AP Calculus AB
Sample Student Responses and Scoring Commentary

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Question 3

(a) \( f(-6) = f(-2) + \int_{-2}^{-6} f'(x) \, dx = 7 - \int_{-6}^{-2} f'(x) \, dx = 7 - 4 = 3 \)
\( f(5) = f(-2) + \int_{-2}^{5} f'(x) \, dx = 7 - 2\pi + 3 = 10 - 2\pi \)

(b) \( f'(x) > 0 \) on the intervals \([-6, -2)\) and \((2, 5)\).
Therefore, \( f \) is increasing on the intervals \([-6, -2]\) and \([2, 5]\).

(c) The absolute minimum will occur at a critical point where \( f'(x) = 0 \)
or at an endpoint.

\[ f'(x) = 0 \Rightarrow x = -2, x = 2 \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-6</td>
<td>3</td>
</tr>
<tr>
<td>-2</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>( 7 - 2\pi )</td>
</tr>
<tr>
<td>5</td>
<td>( 10 - 2\pi )</td>
</tr>
</tbody>
</table>

The absolute minimum value is \( f(2) = 7 - 2\pi \).

(d) \( f''(-5) = \frac{2 - 0}{-6 - (-2)} = -\frac{1}{2} \)

\[ \lim_{x \to 3^-} \frac{f'(x) - f'(3)}{x - 3} = 2 \quad \text{and} \quad \lim_{x \to 3^+} \frac{f'(x) - f'(3)}{x - 3} = -1 \]

\( f''(3) \) does not exist because
\[ \lim_{x \to 3^-} \frac{f'(x) - f'(3)}{x - 3} \neq \lim_{x \to 3^+} \frac{f'(x) - f'(3)}{x - 3} . \]
3. The function $f$ is differentiable on the closed interval $[-6, 5]$ and satisfies $f'(-2) = 7$. The graph of $f'$, the derivative of $f$, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find the values of $f(-6)$ and $f(5)$.

$$f(-6) = \left( \int_{-2}^{-6} f''(x) \, dx \right) + f(-2)$$

$$f(-6) = 3$$

$$f(5) = f(-2) + \int_{-2}^{5} f''(x) \, dx$$

$$f(5) = 10 - 2\pi$$

(b) On what intervals is $f$ increasing? Justify your answer.

$f$ is increasing on $x = (-6, -2)$

$\cup [2, 5]$ since $f' > 0$ on

the interval $x \in (-6, -2) \cup (2, 5)$
(c) Find the absolute minimum value of \( f \) on the closed interval \([-6, 5]\). Justify your answer.

The absolute minimum of \( f \) on \([-6, 5]\) is \( 7-2\pi \), since
\[
f(2) < f(5) < f(6).
\]

Endpoints:
- \( f(-6) = 3 \)
- \( f(5) = 10 - 2\pi \)

Critical points:
- \( f' = 0 \)
- \( f(-2) = 7 \)
- \( f(2) = 7 - 2\pi \)

By EVT

(d) For each of \( f''(-5) \) and \( f''(3) \), find the value or explain why it does not exist.

\( f''(-5) = \frac{-1}{2} \)

\( f''(3) = \text{DNE, as the} \)
\[
\lim_{x \to 3^+} \frac{f(x) - 2}{x - 3} \neq \lim_{x \to 3^-} \frac{f(x) - 2}{x - 3}
\]

Therefore, it is impossible to take a derivative at \( x = 3 \) in \( f' \).
3. The function \( f \) is differentiable on the closed interval \([-6, 5]\) and satisfies \( f(-2) = 7 \). The graph of \( f' \), the derivative of \( f \), consists of a semicircle and three line segments, as shown in the figure above.

(a) Find the values of \( f(-6) \) and \( f(5) \).

\[
\begin{align*}
  f(-6) &= f(-2) - \int_{-6}^{-2} f'(x) \, dx \\
  f(-6) &= 7 - \frac{4 \times 2}{2} = 3 \\
  f(5) &= f(-2) + \int_{-2}^{5} f'(x) \, dx \\
  f(5) &= 7 + \frac{3 \times 2}{2} - \frac{1}{2} \pi \times 2^2 = 10 - 2\pi
\end{align*}
\]

(b) On what intervals is \( f \) increasing? Justify your answer.

Since on intervals \([-6, -2) \) and \((2, 5) \); \( f'(x) > 0 \)

then \( f(x) \) is increasing on intervals \([-6, 2) \) and \((2, 5) \).
(c) Find the absolute minimum value of \( f \) on the closed interval \([-6, 5]\). Justify your answer.

\[ f(x) \text{ has its absolute minimum on either two endpoints and where } f'(x) = 0 \]

According to the graph, \( f'(-2) = f'(2) = 0 \).

\[
\begin{array}{c|c}
 X & f(x) \\
-6 & 3 \\
-2 & 7 \\
2 & 7 - 2\pi \\
5 & 10 - 2\pi \\
\end{array}
\]

According to the table, \( f(x) \) reaches its absolute minimum value \( 7 - 2\pi \) at \( x = 2 \).

