AP® Calculus BC
2016 Scoring Guidelines

Question 1

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

(d) For $0 \le t \le 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

(a) $R'(2) \approx \dfrac{R(3) - R(1)}{3 - 1} = \dfrac{950 - 1190}{3 - 1} = -120$ liters/hr^2

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{units} \end{cases}$

(b) The total amount of water removed is given by $\int_0^8 R(t)\,dt$.

$$\begin{aligned} \int_0^8 R(t)\,dt &\approx 1 \cdot R(0) + 2 \cdot R(1) + 3 \cdot R(3) + 2 \cdot R(6) \\ &= 1(1340) + 2(1190) + 3(950) + 2(740) \\ &= 8050 \text{ liters} \end{aligned}$$

This is an overestimate since R is a decreasing function.

$3 : \begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{estimate} \\ 1 : \text{overestimate with reason} \end{cases}$

(c) $$\begin{aligned} \text{Total} &\approx 50000 + \int_0^8 W(t)\,dt - 8050 \\ &= 50000 + 7836.195325 - 8050 \approx 49786 \text{ liters} \end{aligned}$$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{estimate} \end{cases}$

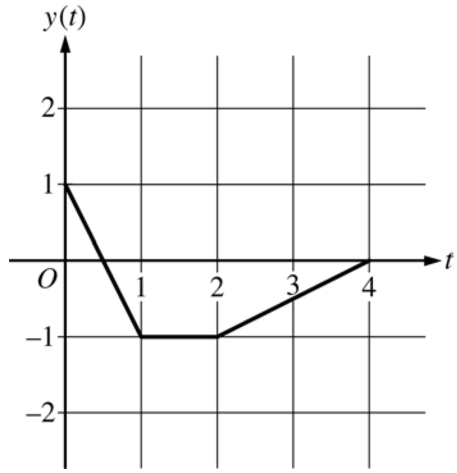
(d) $W(0) - R(0) > 0$, $W(8) - R(8) < 0$, and $W(t) - R(t)$ is continuous.

Therefore, the Intermediate Value Theorem guarantees at least one time t, $0 < t < 8$, for which $W(t) - R(t) = 0$, or $W(t) = R(t)$.

For this value of t, the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank.

$2 : \begin{cases} 1 : \text{considers } W(t) - R(t) \\ 1 : \text{answer with explanation} \end{cases}$

Question 2



At time t, the position of a particle moving in the xy-plane is given by the parametric functions $(x(t), y(t))$, where $\frac{dx}{dt} = t^2 + \sin(3t^2)$. The graph of y, consisting of three line segments, is shown in the figure above. At $t = 0$, the particle is at position $(5, 1)$.

(a) Find the position of the particle at $t = 3$.

(b) Find the slope of the line tangent to the path of the particle at $t = 3$.

(c) Find the speed of the particle at $t = 3$.

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

(a) $x(3) = x(0) + \int_0^3 x'(t)\,dt = 5 + 9.377035 = 14.377$

$y(3) = -\frac{1}{2}$

The position of the particle at $t = 3$ is $(14.377, -0.5)$.

$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

(b) Slope $= \frac{y'(3)}{x'(3)} = \frac{0.5}{9.956376} = 0.05$

1 : slope

(c) Speed $= \sqrt{(x'(3))^2 + (y'(3))^2} = 9.969$ (or 9.968)

$2 : \begin{cases} 1 : \text{expression for speed} \\ 1 : \text{answer} \end{cases}$

(d) Distance $= \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$

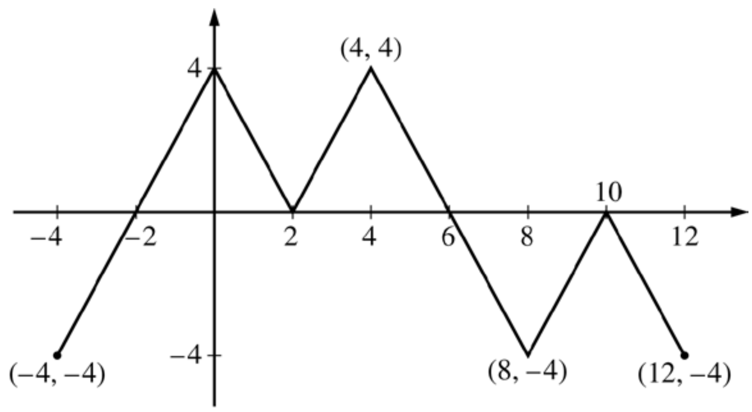
$= \int_0^1 \sqrt{(x'(t))^2 + (-2)^2}\,dt + \int_1^2 \sqrt{(x'(t))^2 + 0^2}\,dt$

