2018

AP® CollegeBoard

AP Calculus BC

Sample Student Responses and Scoring Commentary

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AP® CALCULUS BC
2018 SCORING GUIDELINES

Question 5

(a) Area $= \dfrac{1}{2}\displaystyle\int_{\pi/3}^{5\pi/3}\left(4^2 - (3 + 2\cos\theta)^2\right)d\theta$

$3 : \begin{cases} 1 : \text{constant and limits} \\ 2 : \text{integrand} \end{cases}$

(b) $\dfrac{dr}{d\theta} = -2\sin\theta \Rightarrow \left.\dfrac{dr}{d\theta}\right|_{\theta=\pi/2} = -2$

$r\left(\dfrac{\pi}{2}\right) = 3 + 2\cos\left(\dfrac{\pi}{2}\right) = 3$

$y = r\sin\theta \Rightarrow \dfrac{dy}{d\theta} = \dfrac{dr}{d\theta}\sin\theta + r\cos\theta$

$x = r\cos\theta \Rightarrow \dfrac{dx}{d\theta} = \dfrac{dr}{d\theta}\cos\theta - r\sin\theta$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\pi/2} = \frac{-2\sin\left(\frac{\pi}{2}\right) + 3\cos\left(\frac{\pi}{2}\right)}{-2\cos\left(\frac{\pi}{2}\right) - 3\sin\left(\frac{\pi}{2}\right)} = \frac{2}{3}$$

The slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \dfrac{\pi}{2}$ is $\dfrac{2}{3}$.

— OR —

$y = r\sin\theta = (3 + 2\cos\theta)\sin\theta \Rightarrow \dfrac{dy}{d\theta} = 3\cos\theta + 2\cos^2\theta - 2\sin^2\theta$

$x = r\cos\theta = (3 + 2\cos\theta)\cos\theta \Rightarrow \dfrac{dx}{d\theta} = -3\sin\theta - 4\sin\theta\cos\theta$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\pi/2} = \frac{3\cos\left(\frac{\pi}{2}\right) + 2\cos^2\left(\frac{\pi}{2}\right) - 2\sin^2\left(\frac{\pi}{2}\right)}{-3\sin\left(\frac{\pi}{2}\right) - 4\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)} = \frac{2}{3}$$

The slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \dfrac{\pi}{2}$ is $\dfrac{2}{3}$.

$$3 : \begin{cases} 1 : \dfrac{dy}{d\theta} = \dfrac{dr}{d\theta}\sin\theta + r\cos\theta \\ \quad\text{or } \dfrac{dx}{d\theta} = \dfrac{dr}{d\theta}\cos\theta - r\sin\theta \\ 1 : \dfrac{dy}{dx} = \dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \\ 1 : \text{answer} \end{cases}$$

(c) $\dfrac{dr}{dt} = \dfrac{dr}{d\theta}\cdot\dfrac{d\theta}{dt} = -2\sin\theta\cdot\dfrac{d\theta}{dt} \Rightarrow \dfrac{d\theta}{dt} = \dfrac{dr}{dt}\cdot\dfrac{1}{-2\sin\theta}$

$$\left.\frac{d\theta}{dt}\right|_{\theta=\pi/3} = 3\cdot\frac{1}{-2\sin\left(\frac{\pi}{3}\right)} = \frac{3}{-\sqrt{3}} = -\sqrt{3} \text{ radians per second}$$

$$3 : \begin{cases} 1 : \dfrac{dr}{dt} = \dfrac{dr}{d\theta}\cdot\dfrac{d\theta}{dt} \\ 1 : \dfrac{d\theta}{dt} = \dfrac{dr}{dt}\cdot\dfrac{1}{-2\sin\theta} \\ 1 : \text{answer with units} \end{cases}$$

NO CALCULATOR ALLOWED

5A
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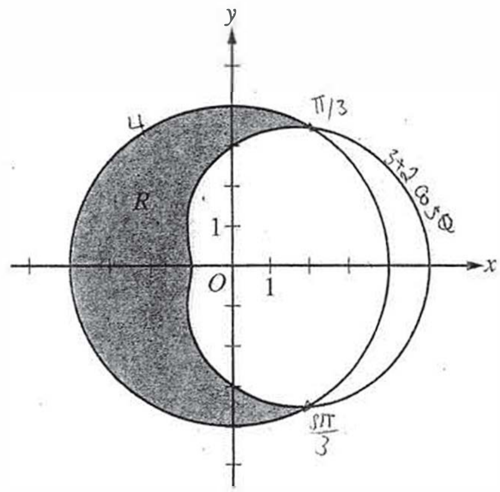


5. The graphs of the polar curves $r = 4$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect at $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.

