

Quantitative Skills and Analysis in AP[®] Physics 1 and 2 Investigations:

A Guide for Teachers

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Acknowledgments

The College Board would like to acknowledge the following individuals for their commitment and dedication toward the completion of this guide:

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Introduction

Experimental physics relies heavily on quantitative skills and analysis. Successful data collection and analysis in the AP Physics laboratory requires many skills, including making accurate and precise measurement of quantities using a range of instruments, converting units, making estimates, carrying out algebraic and statistical calculations, and constructing and interpreting graphs. The information obtained using these skills must then be integrated with reasoning and higher-order thinking skills in order for students to successfully analyze and interpret data and formulate and communicate conclusions.

This guide is intended to provide AP Physics teachers background information covering experimental error and estimation of uncertainties, error analysis, and the creation and analysis of graphic representations of data, knowledge of which is necessary to support students in the development and successful application of the quantitative skills needed in the AP Physics laboratory.

Student Handout

While covered in this guide, experiment and data analysis questions on the AP Physics Exams will not require students to calculate standard deviations, use formal methods to calculate the propagation of error, or carry out a calculated linear best-fit. However students should be able to:

- ▶ Discuss which measurement or variable in a procedure contributes most to overall uncertainty in the final result, and on conclusions drawn from a given data set
- ▶ Recognize that there may be no significant difference between two reported measurements if they differ by less than the smallest division on a scale
- ▶ Reason in terms of percentage error
- ▶ Report results of calculations to a reasonable number of significant digits
- ▶ Construct an estimated best-fit line to data that they plot
- ▶ Articulate the effects of error and error propagation on conclusions drawn from a given data set, and how changing number of measurements, measurement techniques, or precision of measurements would affect the results and conclusions
- ▶ Review and critique an experimental design or procedure and decide whether the conclusions can be justified based on the procedure and the evidence presented

To help teachers support the development of these skills, including some basics on using spreadsheet programs such as Excel or Google Sheets, we have also made available a guide for students that can be printed and handed out for quick reference: the *AP Physics 1 and 2 Lab Investigations: Student Guide to Data Analysis* <https://media.collegeboard.com/digitalServices/pdf/ap/physics-1-2-data-analysis-student-guide.pdf>

Chapter 1: Experimental Error and Uncertainty

In the laboratory neither the measuring instrument nor the measuring procedure is ever perfect; consequently, every experiment is subject to experimental error. A reported result that does not include the experimental error is incomplete. The only numbers that are valid to report without experimental error are discrete quantities that you can count, for example, the number of students in your class. Values measured from a measuring instrument such as a balance or meterstick are not discrete quantities and have experimental error associated with them.

What follows is a discussion of the kinds of measurement error, and how to determine and calculate errors, or **uncertainties**.

Systematic vs. Random Error

Experimental errors are generally classified under two broad categories: systematic errors and random errors.

Systematic errors include errors due to the calibration of instruments and errors due to faulty procedures or assumptions. When reporting results in scientific journals, one might have to go to great lengths to assure that one's metersticks and clocks, for example, have been accurately calibrated against international standards of length and time. However, even if an instrument has been properly calibrated, it can still be used in a fashion that leads to systematically wrong results (either always high or always low).

If instruments are calibrated and used correctly, you can expect accurate results; but even the most basic measurements might include things such as parallax errors in measuring length or human reaction-time errors with a stopwatch, creating inaccuracies in results. Another common example of a systematic error in the physics lab is the assumption that air resistance is not a factor for a falling body, which we do in textbooks all the time, but makes actual results inaccurate.

Random errors include errors of judgment in reading a meter or a scale and errors due to fluctuating experimental conditions. Because no instrument is perfectly precise, even if measurement conditions are not fluctuating, careful measurements of the same quantity by the same person will not yield the same result over multiple trials. For many measurements, environmental fluctuations (e.g., the temperature of the laboratory or the value of the line voltage) will necessarily give different results each time, and there will also be fluctuations caused by the fact that many experimental parameters are not exactly defined.

For example, the width of a table top might be said to be 1 meter, but close examination would show that opposite edges are not precisely parallel and a microscopic examination would reveal that the edges are quite rough. The table does not have a width in the same sense that you can talk about the number of students in your class: you precisely know how many students are missing from class today, but even if you are measuring it correctly (without systematic error), the table's width cannot be precisely known.

