2017

AP® CollegeBoard

AP Calculus BC

Sample Student Responses and Scoring Commentary

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Question 3

(a) $f(-6) = f(-2) + \int_{-2}^{-6} f'(x)\,dx = 7 - \int_{-6}^{-2} f'(x)\,dx = 7 - 4 = 3$

$f(5) = f(-2) + \int_{-2}^{5} f'(x)\,dx = 7 - 2\pi + 3 = 10 - 2\pi$

$3 : \begin{cases} 1 : \text{uses initial condition} \\ 1 : f(-6) \\ 1 : f(5) \end{cases}$

(b) $f'(x) > 0$ on the intervals $[-6, -2)$ and $(2, 5)$.
Therefore, f is increasing on the intervals $[-6, -2]$ and $[2, 5]$.

2 : answer with justification

(c) The absolute minimum will occur at a critical point where $f'(x) = 0$ or at an endpoint.

$f'(x) = 0 \Rightarrow x = -2, x = 2$

x	$f(x)$
-6	3
-2	7
2	$7 - 2\pi$
5	$10 - 2\pi$

The absolute minimum value is $f(2) = 7 - 2\pi$.

$2 : \begin{cases} 1 : \text{considers } x = 2 \\ 1 : \text{answer with justification} \end{cases}$

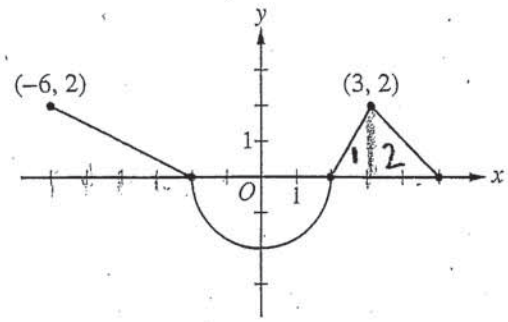
(d) $f''(-5) = \dfrac{2 - 0}{-6 - (-2)} = -\dfrac{1}{2}$

$\lim_{x \to 3^-} \dfrac{f'(x) - f'(3)}{x - 3} = 2$ and $\lim_{x \to 3^+} \dfrac{f'(x) - f'(3)}{x - 3} = -1$

$f''(3)$ does not exist because
$\lim_{x \to 3^-} \dfrac{f'(x) - f'(3)}{x - 3} \neq \lim_{x \to 3^+} \dfrac{f'(x) - f'(3)}{x - 3}.$

$2 : \begin{cases} 1 : f''(-5) \\ 1 : f''(3) \text{ does not exist,} \\ \quad\;\; \text{with explanation} \end{cases}$

3A1 3A1



Graph of f'

3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find the values of $f(-6)$ and $f(5)$.

$$f(-6) = \left(\int_{-2}^{-6} f'(x)\,dx\right) + f(-2)$$

$$f(-6) = 3$$

$$f(5) = f(-2) + \int_{-2}^{5} f'(x)\,dx$$

$$f(5) = 10 - 2\pi$$

(b) On what intervals is f increasing? Justify your answer.

f is increasing on $x = [-6, -2] \cup [2, 5]$, since $f' > 0$ on the interval $x \in [-6, -2) \cup (2, 5)$

Continue question 3 on page 15.

3A2 3A2

NO CALCULATOR ALLOWED

(c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

The absolute minimum of f on $[-6,5]$ is $7-2\pi$, since $f(2) < f(5)$ and $f(-6)$ (the endpoints) and $f(2) < f(-2)$ the other critical point, by EVT

Endpoints

$f(-6) = 3$

$f(5) = 10-2\pi$

critical points

$f' = 0$

$f(-2) = 7$

$f(2) = 7-2\pi$

(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

$$f''(-5) = \frac{-1}{2}$$

$f''(3) =$ DNE, as the

$$\lim_{x\to 3^+} \frac{f'(x)-2}{x-3} \neq \lim_{x\to 3^-} \frac{f'(x)-2}{x-3}$$

Therefore it is impossible to take a derivative at $x=3$ in f'

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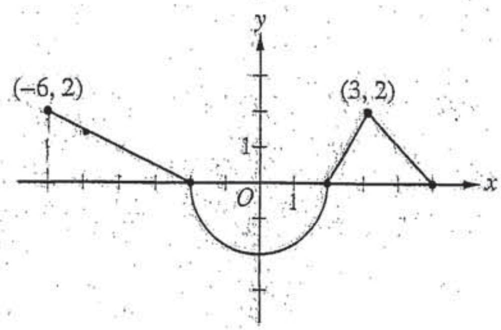
3B1 **NO CALCULATOR ALLOWED** 3B1



Graph of f'

3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find the values of $f(-6)$ and $f(5)$.

