Student Performance Q&A:

2016 AP® Calculus AB and Calculus BC Free-Response Questions

The following comments on the 2016 free-response questions for AP® Calculus AB and Calculus BC were written by the Chief Reader, Stephen Davis of Davidson College. They give an overview of each free-response question and of how students performed on the question, including typical student errors. General comments regarding the skills and content that students frequently have the most problems with are included. Some suggestions for improving student performance in these areas are also provided. Teachers are encouraged to attend a College Board workshop to learn strategies for improving student performance in specific areas.

Question AB1/BC1

What was the intent of this question?

In this problem students were given a function W that models the rate, in liters per hour, at which water is pumped into a tank at time t hours. They were also given a function R that models the rate, in liters per hour, at which water is removed from the tank. W is defined as an exponential function on the interval $0 \le t \le 8,$ and R is differentiable and decreasing on $0 \le t \le 8.$ Selected values of $R(t)$ are given in a table. The amount of water in the tank, in liters, at time $t = 0$ is given. In part (a) students needed to estimate $R'(2)$ by calculating the value of an appropriate difference quotient based on the values in the table. Units of liters/hr^2 are required. In part (b) students needed to use a left Riemann sum approximation for $\int_0^8 R(t)\,dt$ to estimate the total amount of water removed from the tank during the interval $0 \le t \le 8.$ Students needed to use the appropriate function values from the table with the four subintervals $[0, 1],$ $[1, 3],$ $[3, 6],$ and $[6, 8].$ By applying the given information that R is decreasing, students needed to conclude that the left Riemann sum approximation is an overestimate. In part (c) students needed to estimate the total amount of water in the tank at time $t = 8.$ This required adding the amount of water in the tank at time $t = 0$ to the amount of water pumped into the tank during the interval $0 \le t \le 8,$ and then subtracting the overestimate found in part (b). The definite integral $\int_0^8 W(t)\,dt$ gives the amount of water pumped into the tank during the interval $0 \le t \le 8$ and is evaluated using the calculator. In part (d) students needed to apply the Intermediate Value Theorem to $W(t) - R(t).$ This theorem guarantees at least one time t on the interval $0 < t < 8$ for which $W(t) - R(t) = 0$ or $W(t) = R(t).$ For this value of t, the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank.

How well did students perform on this question?

Although problems in context such as this tend to pose challenges for many students, the first three parts of this problem are fairly traditional AP Calculus Exam questions. A well-prepared student was likely to enter the exam on familiar ground. This is reflected in student performance: the mean score was 3.99 for AB students and 5.70 for BC students out of a possible 9 points. These means are the third highest on the exam for Calculus AB and highest on the exam (by a full point) for Calculus BC. Approximately 17 percent of AB students and 4.5 percent of BC students did not earn any points, the second lowest number among the Calculus AB problems and lowest among the Calculus BC problems. This demonstrates the accessible nature of the questions asked in this problem.

The most challenging parts of the problem were those where students need to articulate reasoning that leads to a conclusion: see the third point of part (b) and all of part (d).

What were common student errors or omissions?

In part (a) some students chose the incorrect interval for their difference quotient, or simplified incorrectly, or gave incorrect or no units. A few students used a regression routine on their calculators to model the data and used that model as the basis for their answers in this part. In part (b) some students chose an integral approximation other than a left Riemann sum. Many students had difficulty identifying and explaining why the approximation is an overestimate. Some students incorrectly appealed to supposed concavity of the graph of R. In part (c) a common error was to omit the initial amount of water in the tank (50,000 liters), and some students' carelessness with integral notation (omitting the differential dt) led them astray in evaluating the total. In both parts (b) and (d) some students may have defeated good thinking processes through vague language, referring to "it" or "the function" or "the graph" without explicitly tying the reference to the appropriate function in the problem.

