as the upper bound of the integral instead of something more "sine-ish" like $\frac{\pi}{2}$ because I want *numbers* here. The squaring of the integrand means that differences can't cancel, and there will be exactly one easily computable minimum for $f(A)$. If instead we defined $f(A)$ as $\int_0^1 |\sin(x) - Ax|\,dx$, there would still be a minimum, but it would not be readily computable. The minimum for this definition is $A \approx 0.919$, and that took some work to get.

I also hope that students would think about the following pictures:

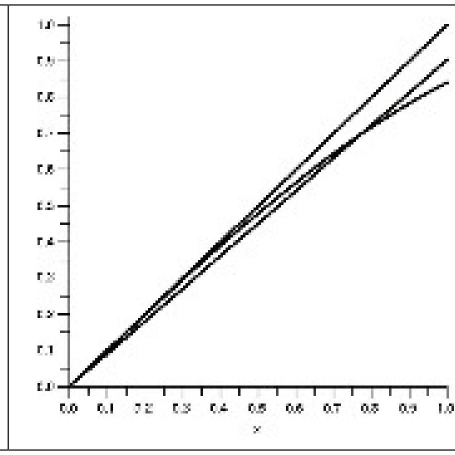


A is too large: The upper chunk of area is too big.	A is *just right.*	A is too small: The lower chunk of area is too big.

Part (c) Sketch sin(x) and the straight lines found in (a) and (b) on the unit interval [0, 1].

Answer: The desired picture is displayed at right.

Comment: This is not a simple problem. The "tangent line" idea includes an approximation theme: The very best closest line to a curve near a point is the tangent line. Making this precise leads to the limit of the slope, etc. But if we change the game and ask: whether there is a simple line that will give, on average, numbers that are closer to a given function over an entire interval, we may well get a different answer. Here we consider sine, the interval [0, 1], and lines through the origin. The answer is 0.90351x, certainly different from x.



Back to Fourier Coefficients

The Taylor polynomials of a function at a point are the partial sums of the Taylor series. These Taylor polynomials are the best—the closest polynomials of their degree to the function near that point. In this way they generalize tangent lines. What about Fourier series?

If $f(x)$ is defined on [0, π], we previously defined the mth Fourier coefficient, a_m, by the formula $a_m = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(mx)dx$. This definition is related to mean square error. The coefficient a_m is the unique A that minimizes $\int_0^{\pi} (f(x) - A\sin(mx))^2\, dx$. To verify this we start with

$$\int_0^{\pi}(f(x) - A\sin(mx))^2 dx = \int_0^{\pi}(f(x))^2 dx - 2A\int_0^{\pi} f(x)\sin(mx)dx + A^2\int_0^{\pi}(\sin(mx))^2 dx$$

$$= \int_0^{\pi}(f(x))^2 dx - 2A\int_0^{\pi} f(x)\sin(mx)dx + A^2\cdot\frac{\pi}{2}$$

This is minimized when

$$\frac{d}{dA}\left(\int_0^{\pi}(f(x))^2 dx - 2A\int_0^{\pi} f(x)\sin(mx)dx + A^2\cdot\frac{\pi}{2}\right) = -2\int_0^{\pi} f(x)\sin(mx)dx + 2A\cdot\frac{\pi}{2} = 0,$$

leading to $A = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(mx)dx$. This clearly provides a minimum for the mean square error by using the second derivative test. The Fourier series of a function satisfies a different minimization criterion from Taylor series. The partial sums of the Fourier series are best mean square approximations to the function.

What does this mean in practice? If someone wants to numerically sample a function or use a function to model the result of an experiment over a range of inputs, the Fourier series or its partial sums are likely to be more useful and more accurate *on average* than Taylor series. So for small perturbations near a known input, Taylor series are good. Over a range of inputs, Fourier series should be considered.

Back to Absolutely Convergent Series

Suppose we have an infinite series $\sum_{j=1}^{\infty} a_j$ that converges absolutely. Then consider the function defined by this Fourier sine series: $f(x) = \sum_{j=1}^{\infty} a_j \sin(jx)$.

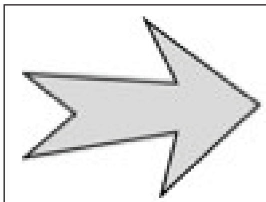
Since $|f(x)| = \left|\sum_{j=1}^{\infty} a_j \sin(jx)\right|$ and values of sine are always at most 1 in absolute value, we have $|f(x)| = \left|\sum_{j=1}^{\infty} a_j \sin(jx)\right| \leq \sum_{j=1}^{\infty} |a_j||\sin(jx)| \leq \sum_{j=1}^{\infty} |a_j|$. So any absolutely convergent series creates a nice function on [0, π]. This function is always continuous, but verifying this assertion is *not* something I would do in any calculus class.