$f(-6) = f(-2) - \int_{-6}^{-2} f'(x)\,dx$

$f(-6) = 7 - \frac{4 \times 2}{2} = \boxed{3}$

$f(5) = f(-2) + \int_{-2}^{5} f'(x)\,dx$

$f(5) = 7 + \frac{3 \times 2}{2} - \frac{1}{2}\pi \times 2^2 = \boxed{10 - 2\pi}$

(b) On what intervals is f increasing? Justify your answer.

Since on intervals of $(-6, 2)$ and $(2, 5)$; $f'(x) > 0$

then $f(x)$ is increasing on intervals $[-6, 2]$ and $[2, 5]$

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Continue question 3 on page 15.

3B2

NO CALCULATOR ALLOWED

3B2

(c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

f(x) has its absolute minimum on either two endpoints and where $f'(x) = 0$

according to the graph: $f'(-2) = f'(2) = 0$

x	f(x)
−6	3
−2	7
2	7 − 2π
5	10 − 2π

according to the table, f(x) reaches its absolute minimum value $\boxed{7-2\pi}$ at $x = 2$

(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

$f''(-5) = \frac{d}{dx} f'(x) \Big|_{x=-5} = \frac{2}{-4} = \boxed{-\frac{1}{2}}$

since $\lim_{x \to 3^-} f''(x) \neq \lim_{x \to 3^+} f''(x)$

then f(x) is not differentiable at $x = 3$

therefore, $f''(3)$ does not exist.

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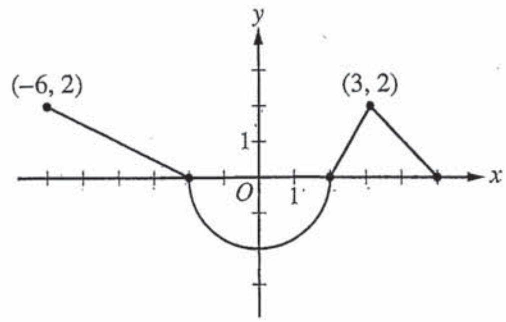

Graph of f'

3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find the values of $f(-6)$ and $f(5)$.

$f(-6) = \frac{2}{5} \times (2) = \frac{4}{5}$

$f(5) = 0$

(b) On what intervals is f increasing? Justify your answer.

From $[-6, -2]$ and $[2, 5]$, f is increasing because the graph of $f'(x)$ is >0 from $[-6, -2)$ and $(2, 5)$

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Continue question 3 on page 15.

3C2

NO CALCULATOR ALLOWED

3C2

(c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

The absolute minimum value of 0 at $x = 2$.
because the graph of f' changes sign from negative to positive at $x = 2$.

(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

$f''(3)$ is an inflection point because f' increases on $[2, 3]$ and decreases on $[2, 4]$.
$f''(-5)$ does not exist because the graph of f' from $[-6, -2]$ has a slope of $\frac{2}{5}$.

Question 3

Overview

In this problem students were given that a function f is differentiable on the interval $[-6, 5]$ and satisfies $f(-2) = 7.$ For $-6 \le x \le 5$, the derivative of f is specified by a graph consisting of a semicircle and three line segments. In part (a) students were asked to find values of $f(-6)$ and $f(5).$ For each of these values, students needed to recognize that the net change in f, starting from the given value $f(-2) = 7,$ can be computed using a definite integral of $f'(x)$ with a lower limit of integration -2 and an upper limit the desired argument of f. These integrals can be computed using properties of the definite integral and the geometric connection to areas between the graph of $y = f'(x)$ and the x-axis. Thus, students needed to add the initial condition $f(-2) = 7$ to the values of the definite integrals for the desired values. [LO 3.2C/EK 3.2C1] In part (b) students were asked for the intervals on which f is increasing, with justification. Since f' is given on the interval $[-6, 5]$, f is differentiable, and thus also continuous, on that interval. Therefore, f is increasing on closed intervals for which $f'(x) > 0$ on the interior. Students needed to use the given graph of f' to see that $f'(x) > 0$ on the intervals $[-6, -2)$ and $(2, 5),$ so f is increasing on the intervals $[-6, -2]$ and $[2, 5],$ connecting their answers to the sign of $f'.$ [LO 2.2A/EK 2.2A1-2.2A2, LO 2.2B/EK 2.2B1] In part (c) students were asked for the absolute minimum value of f on the closed interval $[-6, 5],$ and to justify their answers. Students needed to use the graph of f' to identify critical points of f on the interior of the interval as $x = -2$ and $x = 2.$ Then they can compute $f(-2)$ and $f(2),$ similarly to the computations in part (a), and compare these to the values of f at the endpoints that were computed in part (a). Students needed to report the smallest of these values, $f(2) = 7 - 2\pi$ as the answer. Alternatively, students could have observed that the minimum value must occur either at a point interior to the interval at which f' transitions from negative to positive, at a left endpoint for which f' is positive immediately to the right, or at a right endpoint for which f' is negative immediately to the left. This reduces the options to $f(-6) = 3$ and $f(2) = 7 - 2\pi.$ [LO 2.2A/EK 2.2A1-2.2A2, LO 2.2B/EK 2.2B1, LO 3.3A/EK 3.3A3] In part (d) students were asked to determine values of $f''(-5)$ and $f''(3),$ or to explain why the requested value does not exist. Students needed to find the value $f''(-5)$ as the slope of the line segment on the graph of f' through the point corresponding to $x = -5.$ The point on the graph of f' corresponding to $x = 3$ is the juncture of a line segment of slope 2 on the left with one of slope -1 on the right. Thus, students needed to report that $f''(3)$ does not exist, and explain why the given graph of f' shows that f' is not differentiable at $x = 3.$ Student explanations could be done by noting that the left-hand and right-hand limits at $x = 3$ of the difference quotient $\frac{f'(x) - f'(3)}{x - 3}$ have differing values (2 and $-1,$ respectively), or by a clear description of the relevant features of the graph of f' near $x = 3.$ [LO 1.1A(b)/EK 1.1A3] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, connecting multiple representations, building notational fluency, and communicating.