As noted above, part (d) was especially challenging for students and is likely the reason that this problem had the fewest number of Calculus AB students earning all 9 points of any problem on the exam. Students were averse to constructing a new function, $W(t) - R(t)$, to which the Intermediate Value Theorem could be applied. Nearly all students working on this part chose to view W and R as competing functions with W starting above and finishing below R on the interval $0 \le t \le 8$. With this approach a complete explanation needed to apply the Intermediate Value Theorem twice, to each of W or R (or appeal to the continuity of both W and R). Many students noting that the range of W contains the range of R on the interval failed to note the crucial assumption of continuity for both functions.

Based on your experience of student responses at the AP® Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

A question that asks for an approximation to a derivative value is appealing to the definition of derivative as a limit of difference quotients, so the appropriate response is to find the value of the difference quotient on an interval that most tightly encloses the point in question. Students need to be familiar with the vocabulary of the AP Calculus courses (e.g., "Riemann sum") and read questions carefully so that they answer appropriately. There is little reason for a student to respond in part (b) with a midpoint sum, trapezoidal sum, or right Riemann sum, and such responses did not earn points. Teachers need to reinforce correct notation (e.g., concluding an integral expression with the appropriate differential) and unambiguous references to objects in the problem. Students also need practice using given information to construct a new function that is useful in solving a problem. Finally, when appealing to a major result like the Intermediate Value Theorem to justify a conclusion, either the result should be identified by name, or its hypotheses must be verified.

Question AB2

What was the intent of this question?

In this problem students were given information about a particle moving along the x-axis for time $t \geq 0$. The velocity of the particle is given as a trigonometric function, and the particle is at position $x = 2$ at time $t = 4$. In part (a) students needed to conclude that the particle is slowing down at $t = 4$ because $v(4)$ and $v'(4)$ have different signs. In part (b) students needed to determine when the particle changes direction in the interval $0 < t < 3$, and justify their answer. This required use of the calculator to solve $v(t) = 0$ on $0 < t < 3$. In part (c) students needed to apply the Fundamental Theorem of Calculus to find the position of the particle at time $t = 0$; i.e., $x(0) = x(4) - \int_0^4 v(t)\, dt$. The expression is evaluated using the calculator. In part (d) students needed to find the total distance the particle travels from $t = 0$ to $t = 3$. Students were expected to set up and evaluate $\int_0^3 |v(t)|\, dt$ (or an appropriate sum of definite integrals) using the calculator.

How well did students perform on this question?

Student performance was lower than anticipated, perhaps due to the calculator activity in this problem. The mean score was 3.00 out of a possible 9 points. Approximately 31.5 percent of students did not earn any points, the highest of the 6 problems on the Calculus AB exam.

Students were most successful in parts (a) and (d) but had particular difficulty with the justification for part (b). Part (c) proved the most troublesome, perhaps because the expression included both the initial value and integrating from 4 to 0.

What were common student errors or omissions?

Many students answered part (a) without any supporting reasoning. Although questions on the AP Exam often include an explicit call to explain or show reasoning, the expectation that answers be supported is part of the general instructions on the exam. The explanation must involve both velocity and acceleration at $t = 4$ to be eligible for any credit. Full credit required a correct statement of the signs of $v(4)$ and $v'(4)$.

There were some calculator-specific issues. The TI-Nspire calculators, for example, can have different modes for the graphing and calculator utilities, so that one utility can be in radian mode while the other is in degree mode. Therefore, a student might have a combination of degree-mode and radian-mode answers, depending upon which utility was used for that portion of the problem. In addition, the graphing utility may give values that are accurate to fewer decimal places than the calculator utility, leading to a student possibly not earning points.

Some students conflated the variables t and x, a critical confusion in a context where x denotes position and t denotes time. In part (b) AP Readers could generally follow the student's work, but this led to a disqualifying confusion in part (c). Many students gave an incomplete justification in part (b) such as identifying only where velocity is zero or failing to summarize their calculations in a correct verbal explanation.