Another Homework Problem

Here is an ambitious homework problem that I have assigned, together with some answers. Students can solve this problem, and the problem can provide them with further understanding of convergence of an infinite series. But (as I am sure you realize) very few students will have the background given on the last few pages of this article.

This is a problem about a Fourier *cosine* series. All the corresponding results for Fourier cosine series are true and follow from the same logic as the results for sine series. Some of the formulas in this example are easier to manipulate when cosines are used instead of sines. Please read what's below for verification of this claim.

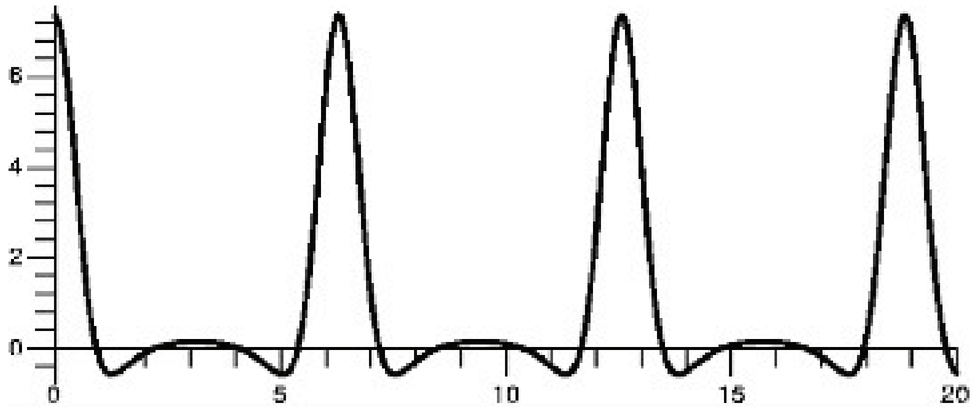
Instructional Admission



I have used the following homework problem in several classes, with varying amounts of success. That the partial sum is always so close to the true function is amazing, and I want students to understand this: A machine can't add up infinitely many numbers, but it certainly can rapidly compute the one-hundredth partial sum. When this partial sum is graphed, it can't be distinguished from the whole infinite series, either on screen or printed out.

Problem stem: Define $f(x)$ by the sum $f(x)=\sum_{n=0}^{\infty}\frac{2^n\cos(nx)}{n!}$. This is not a power series.

Below is a graph of the partial sum $s_{100}(x)=\sum_{n=0}^{100}\frac{2^n\cos(nx)}{n!}$ of the series for $0 \le x \le 20$.



Part (a) Verify that the series defining $f(x)$ converges for all values of x.

Answer: We will prove that the series converges absolutely for all values of x and because absolute convergence implies convergence, the series converges for all values of x. Since $|\cos(\text{anything})|\le 1$, $\left|\frac{2^n\cos(nx)}{n!}\right| \le \frac{2^n}{n!}$. We know that $\sum_{n=0}^{\infty}\frac{2^n}{n!}$ converges and that $\sum_{n=0}^{\infty}\frac{2^n}{n!}=e^2$. Observing the inequalities indicates that $|f(x)|\le e^2$ for all values of x.

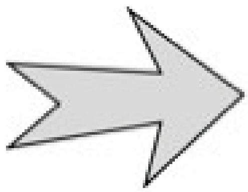
Part (b) Is the apparent periodicity of the function $f(x)$ actually correct? If yes, explain why.

Answer: Yes, $f(x)$ is periodic with period 2π. For integer n, $\cos(n(x + 2\pi)) = \cos(nx + 2n\pi) = \cos(nx)$ since cosine is 2π periodic. So all the terms in the infinite series for $f(x + 2\pi)$ are identical to the terms in the infinite series for $f(x)$.

Part (c) Verify that the actual graph of the function is always within 0.01 of the graph shown. That is, if x is any real number, then $|f(x) - s_{100}(x)| < 0.01$.