Sample: 3A
Score: 9

The response earned all 9 points: 3 points in part (a), 2 points in part (b), 2 points in part (c), and 2 points in part (d). In part (a) the student uses the initial condition $f(-2)$ with an appropriate definite integral $\int_{-2}^{-6} f'(x)\,dx$ to find $f(-6) = 3.$ Thus, the student earned the first and second points. The student uses $f(-2)$ again with an

Question 3 (continued)

appropriate definite integral $\int_{-2}^{5} f'(x)\,dx$ to find $f(5) = 10 - 2\pi$. The student earned the third point. In part (b) the student states two correct and complete intervals, $[-6, -2]$ and $[2, 5]$, where f is increasing. The student justifies the intervals with a discussion of $f' > 0$ for $[-6, -2)$ and $(2, 5)$. The student earned both points. In part (c) the student considers $x = -6, -2, 2,$ and 5 as potential locations for the absolute minimum value. The student earned the first point for considering $x = 2$. The student identifies the absolute minimum value as $7 - 2\pi$. The student justifies by evaluating $f(x)$ at the critical values and endpoints. The student earned the second point. In part (d) the student finds $f''(-5) = -\frac{1}{2}$ and earned the first point. The student states that $f''(3)$ does not exist. The student uses two one-sided limits at $x = 3$ to explain why the derivative of $f'(x)$ does not exist and earned the second point.

Sample: 3B
Score: 6

The response earned 6 points: 3 points in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the student uses the initial condition $f(-2)$ with an appropriate definite integral $\int_{-6}^{-2} f'(x)\,dx$ to find $f(-6) = 3$. Thus, the student earned the first and second points. The student uses $f(-2)$ again with an appropriate definite integral $\int_{-2}^{5} f'(x)\,dx$ to find $f(5) = 10 - 2\pi$. The student earned the third point. In part (b) the student presents two intervals, $[-6, 2)$ and $(2, 5)$. Because $f'(x) < 0$ on $(-2, 2)$, f is decreasing on $[-2, 2]$. The student is not eligible to earn any points because of the presence of an interval containing points where $f'(x) < 0$. Thus, the student did not earn any points. In part (c) the student investigates where $f'(x) = 0$ and identifies $f'(-2)$ and $f'(2)$. The student earned the first point for considering $x = 2$. The student identifies the absolute minimum value as $7 - 2\pi$. The student justifies by evaluating $f(x)$ at the critical values and endpoints. The student earned the second point. In part (d) the student identifies $f''(-5)$ as the derivative of $f'(x)$ at $x = -5$ and finds $f''(-5) = -\frac{1}{2}$. The student earned the first point. The student states that $f''(3)$ does not exist. The student uses two one-sided limits at $x = 3$. The student states that "$f(x)$ is not differentiable at $x = 3$," which contradicts the given statement in the problem that f is differentiable on the closed interval $[-6, 5]$. The student did not earn the second point.

Sample: 3C
Score: 3

The response earned 3 points: no points in part (a), 2 points in part (b), 1 point in part (c), and no points in part (d). In part (a) the student never uses the initial condition, incorrectly evaluates $f(-6)$ as $\frac{4}{5}$, and incorrectly evaluates $f(5)$ as 0. The student earned no points. In part (b) the student states two correct and complete intervals, $[-6, -2]$ and $[2, 5]$, on which f is increasing. The student justifies the intervals with "$f'(x)$ is > 0 from $[-6, -2)$ and $(2, 5)$." The student earned both points. In part (c) the student considers $x = 2$ and earned the first point. The student presents an incorrect answer for the absolute minimum value with an incorrect justification. The student does not evaluate $f(x)$ at the critical values and endpoints in order to determine the

Question 3 (continued)

absolute minimum value. The student did not earn the second point. In part (d) the student incorrectly determines that $f''(-5)$ has a value of $\frac{2}{5}$ and did not earn the first point. The student states that “$f''(3)$ is an inflection point” and does not state that $f''(3)$ does not exist. The student did not earn the second point.