(a) Let R be the shaded region that is inside the graph of $r = 4$ and also outside the graph of $r = 3 + 2\cos\theta$, as shown in the figure above. Write an expression involving an integral for the area of R.

$$R = \frac{1}{2}\int_{\pi/3}^{5\pi/3}\left[(4)^2 - (3+2\cos(\theta))^2\right]d\theta$$

$$R = \frac{1}{2}\int_{\pi/3}^{5\pi/3}\left[16 - (3+2\cos\theta)^2\right]d\theta$$

Continue question 5 on page 19.

5A
2.f2

(b) Find the slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$.

$r = 3+2\cos\theta$

$x = (3+2\cos\theta)\cos\theta$

$y = (3+2\cos\theta)\sin\theta$

$\frac{dy}{d\theta} = (-2\sin\theta\sin\theta) + \cos\theta(3+2\cos\theta)$ $\quad \frac{dy}{dx} = \frac{-2\sin^2\theta + 3\cos\theta + 2\cos^2\theta}{-2\sin\theta\cos\theta - 3\sin\theta - 2\sin\theta\cos\theta}$

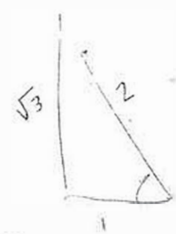
$\frac{dx}{d\theta} = (-2\sin\theta\cos\theta) - \sin\theta(3+2\cos\theta)$ $\quad$ At $\frac{\pi}{2}$

$\frac{-2(1)+0+0}{-2(1)(0)-3(1)-0} = \frac{-2}{-3}$

$\frac{dy}{dx}$ $\boxed{=\frac{2}{3}}$

(c) A particle moves along the portion of the curve $r = 3 + 2\cos\theta$ for $0 < \theta < \frac{\pi}{2}$. The particle moves in such a way that the distance between the particle and the origin increases at a constant rate of 3 units per second. Find the rate at which the angle θ changes with respect to time at the instant when the position of the particle corresponds to $\theta = \frac{\pi}{3}$. Indicate units of measure.

$\frac{dr}{dt} = -2\sin\theta\frac{d\theta}{dt}$



$\frac{dr}{dt} = 3$

$3 = -2\sin\theta\frac{d\theta}{dt}$ $\quad \frac{\sqrt{3}}{2}$

$-\frac{3}{2} = \sin\theta\frac{d\theta}{dt}$

At $\frac{\pi}{3}$ $\quad \frac{-\frac{3}{2}}{\sin\frac{\pi}{3}} = \frac{d\theta}{dt}$ $\quad \frac{-\frac{3}{2}}{\frac{\sqrt{3}}{2}} = \frac{d\theta}{dt}$ $\quad \frac{-6}{2\sqrt{3}} =$ $\boxed{-\sqrt{3} \text{ radians per second}}$

NO CALCULATOR ALLOWED

5B
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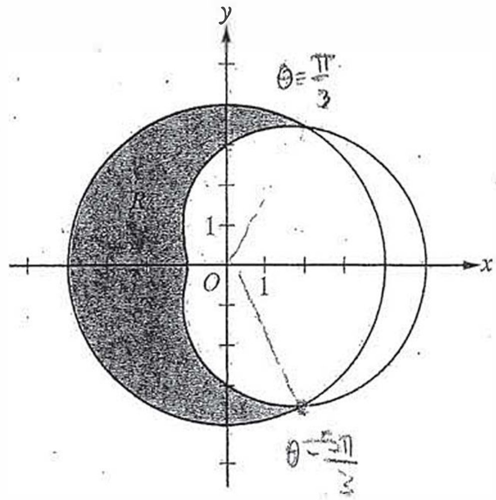

5. The graphs of the polar curves $r = 4$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect at $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.

(a) Let R be the shaded region that is inside the graph of $r = 4$ and also outside the graph of $r = 3 + 2\cos\theta$, as shown in the figure above. Write an expression involving an integral for the area of R.

$$\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} [4-3+2\cos\theta]\,d\theta$$

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NO CALCULATOR ALLOWED

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(b) Find the slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$.

$y = r\sin\theta = 3\sin\theta + (2\cos\theta)\sin\theta$

$x = r\cos\theta = 3\cos\theta + 2\cos^2\theta$

$$\frac{dy}{dx} = \frac{3\cos\theta + 2\cos^2\theta - 2\sin^2\theta}{-3\sin\theta - 4\cos\theta\sin\theta}\Bigg|_{\theta=\frac{\pi}{2}} = \frac{0+0-2}{-3-0}$$

$$\frac{dy}{dx} = \frac{2}{3}$$

(c) A particle moves along the portion of the curve $r = 3 + 2\cos\theta$ for $0 < \theta < \frac{\pi}{2}$. The particle moves in such a way that the distance between the particle and the origin increases at a constant rate of 3 units per second. Find the rate at which the angle θ changes with respect to time at the instant when the position of the particle corresponds to $\theta = \frac{\pi}{3}$. Indicate units of measure.