In part (c) many students tried to solve a general initial value problem via an indefinite integral and failed to realize that a closed-form antiderivative for the given velocity function is not obtainable. This procedure-oriented approach to the solution may stem from an underdeveloped ability to fully understand a definite integral expression such as $\int_4^0 v(t)\, dt$. Other errors arose from difficulty dealing with an initial value at $t = 4$ instead of $t = 0$, carelessness with integral notation (omitting the differential dt, which could lead to the initial value being captured as part of the integrand), or not dealing correctly with a reversal of the limits of integration.

In part (d) the most common error was integrating velocity and obtaining displacement versus integrating speed and obtaining the total distance traveled.

Based on your experience of student responses at the AP® Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

The last two parts of this problem dealt with the connections between velocity-speed and displacement-total distance traveled. Students need to understand these connections. They also need to be able to view $\int_4^0 v(t)\,dt = x(0) - x(4)$ as equivalent to $x(0) = x(4) + \int_4^0 v(t)\,dt$ or $x(0) = x(4) - \int_0^4 v(t)\,dt.$ Students need more practice identifying appropriate instances for calculator-based evaluation. Finally, students need greater facility and familiarity with their calculators to ensure that they are reading the required number of digits to the right of the decimal and that radian mode is used for calculations on the home screen and when graphing.

Question AB3/BC3

What was the intent of this question?

In this problem students were given the graph of f, a piecewise-linear function defined on the interval $[-4, 12].$ A second function g is defined by $g(x) = \int_2^x f(t)\,dt.$ In part (a) students needed to determine whether g has a relative minimum, a relative maximum, or neither at $x = 10,$ and justify their answer. Using the Fundamental Theorem of Calculus, students needed to recognize that $g'(x) = f(x)$ for all x in the interval $[-4, 12].$ Since $g'(10) = f(10) = 0$ and $f(x) \le 0$ for $[8, 12],$ the First Derivative Test may be applied to conclude that there is no relative extremum at $x = 10.$ In part (b) students needed to determine whether the graph of g has a point of inflection at $x = 4,$ and justify their answer. Since $g'(x) = f(x),$ the graph of g has a point of inflection at $x = 4$ because f changes from increasing to decreasing at $x = 4.$ In part (c) students needed to find the absolute minimum value and the absolute maximum value of g on $[-4, 12].$ Since $g'(x) = f(x),$ students were expected to find relative extrema of g by identifying x-values where f changes sign. The absolute extrema occur either at the endpoints of the interval or at the relative extrema. By comparing the values of g at the four candidate x-values, students choose and justify the absolute extrema. Properties of the definite integral and the relation of the definite integral to accumulated area must be used to find the values of g. In part (d) students needed to find all intervals in $[-4, 12]$ for which $g(x) \le 0.$ This part also required properties of the definite integral and the relation of the definite integral to accumulated area.

How well did students perform on this question?

The mean score was 2.74 for AB students and 4.15 for BC students out of a possible 9 points. Approximately 27 percent of AB students and 10 percent of BC students did not earn any points. Calculus AB students did not perform as well on this problem in comparison to a similar problem on the 2015 Calculus AB exam. (Question AB5 in 2015 had a mean of 3.75 out of a possible 9 points with approximately 17 percent of students earning no points.) The key difference for this problem is that students needed to use the Fundamental Theorem of Calculus to obtain that $g'(x) = f(x)$ whereas the 2015 problem specified that the graph is the derivative of the objective function. Making this required connection earned the first point and is prerequisite to correct justifications in parts (a), (b), and (c). In parts (a) and (b) students often had difficulty articulating clear justifications. In part (c) students could identify candidates $x = -2$ and $x = 6$ but had difficulty proceeding further. In part (d) many students earned just 1 of the 2 points.

What were common student errors or omissions?