Implications for Precision and Accuracy

When the systematic errors in an experiment are small, the experiment is said to be **accurate**. Accuracy is a measure of how close you are to the accepted answer.

When the random errors in an experiment are small, the experiment is said to be **precise**. Precision tells you how well you know the answer you have determined; it is how sure you are of your measurement, or alternatively, your uncertainty of your measurement, regardless of whether the measurement is accurate or correct.

For example, if you are doing an experiment to measure the acceleration of a cart on a horizontal track due to an attached falling mass, and you idealize the situation to where there is no friction, the measured acceleration is going to be systematically less than the theoretical acceleration if friction is the only force unaccounted for (the track could also not be level and that would be another systematic error). With systematic errors, the measured result is always different from the expected result in the same way each time (in this example, the measured result is always *less* than the expected result). In addition, there will be random errors in this experiment: if the acceleration is being determined from distances measured on the track, and time measured by stopwatches or photogates, the stopwatches, photogates and metersticks will have some limit to their precision; so each measurement might be a little bit high or a little bit low. Exact measurements cannot be made, and the measurements will randomly fall on either side of the mean result.

Systematic errors are avoidable or can be accounted for, and any time you determine that there is one you should do your best to eliminate it or account for it. Random errors, on the other hand, are reducible in easy ways (using more precise measuring instruments), but are unavoidable. The sensitivity of your measuring instrument determines the ultimate precision of any measurement. Since random errors are unavoidable, you should never use a measuring instrument so crude that these random errors go undetected. You should always use a measuring instrument that is sufficiently sensitive so that duplicate measurements do not yield duplicate results.

The recommended procedure is to report your readings of analog scales and meters by estimating the last digit as best you can. Digital meters will generally come with a stated precision by the manufacturer.

Significant Digits

When doing data analysis calculations, calculators and computers typically give more digits than are significant. A typical calculator may give an eight or ten digit answer. Students need to think about their answers, and think about how many of those digits are actually meaningful, or **significant**. Experimentally measured numbers need to be expressed to the correct number of significant digits, determined by the error, or the precision, to which they are known.

When counting significant digits, leading zeroes are not significant:

$0.0045 \text{ m} = 4.5 \times 10^{-3} \text{ m} = 4.5 \text{ mm}$ (each of these measures have two significant figures). But trailing zeroes are significant, and tell us how precise something is: 0.5 s is less precise than 0.500 s . In this example, the first (one significant digit in the tenths place) implies that the time measurement is known within about a tenth of a second, while the second (three significant digits, with the least significant being in the thousandths place) implies it is known to about a millisecond.

When you quantify random error in a measurement, as you should always do in the lab, the error tells us about how unsure you are of each digit. Since the error, or precision, is the uncertainty in the measurement, it determines the number of significant digits in the measurement.

Let's say you determine with some calculations the speed of a student walking at constant speed in the lab. Your calculator may give you eight digits, so you find that $v = 3.0342842 \pm 0.17092203 \text{ m/s}$. What this uncertainty tells you is that you are unsure of the tenths place to ± 0.1 , of the hundredths place to ± 0.07 , and so on. If you are already unsure of the tenths place (the most significant digit in this uncertainty), then generally you need not be concerned with an additional uncertainty in a less significant digit such as the hundredths place. The biggest uncertainty is in the largest decimal place, or most significant digit of the uncertainty, so the rest of the error is extraneous. Since you should always round errors to one significant digit, you are left with $v = 3.0342842 \pm 0.2 \text{ m/s}$.