Answer: Again the observation $|\cos(\text{anything})|\le 1$ will simplify our work. We know $|f(x)-s_{100}(x)|=\left|\sum_{n=0}^{\infty}\frac{2^n\cos(nx)}{n!}-\sum_{n=0}^{100}\frac{2^n\cos(nx)}{n!}\right|=\left|\sum_{n=101}^{\infty}\frac{2^n\cos(nx)}{n!}\right|\le\sum_{n=101}^{\infty}\frac{2^n}{n!}$. This infinite tail can be overestimated by a geometric series because the ratio between successive terms of the tail series is $\frac{2}{n+1}$ (the tops of the tail series terms are powers of 2, and the bottom are factorials). Here $n \ge 101$, so the ratio is at most $\frac{2}{102}=\frac{1}{51}$. So the tail series is less than the geometric series with $a = \frac{2^{101}}{(101)!}$ and $r = \frac{1}{51}$. This is $\sum_{n=0}^{\infty} \frac{2^{101}}{(101)!}\left(\frac{1}{51}\right)^n = \frac{\frac{2^{101}}{(101)!}}{1-\left(\frac{1}{51}\right)}$, which has an approximate value of $0.27 \cdot 10^{-129}$. This result is much smaller than 0.01, so the graph shown is very much like the true graph.

Not for Everyone



What’s below is a discussion showing that the function itself, the sum of the whole series, can be written as a neat formula using well-known functions. This result is not obvious, and I thank my colleague, Professor Amy Cohen, for helping me with this computation. I have shown calculus students (interested students, outside of standard class time) these details. I would not do this to a whole class.

What is This Function?

Step 1

Euler’s formula states that $e^{ix} = \cos(x) + i\sin(x)$.

Step 2

If we substitute nx for x, we see that $e^{inx} = \cos(nx) + i\sin(nx)$.

Step 3

This problem is about the series $\sum_{n=0}^{\infty} \frac{2^n \cos(nx)}{n!}$. But the preceding step makes me want to consider the following: $\left(\sum_{n=0}^{\infty} \frac{2^n \cos(nx)}{n!}\right) + i\left(\sum_{n=0}^{\infty} \frac{2^n \sin(nx)}{n!}\right)$.

Step 4

So we are looking at $\sum_{n=0}^{\infty} \frac{2^n(\cos(nx)+i\sin(nx))}{n!}$, which is equal to $\sum_{n=0}^{\infty} \frac{2^n\left(e^{inx}\right)}{n!}$.

Step 5

But $e^{inx} = \left(e^{ix}\right)^n$, so this is the series $\sum_{n=0}^{\infty} \frac{2^n\left(e^{ix}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{\left(2e^{ix}\right)^n}{n!}$.

Step 6

The exponential function *is* $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$. The series we are considering has $A = 2e^{ix}$, so the sum of the series must be $e^{\left(2e^{ix}\right)}$.

Step 7

Euler's formula states that this is $e^{2(\cos(x)+i\sin(x))} = e^{2\cos(x)+2i\sin(x)}$.

Step 8

The exponential function converts addition to multiplication, and therefore we know $e^{2\cos(x)+2i\sin(x)} = e^{2\cos(x)}e^{2i\sin(x)}$.

Step 9

Look at $e^{2i\sin(x)} = e^{i(2\sin(x))}$. Use Euler's formula again, replacing the x in the original formula with $2\sin(x)$. The result is $e^{i(2\sin(x))} = \cos(2\sin(x)) + i\sin(2\sin(x))$.

Step 10

The series sum is

$e^{2\cos(x)}e^{i(2\sin(x))} = e^{2\cos(x)}\left(\cos(2\sin(x)) + i\sin(2\sin(x))\right) = e^{2\cos(x)}\cos(2\sin(x)) + ie^{2\cos(x)}\sin(2\sin(x))$.

Step 11

Compare the results of Step 3 and the preceding step. The same quantities are being described. The "real parts" (the quantities without *i*) should be the same, so therefore (but not *clearly*, definitely not clearly): $\sum_{n=0}^{\infty} \frac{2^n \cos(nx)}{n!} = e^{2\cos(x)}\cos(2\sin(x))$

Indeed, a graph of $e^{2\cos(x)}\cos(2\sin(x))$ looks the same as the graph that began the problem.

Just One More Example...

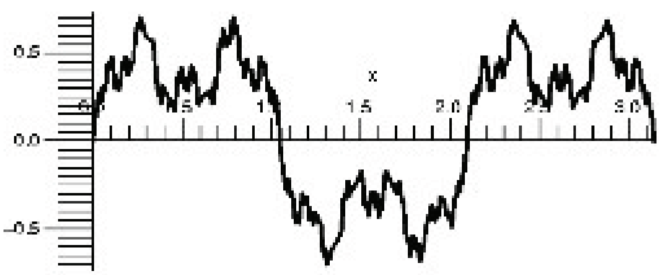
Let's take a specific convergent positive series: $\sum_{j=1}^{\infty} \frac{1}{2^j}$. This is so nice that I even can tell you its sum, which is 1. We know if we put this series together with sines we'll get a result that converges, and always has a value of between –1 and 1. The frequencies *inside* the sines do not affect convergence, since the values *outside* of the sine functions will still be between –1 and 1. Because absolute convergence implies convergence, nothing will go wrong….