$= 2.237871 + 2.112003 = 4.350$ (or 4.349)

$3 : \begin{cases} 1 : \text{expression for distance} \\ 1 : \text{integrals} \\ 1 : \text{answer} \end{cases}$

Question 3

The figure above shows the graph of the piecewise-linear function f. For $-4 \le x \le 12$, the function g is defined by $g(x) = \int_2^x f(t)\,dt$.



Graph of f

(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \le x \le 12$. Justify your answers.

(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.

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1 : $g'(x) = f(x)$ in (a), (b), (c), or (d)

(a) The function g has neither a relative minimum nor a relative maximum at $x = 10$ since $g'(x) = f(x)$ and $f(x) \le 0$ for $8 \le x \le 12$.

1 : answer with justification

(b) The graph of g has a point of inflection at $x = 4$ since $g'(x) = f(x)$ is increasing for $2 \le x \le 4$ and decreasing for $4 \le x \le 8$.

1 : answer with justification

(c) $g'(x) = f(x)$ changes sign only at $x = -2$ and $x = 6$.

x	$g(x)$
-4	-4
-2	-8
6	8
12	-4

On the interval $-4 \le x \le 12$, the absolute minimum value is $g(-2) = -8$ and the absolute maximum value is $g(6) = 8$.

4 : $\begin{cases} 1 : \text{considers } x = -2 \text{ and } x = 6 \text{ as candidates} \\ 1 : \text{considers } x = -4 \text{ and } x = 12 \\ 2 : \text{answers with justification} \end{cases}$

(d) $g(x) \le 0$ for $-4 \le x \le 2$ and $10 \le x \le 12$.

2 : intervals

Question 4

Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

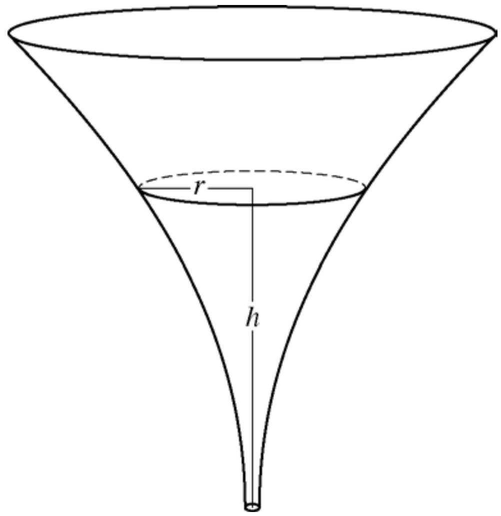
(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x\to -1}\left(\frac{g(x)-2}{3(x+1)^2}\right)$. Show the work that leads to your answer.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