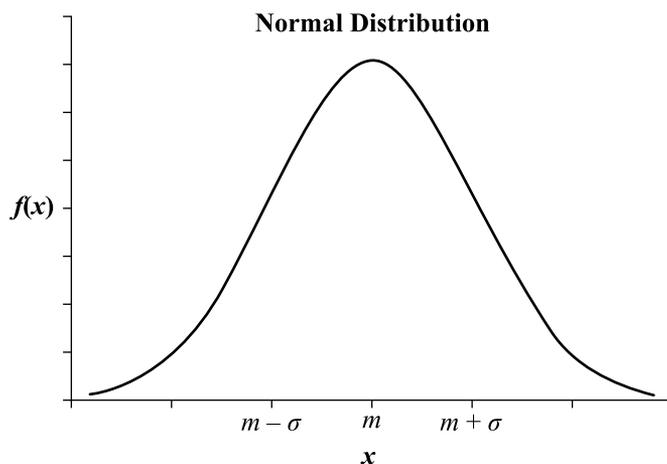
In addition, now that you've determined that having a measurement in the hundredths place is not very meaningful, you need to *round the measurement to the same decimal place as the uncertainty*; so the correct answer is $v = 3.0 \pm 0.2 \text{ m/s}$. Note that the rounding of the measurement is to the same decimal place as the error, not the same number of significant digits, so in this case the measurement has two significant digits while the error has one.

Distribution of Measurements

Given the pervasiveness of random error, one of the tasks of the experimentalist is to estimate the probability that someone who performs an apparently identical experiment will obtain a different result. Subsequent measurements will not be expected to give the same answer, but multiple measurements will be distributed in such a way that you can make a good estimate of what you think is the correct answer. If you were to make a lot of measurements of one quantity, and the error was truly random, you would expect to make as many measurements that are higher than the correct value as measurements that are lower, and to obtain fewer measurements that are further away from the correct value than those that are closer.

For example, assume lab groups from your class are at an amusement park trying to measure the speed of a roller coaster car at a certain point on its track, and the average of all their measurements is 21.5 m/s. You would not expect any of the individual measured values to necessarily be the same due to random errors, but you would expect it to be as likely that a student measures 22 m/s as 21 m/s. On the other hand, you would expect it to be a lot less likely for them to measure a value much farther away – such as 15 m/s – and to see fewer of those measurements than measurements that are closer to the average.

If many measurements are made, you can expect to see a pattern emerge. For example, if you were to plot the frequency of measurements vs. the value of the measurement, you would expect to get a distribution that approximates the following:



This distribution occurs so frequently it is referred to as the normal (or Gaussian) distribution. Remember, we are talking about making multiple measurements of a single quantity here – x is simply the value of the measurement you make (e.g. the width of the table) and $f(x)$ is the number of times you come up with a particular number. The distribution of answers will usually look like the normal distribution no matter what you are measuring, as long as you make a large number of measurements.

Note that if you make only two or three measurements, there is no way you could get a distribution of measurements that looks this way; but once you take about ten or more measurements that are influenced entirely by random error, the normal distribution starts to emerge. Statistical theory provides the means to determine an equation describing the normal distribution:

$$f(x) = \frac{e^{-\left[\frac{(x-m)^2}{2\sigma^2}\right]}}{\sqrt{2\sigma^2}}$$

[Equation 1]

This equation is not one you should expect students to use, but the parameters m and σ are important to understand. The function is characterized by these two parameters: the **mean**, m , which tells us where the peak of the curve falls along the x axis, and the **standard deviation**, σ , which is a measure of how wide the curve is. The mean is your best guess of the correct value if only random errors impact the measurements, and the standard deviation is a measure of how close you expect any given measurement to be to the mean.

Chapter 2: Analysis of Uncertainty (Error Analysis)

Estimates of the Mean and Standard Deviation

So what does statistical theory tell us about being able to determine the value of a measurement? The method is different depending on whether you take a very large number of measurements (thousands), or just a few. It is often the case that when making scientific measurements, the number of measurements is insufficient to get the exact normal distribution that would allow you to precisely determine a measured value. Given a sample of a finite number of N measurements, you should **estimate** the mean and standard deviation of a series of measurements of values denoted x_i . The estimate of the mean, M , is given by a familiar procedure:

$$M = \sum_{i=1}^N \frac{x_i}{N}$$

[Equation 2]

Similarly, when you have a set of measurements from a sample, the estimate of the standard deviation, S , is:

$$S = \sqrt{\sum_{i=1}^N \frac{(x_i - M)^2}{N - 1}}$$

[Equation 3]

Of course, most data analysis programs such as Excel® or Logger Pro® have these functions built in so that you do not have to do the sums manually.

Estimating Precision

Precision is a measure of how uncertain you are about your measurement given random fluctuations.