Here is what can go wrong. This wrongness disturbed many mathematicians in the nineteenth century. I’ll select the frequencies inside the sines in a strange way. Here we go: Define the function f by $f(x)=\sum_{j=1}^{\infty}\frac{1}{2^j}\sin(3^j x)$. Some pictures of partial sums of this series and of the derivatives of these partial sums follow. Examine the pictures closely. Pay close attention to the vertical scales of the graphs, because they are nearly *unbelievable!*

The Tenth Partial Sum and Its Derivative

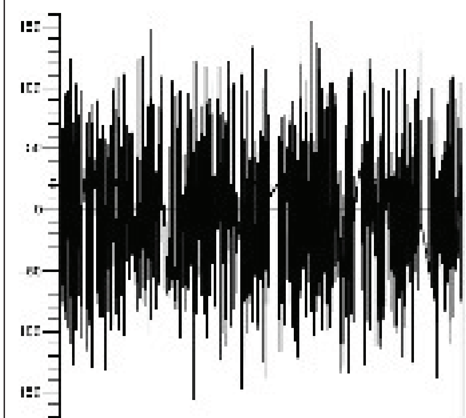
Graph of

$$\sum_{j=1}^{10}\tfrac{1}{2^j}\sin(3^j x)$$



Graph of

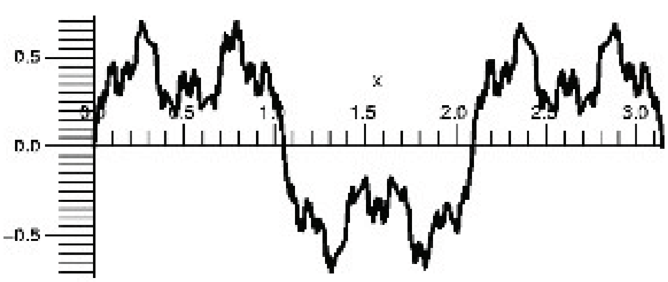
$$\frac{d}{dx}\left(\sum_{j=1}^{10}\frac{\sin\left(3^j x\right)}{2^j}\right)=\sum_{j=1}^{10}\frac{3^j}{2^j}\cos\left(3^j x\right)$$



The Twentieth Partial Sum and Its Derivative

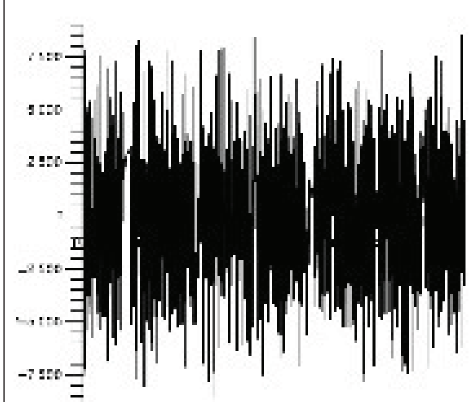
Graph of

$$\sum_{j=1}^{20}\tfrac{1}{2^j}\sin(3^j x)$$



Graph of

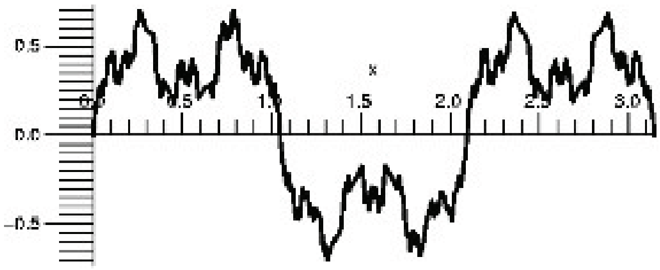
$$\frac{d}{dx}\left(\sum_{j=1}^{20}\frac{\sin\left(3^j x\right)}{2^j}\right)=\sum_{j=1}^{20}\frac{3^j}{2^j}\cos\left(3^j x\right)$$