(d) For each of \( f''(-5) \) and \( f''(3) \), find the value or explain why it does not exist.

\[ f''(-5) = \frac{d}{dx} f'(x) \bigg|_{x=-5} = \frac{2}{-4} = -\frac{1}{2} \]

Since \( \lim_{x \to 3^-} f''(x) \neq \lim_{x \to 3^+} f''(x) \),

then \( f(x) \) is not differentiable at \( x = 3 \).

Therefore, \( f''(3) \) does not exist.
3. The function \( f \) is differentiable on the closed interval \([-6, 5]\) and satisfies \( f(-2) = 7 \). The graph of \( f' \), the derivative of \( f \), consists of a semicircle and three line segments, as shown in the figure above.

(a) Find the values of \( f(-6) \) and \( f(5) \).

\[
\begin{align*}
 f'(-6) &= \frac{2}{5} \times (2) = \frac{4}{5} \\
 f'(5) &= 0
\end{align*}
\]

(b) On what intervals is \( f \) increasing? Justify your answer.

From \([-6, -2]\) and \([2, 5]\), \( f \) is increasing because the graph of \( f'(x) \) is \( > 0 \) from \((-6, -2)\) and \((2, 5)\).
(c) Find the absolute minimum value of \( f \) on the closed interval \([-6, 5]\). Justify your answer.

The absolute minimum value of \( f \) at \( x = 2 \) because the graph of \( f' \) changes sign from negative to positive at \( x = 2 \).

(d) For each of \( f''(-5) \) and \( f''(3) \), find the value or explain why it does not exist.

\( f''(3) \) is an inflection point because \( f' \) increases on \([2, 3]\)
and decreases on \([2, 4]\).

\( f''(-5) \) does not exist because the graph of \( f' \) from \([-6, -2]\) has a slope of \( \frac{7}{5} \).
Overview

In this problem students were given that a function $f$ is differentiable on the interval $[-6, 5]$ and satisfies $f(-2) = 7$. For $-6 \le x \le 5$, the derivative of $f$ is specified by a graph consisting of a semicircle and three line segments. In part (a) students were asked to find values of $f(-6)$ and $f(5)$. For each of these values, students needed to recognize that the net change in $f$, starting from the given value $f(-2) = 7$, can be computed using a definite integral of $f'(x)$ with a lower limit of integration $-2$ and an upper limit the desired argument of $f$. These integrals can be computed using properties of the definite integral and the geometric connection to areas between the graph of $y = f'(x)$ and the $x$-axis. Thus, students needed to add the initial condition $f(-2) = 7$ to the values of the definite integrals for the desired values. [LO 3.2C/EK 3.2C1] In part (b) students were asked for intervals on which $f'$ is increasing, with justification. Since $f'$ is given on the interval $[-6, 5]$, $f$ is differentiable, and thus also continuous, on that interval. Therefore, $f$ is increasing on closed intervals for which $f'(x) > 0$ on the interior. Students needed to use the given graph of $f'$ to see that $f'(x) > 0$ on the intervals $[-6, -2)$ and $(2, 5)$, so $f$ is increasing on the intervals $[-6, -2]$ and $[2, 5]$, connecting their answers to the sign of $f'$. [LO 2.2A/EK 2.2A1-2.2A2, LO 2.2B/EK 2.2B1] In part (c) students were asked for the absolute minimum value of $f$ on the closed interval $[-6, 5]$, and to justify their answers. Students needed to use the graph of $f'$ to identify critical points of $f$ on the interior of the interval as $x = -2$ and $x = 2$. Then they can compute $f(-2)$ and $f(2)$, similarly to the computations in part (a), and compare these to the values of $f$ at the endpoints that were computed in part (a). Students needed to report the smallest of these values, $f(2) = 7 - 2\pi$ as the answer. Alternatively, students could have observed that the minimum value must occur either at a point interior to the interval at which $f'$ transitions from negative to positive, or at a right endpoint for which $f'$ is negative immediately to the left. This reduces the options to $f(-6) = 3$ and $f(2) = 7 - 2\pi$. [LO 2.2A/EK 2.2A1-2.2A2, LO 2.2B/EK 2.2B1, LO 3.3A/EK 3.3A3] In part (d) students were asked to determine values of $f''(-5)$ and $f''(3)$, or to explain why the requested value does not exist. Students needed to find the value $f''(-5)$ as the slope of the line segment on the graph of $f'$ through the point corresponding to $x = -5$. The point on the graph of $f'$ corresponding to $x = 3$ is the juncture of a line segment of slope 2 on the left with one of slope $-1$ on the right. Thus, students needed to report that $f''(3)$ does not exist, and explain why the given graph of $f'$ shows that $f'$ is not differentiable at $x = 3$. Student explanations could be done by noting that the left-hand and right-hand limits at $x = 3$ of the difference quotient $\frac{f'(x) - f'(3)}{x - 3}$ have differing values (2 and $-1$, respectively), or by a clear description of the relevant features of the graph of $f'$ near $x = 3$. [LO 1.1A(b)/EK 1.1A3] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, connecting multiple representations, building notational fluency, and communicating.