The most common student error was a failure to communicate the connection that $g'(x) = f(x)$. Although it was preferable for the student to make this connection explicitly, implicit connections were accepted. This included using f in a context where g' is appropriate. However, many students confined their discussion to the behavior of g without any mention of f. In part (b) students incorrectly viewed the presence of a point of inflection as equivalent with a zero value for the second derivative. In part (c) a common error was to not consider the endpoints of the interval. Many students identified the location of the minimum and maximum without calling out the maximum and minimum values as requested. In part (d) common errors included omitting endpoints in the intervals, truncating the interval $[-4, 2]$ to either $[-4, -2]$ or $[-2, 2]$, or answering the question for where $f(x) \le 0$ instead of where $g(x) \le 0$.

Based on your experience of student responses at the AP® Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

This problem investigated properties of a function defined by a definite integral, $g(x) = \int_2^x f(t)\,dt$. Students need practice with functions defined in this way and, in particular, how to leverage the Fundamental Theorem of Calculus result that states that $\frac{d}{dx}\left(\int_2^x f(t)\,dt\right) = f(x)$ to connect properties of f with properties of g.

Three parts of this problem specifically directed students to justify their answers, so AP Readers had ample opportunity to see the many ways students struggle to produce cogent explanations in support of their answers. One aspect of this struggle is vagueness of expression, either leaving key assumptions unmentioned or generically referencing “it” or “the function” instead of referencing a specific function by name.

Question AB4

What was the intent of this question?

In this problem students were presented with a first-order separable differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$. In part (a) students needed to sketch a slope field at six points in the xy-plane provided: $(0, 0)$, $(0, 1)$, $(0, 2)$, $(2, 0)$, $(2, 1)$, and $(2, 2)$. In part (b) students were given that $y = f(x)$ is the particular solution to the differential equation with the initial condition $f(2) = 3$. Students needed to write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$, where the slope is computed using the given $\frac{dy}{dx}$. The value of $f(2.1)$ is approximated using the tangent line. In part (c) students were expected to use separation of variables to find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 3$.

How well did students perform on this question?

Students scored well on this problem, particularly in parts (a) and (b). The mean score was 4.68 out of a possible 9 points. Approximately 10.5 percent of students did not earn any points, whereas 50 percent of students earned at least 5 points. The mechanics of parts (a) and (b) are fairly straightforward, and students generally made the appropriate substitution of x- and y-values into the expression for $\frac{dy}{dx}$ and interpreted the value that results. Students had more difficulty with the antidifferentiation step and ensuing steps for part (c).

What were common student errors or omissions?

In part (a) some students interpreted zero slope as corresponding to a vertical line segment. In part (b) some students used the incorrect point for their tangent line: either $(2, 2)$, $(3, 3)$, or one of the six points used in part (a). Other students felt the need to find the solution to the given differential equation corresponding to $f(2) = 3$ and then finding a tangent line to that function. This approach uses valuable time and introduces many opportunities for computational errors.

Students made algebraic errors at both beginning and end steps of part (c). Many incorrectly separated variables at the beginning as $\frac{dy}{y^2} = (x - 1)\,dx$ or as $y^2\,dy = \frac{1}{x - 1}\,dx.$ Students also experienced algebra challenges in solving for y at the end of the process. Antidifferentiation errors included involving a logarithm function in an antiderivative of $\frac{1}{y^2}.$ Finally, some students omitted the constant of integration or were capricious in where it appeared in their solution process.

Based on your experience of student responses at the AP® Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

Based on part (b) performance, students need practice in recognizing how given information can be used to solve a problem without going back to step one of a familiar process.

Based on part (c) performance, many students could profit from more antidifferentiation practice so that they recognize that not every function expressed using a fraction (e.g., $\frac{1}{y^2}$) is the derivative of a logarithmic expression. Students also need to build appreciation for the notation and reasoning behind the steps of the process for solving a separable differential equation initial value problem. As students acquire meaning behind the mechanics of this multi-step process, they are less apt to stumble into errors in mathematical communication.

Question AB5/BC5

What was the intent of this question?