The Thirtieth Partial Sum and Its Derivative

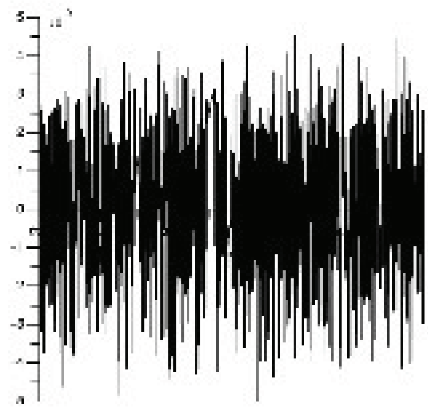
Graph of

$$\sum_{j=1}^{30} \tfrac{1}{2^j} \sin(3^j x)$$



Graph of

$$\frac{d}{dx}\left(\sum_{j=1}^{30} \frac{\sin(3^j x)}{2^j}\right) = \sum_{j=1}^{30} \frac{3^j}{2^j} \cos(3^j x)$$



Discussion of the Pictures

Let's deal with the three graphs on the left. They hardly differ. This is because their sum is a function *f*, and the error is easy to estimate. We defined *f* by the equation $f(x) = \sum_{j=1}^{\infty} \frac{1}{2^j} \sin(3^j x)$. So we can write $\sum_{j=1}^{\infty} \frac{1}{2^j} \sin(3^j x) = \sum_{j=1}^{10} \frac{1}{2^j} \sin(3^j x) + \sum_{j=11}^{\infty} \frac{1}{2^j} \sin(3^j x)$.
The error in the first graph of the partial sums is at most
$\sum_{j=11}^{\infty} \frac{1}{2^j} |\sin(3^j x)| \leq \sum_{j=11}^{\infty} \frac{1}{2^j} = \frac{1}{2^{10}} < .001$. The error in the second graph is at most
$\frac{1}{2^{20}}$<0.000001, and the error in the third graph is $\frac{1}{2^{30}}$<0.000000001. Since the partial sums are approximations of the same limit, their graphs are identical (at least to my eyes, on this scale).

The graphs on the right have a very different aspect. I had the graphs created using Maple and this program is quite accurate. Please consider the vertical scales carefully, and deduce information about the ranges of the derivatives of these partial sums. The derivative of the tenth partial sum seems to have values between –150 and 150. The derivative of the twentieth partial sum seems to have values between –8,000 and 8,000. The derivative of the thirtieth partial sum seems to have values between –450,000 and 450,000. These graphs are *not* getting closer to one another, and they are *not* "stabilizing" in any sense. They actually seem to be getting more chaotic. What's happening?

A Continuous Function That Is Differentiable at No Point

When we try to differentiate $\sum_{j=1}^{\infty} \frac{1}{2^j} \sin(3^j x)$ using the rules we love to follow, the chain rule leaves us with an unappealing multiplier, $\frac{3^j}{2^j}$, of the function $\cos(3^j x)$. While the values of cosine occupy [–1, 1], this multiplier gets large very rapidly. Things do *not* cancel out, and we have an example of a function that is continuous at every point and differentiable at no point! Verification of the last claim is straightforward but tedious.

That such functions exist was acknowledged only with great reluctance by many authorities in mathematics during the nineteenth century. Although the example given here is usually attributed to Weierstrass or Riemann, Bolzano had similar ideas and examples several decades earlier. (Bolzano also seems to have been the first person to give a statement resembling the version of the intermediate value theorem we currently use [1817]).

The idea of a continuous nowhere differentiable function was very distasteful, and the examples were widely viewed as artificial and absurdly theoretical. Most of the academic mathematicians of the late nineteenth century were quite reluctant to accept the validity of the results. But these functions really describe natural phenomena. In fact, some observations done many years earlier were consistent with the graphs shown above.

In 1827, biologist Robert Brown microscopically observed the motion of pollen in water. The pollen seemed to jump bizarrely, in strange and jagged paths—very different from the smooth motion of, say, a cannon ball in a parabolic trajectory. Observation of dust particles gave the same sort of results, so the motion couldn’t be attributed to some sort of life force in the pollen grains. This movement of the particles was named *Brownian motion*. The doctoral thesis of Bachelier in 1900 connected Brownian motion with variations in stock and option markets. Such jagged graphs typically appear in many financial reports. One of Einstein’s famous results of 1905 explained Brownian motion using probability — the particles of dust move as a result of random molecular collisions, and the molecular motion is what we perceive as heat. The paths typically are *not* smooth curves, and are usually *not* differentiable.

In the last 20 or 30 years, Brownian motion and related topics have been extensively studied by mathematicians and physicists, and nondifferentiable functions are standard tools in mathematical finance and other applications.

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