$\frac{dr}{dt} = 3 \quad \frac{d\theta}{dt} = ? \quad \theta = \frac{\pi}{3} \quad r = 4$

$$\frac{dr}{dt} = -2\sin\theta\,\frac{d\theta}{dt}$$

$$3 = -2\left(\frac{1}{2}\right)\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = -3 \text{ rad/sec}$$

NO CALCULATOR ALLOWED

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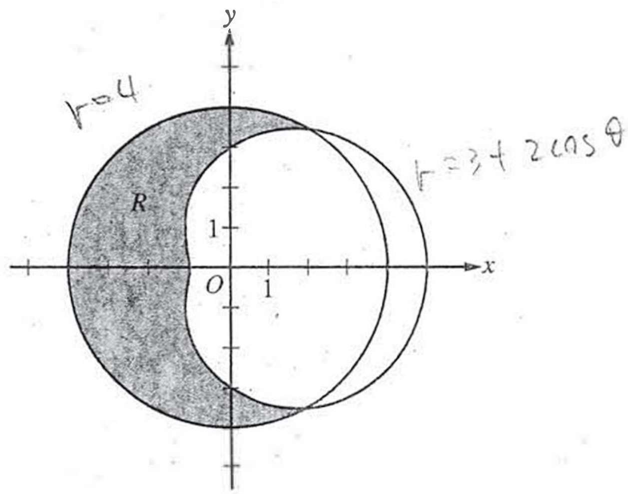

5. The graphs of the polar curves $r = 4$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect at $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.

(a) Let R be the shaded region that is inside the graph of $r = 4$ and also outside the graph of $r = 3 + 2\cos\theta$, as shown in the figure above. Write an expression involving an integral for the area of R.

$$\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \left(4 - (3+2\cos\theta)\right)^2 \, d\theta$$

Continue question 5 on page 19.

NO CALCULATOR ALLOWED

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(b) Find the slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$.

$\frac{dr}{d\theta} = -2\sin\theta \Big|_{\theta=\frac{\pi}{2}}$

$= -2.$

(c) A particle moves along the portion of the curve $r = 3 + 2\cos\theta$ for $0 < \theta < \frac{\pi}{2}$. The particle moves in such a way that the distance between the particle and the origin increases at a constant rate of 3 units per second. Find the rate at which the angle θ changes with respect to time at the instant when the position of the particle corresponds to $\theta = \frac{\pi}{3}$. Indicate units of measure.

$\frac{dr}{dt} = 3$

$\frac{dr}{dt} = -2\sin\theta \frac{d\theta}{dt}$

$3 = -2\sin\frac{\pi}{3} \frac{d\theta}{dt}$

$3 = -\sqrt{3} \frac{d\theta}{dt}$

$\frac{d\theta}{dt} = -\frac{3}{\sqrt{3}}$

$= -\frac{9}{3} = \boxed{-3 \text{ rad/sec}}$

GO ON TO THE NEXT PAGE.

Question 5

Overview

In this problem a polar graph is provided for polar curves $r = 4$ and $r = 3 + 2\cos\theta$. It was given that the curves intersect at $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$. In part (a) students were asked for an integral expression that gives the area of the region R that is inside the graph of $r = 4$ and outside the graph of $r = 3 + 2\cos\theta$. A correct response should resource the formula for the area of a simple polar region as half of a definite integral of the square of the radius function. The area of R is given by $\frac{1}{2}\int_{\pi/3}^{5\pi/3} 4^2\, d\theta - \frac{1}{2}\int_{\pi/3}^{5\pi/3} (3 + 2\cos\theta)^2\, d\theta = \frac{1}{2}\int_{\pi/3}^{5\pi/3} \left(4^2 - (3 + 2\cos\theta)^2\right) d\theta.$

In part (b) students were asked for the slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$. A correct response should deal with the conversion between polar and rectangular coordinate systems given by $y = r\sin\theta$ and $x = r\cos\theta$, differentiate these with respect to θ using the product rule, and find the slope of the line tangent to the graph as the value of $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ at $\theta = \frac{\pi}{2}$. In part (c) the motion of a particle along the portion of the curve $r = 3 + 2\cos\theta$ for $0 < \theta < \frac{\pi}{2}$ is such that the distance between the particle and the origin increases at a constant rate of 3 units per second. Students were asked for the rate at which the angle θ changes with respect to time at the instant when the position of the particle corresponds to $\theta = \frac{\pi}{3}$ and to indicate units of measure. A correct response should use the chain rule to relate the rates of r and θ with respect to time t: $\frac{dr}{dt} = -2\sin\theta \cdot \frac{d\theta}{dt}$. Recognizing that $\frac{dr}{dt} = 3$ from the problem statement, it follows that $\left.\frac{d\theta}{dt}\right|_{\theta=\pi/3} = -\sqrt{3}$ radians per second.