In this problem students were presented with a funnel of height 10 inches and circular cross sections. At height h the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right),$ where r and h are measured in inches. In part (a) students needed to find the average value of the radius of the funnel. This required evaluating $\frac{1}{10}\int_0^{10} \frac{1}{20}\left(3 + h^2\right) dh$ by finding an antiderivative. In part (b) students needed to find the volume of the funnel. By incorporating the fact that the cross sections are circular, the students were expected to set up and evaluate an integral of the form $\pi\int_0^{10} r^2\,dh = \pi\int_0^{10} \left(\left(\frac{1}{20}\right)\left(3 + h^2\right)\right)^2 dh.$ In part (c) students were given that the funnel contains liquid that is draining from the bottom. When the height of the liquid is 3 inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ in/sec. Students were expected to find the rate at which the height is changing at this instant. To solve this related rates problem, students needed to use $r = \frac{1}{20}\left(3 + h^2\right)$ and take the derivative with respect to t.

How well did students perform on this question?

The mean score was 2.50 for AB students and 4.19 for BC students out of a possible 9 points. Approximately 31 percent of AB students and 13 percent of BC students did not earn any points, and about 57 percent of the AB students earned just 2 or fewer points. As these data show, Calculus AB student performance was disappointing, scoring lowest on this problem on the exam. Most students recognized that parts (a) and (b) involved definite integrals, but many had trouble building a correct integral and/or evaluating their integrals. Similarly, most students recognized part (c) as a related rates problem but had difficulty completing the task.

What were common student errors or omissions?

Overall, students made many arithmetic and algebra errors, especially dealing with the fraction in the expression for r. Some students miscopied the expression for r as $r = \frac{1}{20}(3 + h)^2$. In part (a) many students keyed on the word "average," but then attempted to find an average rate of change via a difference quotient or to simply average the two values $r(0)$ and $r(10)$. Others had a correct integrand but did not divide by 10 to obtain the average value. In part (b) students were challenged by the algebra of squaring the expression for r. Some students attempted to simplify the problem by assuming the funnel was a cone, sphere, or cylinder. It appears that many students had difficulty seeing that part (b) is asking for a volume of revolution about the vertical axis through the center of the funnel. In both parts (a) and (b) many students wrote an incorrect differential for their integrals, hinting at a lack of understanding of integral notation. Also in these two parts many students gave an incorrect antiderivative, often as the result of an errant attempt at substitution.

In part (c) students made chain rule errors, omitting the crucial factor $\frac{dh}{dt}$. Some students also had trouble differentiating $\frac{1}{20}\left(3 + h^2\right)$ with respect to h. Many of those who did correctly apply the chain rule failed to interpret the radius decreasing at the rate of $\frac{1}{5}$ inch per second to indicate that $\frac{dr}{dt}$ is negative.

Based on your experience of student responses at the AP® Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

Teachers should hold students accountable to correctly manipulate algebraic and arithmetic expressions. It is disappointing when what appears as good conceptual understanding of calculus is diminished by poor algebra and arithmetic skills. Students also need to read carefully, e.g., to determine what sort of "average" is requested. Students need practice computing a volume in context, especially one where the cross sections are naturally presented as horizontal. Teachers can also build greater notational fluency by reinforcing the role of the variable in a differential, whether as part of an integral or of a derivative. Appropriate Leibniz notation for a derivative can be a helpful cue to appropriate applications of the chain rule.

Question AB6

What was the intent of this question?

In this problem students were given two general functions, f and g, that have continuous second derivatives. A table is presented with values of the functions and their derivatives at selected values of x. In part (a) students needed to find the equation of the line tangent to the graph of k at $x = 3$, where k is defined by $k(x) = f(g(x))$. This required application of the chain rule and use of values from the table to compute $k'(3) = f'(g(3)) \cdot g'(3)$ and $k(3) = f(g(3))$. In part (b) students were given $h(x) = \frac{g(x)}{f(x)}$ and asked to compute $h'(1)$. Students were expected to use the quotient rule and values from the table. Alternately, the product rule and chain rule can be applied to $h(x) = g(x) \cdot (f(x))^{-1}$. In part (c) students needed to evaluate the definite integral $\int_1^3 f''(2x)\,dx$. Using substitution of variables and applying the Fundamental Theorem of Calculus, students were expected to find an antiderivative involving f' and evaluate using values from the table.