Sample: 3A
Score: 9

The response earned all 9 points: 3 points in part (a), 2 points in part (b), 2 points in part (c), and 2 points in part (d). In part (a) the student uses the initial condition $f(-2)$ with an appropriate definite integral $\int_{-2}^{5} f'(x) \, dx$ to find $f(-6) = 3$. Thus, the student earned the first and second points. The student uses $f(-2)$ again with an
Question 3 (continued)

appropriate definite integral \( \int_{-2}^{5} f'(x) \, dx \) to find \( f(5) = 10 - 2\pi \). The student earned the third point. In part (b) the student states two correct and complete intervals, \([-6, -2]\) and \([2, 5]\), where \( f \) is increasing. The student justifies the intervals with a discussion of \( f' > 0 \) for \([-6, -2]\) and \((2, 5)\). The student earned both points. In part (c) the student considers \( x = -6, -2, 2, \) and \( 5 \) as potential locations for the absolute minimum value. The student earned the first point for considering \( x = 2 \). The student identifies the absolute minimum value as \( 7 - 2\pi \). The student justifies by evaluating \( f(x) \) at the critical values and endpoints. The student earned the second point. In part (d) the student finds \( f''(-5) = -\frac{1}{2} \) and earned the first point. The student states that \( f''(3) \) does not exist. The student uses two one-sided limits at \( x = 3 \) to explain why the derivative of \( f'(x) \) does not exist and earned the second point.

Sample: 3B
Score: 6

The response earned 6 points: 3 points in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the student uses the initial condition \( f(-2) \) with an appropriate definite integral \( \int_{-6}^{5} f'(x) \, dx \) to find \( f(-6) = 3 \). Thus, the student earned the first and second points. The student uses \( f(-2) \) again with an appropriate definite integral \( \int_{-2}^{5} f'(x) \, dx \) to find \( f(5) = 10 - 2\pi \). The student earned the third point. In part (b) the student presents two intervals, \([-6, 2]\) and \((2, 5)\). Because \( f'(x) < 0 \) on \((-2, 2)\), \( f \) is decreasing on \([-2, 2]\). The student is not eligible to earn any points because of the presence of an interval containing points where \( f'(x) < 0 \). Thus, the student did not earn any points. In part (c) the student investigates where \( f'(x) = 0 \) and identifies \( f'(-2) \) and \( f'(2) \). The student earned the first point for considering \( x = 2 \). The student identifies the absolute minimum value as \( 7 - 2\pi \). The student justifies by evaluating \( f(x) \) at the critical values and endpoints. The student earned the second point. In part (d) the student identifies \( f''(-5) \) as the derivative of \( f'(x) \) at \( x = -5 \) and finds \( f''(-5) = -\frac{1}{2} \). The student earned the first point. The student states that \( f''(3) \) does not exist. The student uses two one-sided limits at \( x = 3 \). The student states that “\( f(x) \) is not differentiable at \( x = 3 \),” which contradicts the given statement in the problem that \( f \) is differentiable on the closed interval \([-6, 5]\). The student did not earn the second point.

Sample: 3C
Score: 3

The response earned 3 points: no points in part (a), 2 points in part (b), 1 point in part (c), and no points in part (d). In part (a) the student never uses the initial condition, incorrectly evaluates \( f(-6) \) as \( \frac{4}{5} \), and incorrectly evaluates \( f(5) \) as 0. The student earned no points. In part (b) the student states two correct and complete intervals, \([-6, -2]\) and \([2, 5]\), on which \( f \) is increasing. The student justifies the intervals with “\( f'(x) > 0 \) from \([-6, -2]\) and \((2, 5)\).” The student earned both points. In part (c) the student considers \( x = 2 \) and earned the first point. The student presents an incorrect answer for the absolute minimum value with an incorrect justification. The student does not evaluate \( f(x) \) at the critical values and endpoints in order to determine the
absolute minimum value. The student did not earn the second point. In part (d) the student incorrectly determines that $f''(-5)$ has a value of $\frac{2}{5}$ and did not earn the first point. The student states that “$f''(3)$ is an inflection point” and does not state that $f''(3)$ does not exist. The student did not earn the second point.