(a) $\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx} = 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right)$	$2 : \frac{d^2y}{dx^2}$ in terms of x and y
(b) $\left.\frac{dy}{dx}\right\vert_{(x,\,y)=(-2,\,8)} = (-2)^2 - \frac{1}{2}\cdot 8 = 0$ $\left.\frac{d^2y}{dx^2}\right\vert_{(x,\,y)=(-2,\,8)} = 2(-2) - \frac{1}{2}\left((-2)^2 - \frac{1}{2}\cdot 8\right) = -4 < 0$ Thus, the graph of f has a relative maximum at the point $(-2, 8)$.	2 : conclusion with justification
(c) $\lim_{x\to -1}(g(x) - 2) = 0$ and $\lim_{x\to -1} 3(x+1)^2 = 0$ Using L'Hospital's Rule, $\lim_{x\to -1}\left(\frac{g(x)-2}{3(x+1)^2}\right) = \lim_{x\to -1}\left(\frac{g'(x)}{6(x+1)}\right)$ $\lim_{x\to -1} g'(x) = 0$ and $\lim_{x\to -1} 6(x+1) = 0$ Using L'Hospital's Rule, $\lim_{x\to -1}\left(\frac{g'(x)}{6(x+1)}\right) = \lim_{x\to -1}\left(\frac{g''(x)}{6}\right) = \frac{-2}{6} = -\frac{1}{3}$	$3 : \begin{cases} 2 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$
(d) $h\left(\frac{1}{2}\right) \approx h(0) + h'(0)\cdot\frac{1}{2} = 2 + (-1)\cdot\frac{1}{2} = \frac{3}{2}$ $h(1) \approx h\left(\frac{1}{2}\right) + h'\left(\frac{1}{2}\right)\cdot\frac{1}{2} \approx \frac{3}{2} + \left(-\frac{1}{2}\right)\cdot\frac{1}{2} = \frac{5}{4}$	$2 : \begin{cases} 1 : \text{Euler's method} \\ 1 : \text{approximation} \end{cases}$

Question 5



The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $0 \le h \le 10$. The units of r and h are inches.

(a) Find the average value of the radius of the funnel.

(b) Find the volume of the funnel.

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

(a) Average radius $= \frac{1}{10}\int_0^{10} \frac{1}{20}\left(3 + h^2\right) dh = \frac{1}{200}\left[3h + \frac{h^3}{3}\right]_0^{10}$

$$= \frac{1}{200}\left(\left(30 + \frac{1000}{3}\right) - 0\right) = \frac{109}{60} \text{ in}$$

$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi \int_0^{10} \left(\left(\frac{1}{20}\right)\left(3 + h^2\right)\right)^2 dh = \frac{\pi}{400}\int_0^{10}\left(9 + 6h^2 + h^4\right) dh$

$$= \frac{\pi}{400}\left[9h + 2h^3 + \frac{h^5}{5}\right]_0^{10}$$

$$= \frac{\pi}{400}\left(\left(90 + 2000 + \frac{100000}{5}\right) - 0\right) = \frac{2209\pi}{40} \text{ in}^3$$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(c) $\frac{dr}{dt} = \frac{1}{20}(2h)\frac{dh}{dt}$

$$-\frac{1}{5} = \frac{3}{10}\frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{1}{5} \cdot \frac{10}{3} = -\frac{2}{3} \text{ in/sec}$$

$3 : \begin{cases} 2 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$

Question 6

The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \ge 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

(a) $f(1) = 1,\ f'(1) = -\frac{1}{2},\ f''(1) = \frac{1}{2^2},\ f'''(1) = -\frac{2}{2^3}$

$$f(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{2^2 \cdot 2}(x-1)^2 - \frac{1}{2^3 \cdot 3}(x-1)^3 + \cdots + \frac{(-1)^n}{2^n \cdot n}(x-1)^n + \cdots$$

$4 : \begin{cases} 1 : \text{first two terms} \\ 1 : \text{third term} \\ 1 : \text{fourth term} \\ 1 : \text{general term} \end{cases}$

(b) $R = 2$. The series converges on the interval $(-1, 3)$.

When $x = -1$, the series is $1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$.
Since the harmonic series diverges, this series diverges.

When $x = 3$, the series is $1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots$.
Since the alternating harmonic series converges, this series converges.

Therefore, the interval of convergence is $-1 < x \le 3$.

$2 : \begin{cases} 1 : \text{identifies both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$

(c) $f(1.2) \approx 1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.2)^2 = 1 - 0.1 + 0.005 = 0.905$

1 : approximation

(d) The series for $f(1.2)$ alternates with terms that decrease in magnitude to 0.

$$|f(1.2) - T_2(1.2)| \le \left| \frac{-1}{2^3 \cdot 3}(0.2)^3 \right| = \frac{1}{3000} \le 0.001$$

$2 : \begin{cases} 1 : \text{error form} \\ 1 : \text{analysis} \end{cases}$