How well did students perform on this question?

Students performed well on this problem, especially considering that it was the last problem on the exam and that students were so challenged by Question 5. The mean score was 4.56 out of a possible 9 points. Approximately 18 percent of students did not earn any points. By contrast about 54 percent of students earned 5 or more points. In general students were able to make progress on each part of the problem.

What were common student errors or omissions?

In part (a) the most common error was to differentiate incorrectly, typically arriving at $k'(3)$ expressed as either $f'(g(3))$ or $f'(g'(3))$. In part (b) some students had trouble applying a quotient rule learned for the generic quotient $\frac{f(x)}{g(x)}$ to this particular instance expressed with $h(x) = \frac{g(x)}{f(x)}$. Some students, perhaps challenged by abstraction, replaced f and g by explicit formulas, usually linear functions, before differentiating. Other errors tended to be in numeric simplification although this was not required. Also common here was to write $(-6)^2$ as -6^2. In part (c) students mishandled the role of "2" in the antiderivative, either by ignoring it or by multiplying by 2 rather than dividing by 2. Some students who used a correct u-substitution for the antiderivative had difficulty in evaluating the antiderivative at appropriate limits.

Based on your experience of student responses at the AP® Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

This problem was another instance where notational fluency is important to student success. To build one aspect of this fluency, teachers can provide students with multiple experiences involving abstract functions. Teachers can also reinforce that numeric answers do not need to be simplified on the AP Exam, so students can avoid the peril of simplification errors. Teachers should encourage students to produce clear and organized presentations of their solutions and give students opportunities to practice and develop this skill.

Question BC2

What was the intent of this question?

In this problem students were given information about the motion of a particle in the *xy*-plane. The position of the particle is defined as $(x(t), y(t))$, where $\frac{dx}{dt} = t^2 + \sin\left(3t^2\right)$. The graph of y, consisting of three line segments, and the position of the particle at $t = 0$ were also given. In part (a) students needed to find the position of the particle at $t = 3$. Students were expected to evaluate $5 + \int_0^3 \left(t^2 + \sin\left(3t^2\right)\right) dt$ using the calculator to find $x(3)$. The value of $y(3)$ can be read from the graph. In part (b) students needed to compute the slope of the line tangent to the particle's path at $t = 3$ by evaluating $\frac{y'(3)}{x'(3)}$. The value for $y'(3)$ is found by computing the slope of the line segment from $t = 2$ to $t = 4$, and the value for $x'(3)$ is found using the calculator. In part (c) students needed to find the speed of the particle at $t = 3$. Students were expected to write the expression for speed, $\sqrt{(x'(3))^2 + (y'(3))^2}$, and use the calculator to compute this value. In part (d) students needed to find the total distance traveled from $t = 0$ to $t = 2$. Although the general formula for total distance traveled is $\int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$, students were expected to write the total distance traveled as the sum of two integrals since $y'(t) = -2$ from $t = 0$ to $t = 1$, and $y'(t) = 0$ from $t = 1$ to $t = 2$. The expression $\int_0^1 \sqrt{(x'(t))^2 + (-2)^2}\, dt + \int_1^2 \sqrt{(x'(t))^2 + (0)^2}\, dt$ is evaluated using the calculator.

How well did students perform on this question?

Students with experience with parametric particle motion problems did well overall on parts (a), (b), and (c). In part (d) many students were able to produce a correct integral expression for the total distance traveled but had difficulty dealing with the change in the graph of y at $t = 1$. The mean score was 4.14 out of a possible 9 points. Approximately 16 percent of students did not earn any points.

What were common student errors or omissions?