If you are making a single measurement with an analog instrument, the precision can be estimated from how well you think you can read the instrument. Typically that means estimating half of the smallest division. For example, using a meter stick with millimeter divisions, for a single measurement you should be able to estimate the measurement to the nearest half a millimeter. If you are using a digital measuring device, which gives an exact measurement as output, the precision should be given in the instruction manual.

But what if you are making multiple measurements of a single quantity? Ask yourself: “What is the probability that my estimate of the mean, M , based upon a small number of measurements, will fall close to the true mean, m , which is based on a large (ideally, infinite) number of measurements?” What you want to estimate here is not the standard deviation (spread) of the individual measurements, but the standard deviation of estimated means. For example, if multiple lab groups were to do similar experiments and each took a relatively small number of data points, none would be expected to measure the true mean, m , but each would be expected to measure a different estimated mean M .

According to statistical theory, if you know how much the estimated mean values are expected to be spread out, then you know the correct value is likely to be in that spread. The estimated spread of the estimated mean measured values is called the **standard error**, which is abbreviated SE .

Statistical theory also tells us that a good estimate of the standard deviation in measured mean values, SE , is:

$$SE = \frac{S}{\sqrt{N}}$$

[Equation 4]

Notice that the estimate of the mean, M , and standard deviation, S , might not change appreciably as the number of measurements, N , is increased, because as the numerator increases proportionally, so does the denominator (the wider the spread of values x_i , the more are needed before M and S become approximately constant with additional measurements). Alternatively note that the standard error, SE , gets smaller with larger N .

The standard error is a measure of the precision, or uncertainty, when you take multiple measurements of a quantity. You can state it as absolute uncertainty or as a percent uncertainty. For example, if you measure the acceleration of your cart on a track system to be $a = 1.21 \text{ m/s}^2$, and you determine the standard error to be $SE = 0.06 \text{ m/s}^2$, then the acceleration can be written with uncertainty to be $a = 1.21 \pm 0.06 \text{ m/s}^2$, or $a = 1.21 \text{ m/s}^2 \pm 7\%$.

Methods of Reducing Random Errors

Since SE gets smaller as the number of measurements, N , gets larger, you can always reduce your error in an experiment by making additional measurements. It is important to know that this estimate of the standard error is an approximation that is only valid for small N and does not mean that the standard error goes to zero as N gets large. There is always a finite, non-zero uncertainty for every measurement.

In addition, note that if the measuring instrument is so crude as to give the same value every time, with no standard deviation, that does not mean your measurement is infinitely precise, it means that your measuring instrument is either not precise enough or is being used incorrectly. You can always reduce random errors by choosing an appropriate measuring instrument for the experiment. A more precise instrument will generally help to reduce random errors, but it must be appropriate to the purpose: if you are measuring the length of a room, a more precise caliper will probably create more error than a less precise tape measure since the caliper was not designed to measure as large a distance.

Since typically stopwatches used in labs are of similar precision, when making a time measurement it is usually better to measure for a longer time in order to reduce human reaction time errors and precision errors. For example, if you are measuring the speed of a platform rotating at constant speed, it is better to time how long it takes to rotate for 10 rotations (and appropriately divide by 10) than just one rotation.

On the other hand, if you are doing an experiment that involves changes over time, you need to balance measuring for a sufficient length of time versus how much the other changes will impact the experiment.

Propagation of Error

Let's say you want to measure the area of a tabletop and report the result with uncertainty, since you cannot make an exact measurement.

Assume that you have measured the length as $L = 1.763 \pm 0.004$ m, and the width as $W = 0.759 \pm 0.003$ m, where the uncertainties in the length and width come from standard error calculations. To report the area of the table with uncertainty, you must allow for the possibility that errors in the independent measurements of L and W might offset one another. The correct procedure according to statistical theory is as follows:

First, calculate the uncertainty in A caused by the uncertainty in L only, using the mean value of W but the maximum value of L . The uncertainties in a quantity are denoted by the Greek letter Δ . Call the resulting uncertainty ΔA_L :

$$\Delta A_L = (L + \Delta L) * W - L * W.$$

Note that in this case, you are using $A = L * W$, but this can be applied to any function of one or more variables.