For part (a) see LO 3.4D/EK 3.4D1 (BC). For part (b) see LO 2.1C/EK 2.1C7 (BC), LO 2.2A/EK 2.2A4 (BC), LO 2.3B/EK 2.3B1. For part (c) see LO 2.1C/EK 2.1C7 (BC), LO 2.2A/EK 2.2A4 (BC), LO 2.3C/EK 2.3C2. This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, connecting multiple representations, building notational fluency, and communicating.

Sample: 5A
Score: 9

The response earned all 9 points: 3 points in part (a), 3 points in part (b), and 3 points in part (c). In part (a) the response earned the first point with the constant of $\frac{1}{2}$ and the limits of integration of $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ in line 1. The response would have earned the 2 integrand points with $\left[(4)^2 - (3 + 2\cos(\theta))^2\right]$ in line 1 with no simplification. In this case, the correct simplification in line 2 earned the points. In part (b) the response earned the first point for one of the following: $\frac{dx}{d\theta}$ is computed in line 5 on the left, and $\frac{dy}{d\theta}$ is computed in line 4 on the left. The response earned the second point with the assembly of $\frac{dy}{dx}$ in line 1 on the right. The response would have earned the third point with the expression $\frac{-2(1) + 0 + 0}{-2(1)(0) - 3(1) - 0}$ on the right with no simplification. In this

Question 5 (continued)

case, the correct simplification to $\frac{2}{3}$ earned the point. In part (c) the response earned the first point for application of the chain rule with $\frac{dr}{dt} = -2\sin\theta\frac{d\theta}{dt}$ in line 1. Although the response substitutes 3 for $\frac{dr}{dt}$ and $\frac{\pi}{3}$ for θ before isolating $\frac{d\theta}{dt}$, the equation $\frac{-\frac{3}{2}}{\sin\frac{\pi}{3}} = \frac{d\theta}{dt}$ in the last line on the left earned the second point. The response earned the third point with $-\sqrt{3}$ radians per second in the last line on the right.

Sample: 5B
Score: 6

The response earned 6 points: 1 point in part (a), 3 points in part (b), and 2 points in part (c). In part (a) the response earned the first point with the constant of $\frac{1}{2}$ and the limits of integration of $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ in line 1. The response did not earn either of the integrand points because the integrand is incorrect. In part (b) the response earned the first point for one of the following: $\frac{dy}{d\theta}$ is computed in the numerator of line 3, and $\frac{dx}{d\theta}$ is computed in the denominator of line 3. The response earned the second point with the assembly of $\frac{dy}{dx}$ in line 3. The response would have earned the third point with the expression $\frac{0+0-2}{-3-0}$ in line 3 with no simplification. In this case, the correct simplification to $\frac{2}{3}$ earned the point. In part (c) the response earned the first point for application of the chain rule with $\frac{dr}{dt} = -2\sin\theta\frac{d\theta}{dt}$ in line 2. The response earned the second point because although $\sin\frac{\pi}{3}$ is incorrectly evaluated as $\frac{1}{2}$ in line 3, $\frac{d\theta}{dt}$ is isolated correctly with a consistent result in the equation $\frac{d\theta}{dt} = -3$ in the last line. The response did not earn the third point because the evaluation of $\sin\frac{\pi}{3}$ as $\frac{1}{2}$ makes the response not eligible for the third point. The units are correct.

Sample: 5C
Score: 3

The response earned 3 points: 1 point in part (a), no points in part (b), and 2 points in part (c). In part (a) the response earned the first point with the constant of $\frac{1}{2}$ and the limits of integration of $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. The response did not earn either of the integrand points because the integrand is incorrect. In part (b) the response did not earn any points. The response did not earn the first point because neither $\frac{dx}{d\theta}$ nor $\frac{dy}{d\theta}$ is presented. Therefore, the response is not eligible for the second and third points. In part (c) the response earned the first point for application of the chain rule with $\frac{dr}{dt} = -2\sin\theta\frac{d\theta}{dt}$ in line 1 on the right. Although the response substitutes 3 for

Question 5 (continued)

$\frac{dr}{dt}$ and $\frac{\pi}{3}$ for θ before isolating $\frac{d\theta}{dt}$, the equation $\frac{d\theta}{dt} = -\frac{3}{\sqrt{3}}$ in line 4 on the right earned the second point. The response did not earn the third point because the answer is incorrect. The units are correct.