Common errors overall arose from confusing x and/or y with their derivatives with respect to t and from misinterpreting information from the graph of y. There were also many instances of "linkage" errors where students placed an equal sign between two steps of their solutions, thus claiming equality between unequal values. In part (a) some students evaluated $\frac{dx}{dt}$ at $t = 3$ rather than integrating $\frac{dx}{dt}$ from $t = 0$ to $t = 3$. Students omitting the differential from an integral expression incorrectly captured the initial condition within the integrand. For $y(3)$ some students integrated the given function for y from $t = 0$ to $t = 3$ rather than just determining the value from the graph. In part (b) common errors included calculating the slope using $\frac{x'(t)}{y'(t)}$ or $\frac{y'(t)}{x''(t)}$ or reporting the negative reciprocal of the correct answer as the slope of the tangent line. Errors in part (c) arose mostly from incorrect formulas for speed.

In part (d) most students did not know how to handle the change in the graph for y at $t = 1$. Some students averaged the two slopes and used a linear function for $y'(t)$. Some replaced the integrand with the numeric value for the speed of the particle at $t = 3$ as computed in part (c). Other students moved the square root on the integrand outside the integral.

Although this was a calculator-active problem, there were relatively few calculator-specific issues. Teachers appear to have done a good job of training students to retain intermediate values in the calculator memory in order to produce more accurate final answers. There were very few instances of students working with calculators set in degree mode. If anything, the most common calculator error was in not employing the calculator when it was needed.

Based on your experience of student responses at the AP® Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

Teachers can help broaden students' experiences with functions expressed in a variety of ways, whether numerically, analytically, graphically, or verbally, and with applying given information appropriately within a specific situation. Students also need to develop better skills in communicating mathematics. This includes general issues such as the correct use of the equals sign and specific issues such as the student writing an indefinite integral and treating it as a definite integral with inferred limits of integration. Teachers can also work with students to develop a sense of when a calculator is a useful tool and when it is necessary in problem solutions.

Question BC4

What was the intent of this question?

In this problem students were presented with a differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y.$ In part (a) students needed to find the second derivative $\frac{d^2y}{dx^2}$ in terms of x and y using implicit differentiation. In part (b) students were given that $y = f(x)$ is the particular solution to the differential equation whose graph passes through $(-2, 8).$ Students needed to use both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ to determine if the graph of f has a relative minimum, a relative maximum, or neither at the point $(-2, 8),$ and justify their answer. In part (c) students were given that $y = g(x)$ is the particular solution to the differential equation with $g(-1) = 2.$ Students needed to compute a limit. After indicating that L'Hospital's Rule is required, students were expected to apply L'Hospital's Rule twice. In part (d) students were given that $y = h(x)$ is the particular solution to the differential equation with $h(0) = 2.$ Students needed to use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1).$

How well did students perform on this question?

Students scored well on this problem with the second highest mean on the exam and highest mean among the three BC-only problems. The mean score was 4.68 out of a possible 9 points. Approximately 10 percent of students did not earn any points, while 43 percent of students earned 6 or more points.

In part (a) most students could enter the problem with a correct derivative expression. In part (b) students who used the second derivative from part (a) did so with good success, but many students constrained themselves to using the first derivative only without success. In part (c) many students were able to earn at least 2 of the 3 points. In part (d) poor organization and communication got in the way of what may otherwise have earned students some points.

What were common student errors or omissions?

In part (a) a common error was to differentiate the y portion of $\frac{dy}{dx}$ with respect to y instead of with respect to x. Other students did not complete the part, stopping with a second derivative that involved $\frac{dy}{dx}$. Still other students made the correct substitution for $\frac{dy}{dx}$ but went further and made an algebra error.

In part (b) many students chose the wrong tool, the First Derivative Test, over the much more appropriate Second Derivative Test. Although it is possible to use the First Derivative Test to justify a correct conclusion here, this necessitates analyzing a first derivative whose given form involves y to determine the sign of a particular solution's derivative across an interval. This argument is very difficult to make.

A common error in part (c) was to mishandle or omit correct limit notation. When applying a calculus theorem such as the Intermediate Value Theorem or L'Hospital's Rule, students are expected either to verify the conditions for applying the theorem or to identify the theorem by name. Many students fell short of earning both of the first 2 points in this part by omitting such documentation of their solution process.

In part (d) an application of Euler's method involves the combination of many elements. Students' poor communication and organization of the various components of this method led to incomplete documentation of the method and/or errors in their calculations.

Based on your experience of student responses at the AP® Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

Teachers can encourage students to expand their toolkit for relative extremum problems to include the Second Derivative Test, and develop strategies for choosing the most appropriate tool for a given problem. Students also need continual practice on the use of quality mathematical expressions and notation. When evaluating a limit, correct limit notation should be present at the appropriate stages of the evaluation. When applying a result such as L'Hospital's Rule, student work should indicate that appropriate hypotheses are satisfied. When employing a complicated process such as Euler's method, students need to organize and clearly label their work. It is not just sufficient for students to "know" how to solve problems; they also need to be able to clearly communicate their solution processes.

Question BC6

What was the intent of this question?

In this problem students were presented with a function f that has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. The values of f, the first derivative of f, and nth derivatives of f at $x = 1$ are given. In part (a) students needed to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$. In part (b) students were given the radius of convergence and asked to find the interval of convergence. Since the series is centered about $x = 1$ with a radius of convergence of 2, students were expected to determine if the series converges at the endpoints $x = -1$ and $x = 3$. Students needed to use knowledge of the harmonic series and alternating harmonic series. In part (c) students were expected to use the Taylor series found in part (a) to represent $f(1.2)$ as an alternating series, and then use the first three nonzero terms of the alternating series to approximate $f(1.2)$. In part (d) students were expected to show that the approximation from part (c) is within 0.001 of the exact value of $f(1.2)$. The error of this approximation is bounded by the magnitude of the fourth term of the series for $f(1.2)$.

How well did students perform on this question?

Student performance on this problem was fairly typical for a series problem. The mean score was 3.82 out of a possible 9 points. This is the lowest mean on the Calculus BC exam. Approximately 23 percent of students did not earn any points on this problem.

Students generally performed well on parts (a) and (c). In part (b) they could identify the endpoints of the interval of convergence but had difficulty analyzing endpoint behavior for convergence. In part (d) about half of the students could identify a correct form for the error bound, but many had difficulty resolving the needed inequality.

What were common student errors or omissions?

In part (a) common errors were to omit the factorial portion of the derivative expression or to make arithmetic mistakes. Some students omitted the general term, and others made algebraic errors in simplifying this term, particularly when dealing with the factorial expression.

In part (b) some students unnecessarily used a ratio test to find the endpoints of the interval, rather than to recognize that these endpoints are readily obtainable from the given information. Many students either omitted consideration of endpoint convergence or gave an incorrect analysis.

In part (c) some students included four terms in their approximation rather the specified three, and some others made arithmetic errors.

In part (d) some students attempted to apply the Lagrange error bound, either coming up short or making unwarranted assumptions in an attempt to get the desired result. For students employing the alternating series error bound, a common mistake was to forget or mishandle applying absolute value to the term that gives the error bound. Other errors involved inappropriate insertions of 0.001 within their error bound calculations.

Based on your experience of student responses at the AP® Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

Teachers can help students recognize how given information can be used without having to regress a problem to a common starting point. In this instance, the given information about the center and radius of convergence makes the endpoints of the interval of convergence readily available. Starting a problem “in the middle” is good practice for students and helps them understand concepts like center, radius, and interval of convergence beyond a rote process memorized to produce these results. This also will allow for more focus on endpoint analysis as necessary in determining an interval of convergence. Students also need to expand their toolkit of error bounds to include the alternating series error bound in addition to the Lagrange error bound. Teachers can give students opportunities to investigate when competing methods are applicable and how to select the best method for a given situation. Finally, students need practice with the algebra of inequalities, in particular how this algebra is applied in the context of an error bound.