In our example, this result is:

$$\Delta A_L = 1.767 \text{ m} * 0.759 \text{ m} - 1.763 \text{ m} * 0.759 \text{ m} = 0.003 \text{ m}^2.$$

Next, calculate the uncertainty in A caused by the uncertainty in W only, using the mean value of L in the calculation. Call this result ΔA_W :

$$\Delta A_W = L * (W + \Delta W) - L * W.$$

In our example, this result is:

$$\Delta A_W = 1.763 \text{ m} * 0.762 \text{ m} - 1.763 \text{ m} * 0.759 \text{ m} = 0.005 \text{ m}^2.$$

Statistics tells us to combine the uncertainties to find that the net uncertainty in area is given by the following: $\Delta A = \sqrt{\Delta A_L^2 + \Delta A_W^2}$.

Thus in our example, $\Delta A = \sqrt{0.003^2 + 0.005^2} = 0.006 \text{ m}^2$,

so that the determined area with correct uncertainty is $A = 1.338 \pm 0.006 \text{ m}^2$.

In general, for a function $f(G, H)$ of two measured variables with means G and H , and uncertainties ΔG and ΔH , the error in f due to G is:

$$\Delta f_G = f(G + \Delta G, H) - f(G, H)$$

[Equation 5]

and the error in f due to H is:

$$\Delta f_H = f(G, H + \Delta H) - f(G, H)$$

[Equation 6]

The error in f is then:

$$\Delta f = \sqrt{\Delta f_G^2 + \Delta f_H^2}$$

[Equation 7]

The function can be of one, two, three or more variables and can involve any arithmetic operation.

Measurements of a Dependent Variable

When measuring, for example, the constant acceleration of an object, since $v = v_0 + at$, you could choose to start the object from rest ($v_0 = 0$) and then measure the velocity, v , after the same amount of time, each time, and average all your values of v / t to get a value of a . This method may lack accuracy and precision versus measuring the velocity as a function of time for varying times.

When taking measurements, you can often reduce error by varying an independent variable and measuring a dependent variable, instead of measuring the same thing over and over again. By varying the time over which you take data, you are exploring a range of values, and a more complete graphical analysis can reduce the random and systematic errors in the experiment.

The Confidence Interval

As stated earlier, the standard error, SE , is a measure of precision. Again, statistical theory tells us what the standard error means, and it can be used to express confidence in your estimate of the mean as follows:

If you add or subtract the standard error to the mean, you will get a possible range of values. This is usually written as the interval $M \pm SE$. According to statistics, if you are only making a very small number of measurements, you should expect the likelihood that the true mean (the “correct” value) is in that interval is 60%. The interval determined by $M \pm SE$ is called a “60% confidence interval”.

For your purposes, 60% confidence might not be good enough. So you can choose to make the interval bigger, making it more likely that the correct value is contained. If you add or subtract two standard errors to the mean, $M \pm 2 * SE$, for small numbers of measurements, you should expect the “correct” value to be in that interval 90% of the time, or with a confidence of 90%. Obviously, this can be taken to extremes, and you can keep adding more SE 's to your range of precision. You will be more confident that you have the correct answer in your range, but your answer with uncertainty will get less precise. You need to balance needs of precision with needs of confidence, and that is typically choosing to report results to 60 or 90 percent confidence.

Your answer is accurate if its confidence interval encompasses the expected value. For example, if the expected value is 9.8, a measured value of 9.9 ± 0.2 is accurate. Of course, you cannot know if it is accurate if you do not have an expected answer. You would assume accuracy, but still only to the confidence level that you are measuring, if you believe you have accounted for all systematic errors.

For some measurements, there are expected values for what you are trying to measure. Assume you are using 90% confidence: if you do have an expected value, and if your 90% confidence interval for the measured quantity does not agree with the expected value, then you should investigate the accuracy and the systematic errors that may have been present in your experiment. If there are no systematic errors, it is possible for the answer to be inaccurate. Statistics state that 10% of correctly performed measurements will give an incorrect estimate of the mean within the 90% confidence interval.

Comparing Results to Each Other or to an Expected Value

Percent Difference and Percent Error

If two lab groups measure two different values for an experimental quantity, they may be interested in how their results compare to each other. This is often expressed as a **percent difference**, defined as the absolute value of the difference divided by the mean times 100:

$$\text{Percent difference} = \left| \frac{\text{value 1} - \text{value 2}}{\frac{1}{2}(\text{value 1} + \text{value 2})} \right| \times 100$$

[Equation 8]

When you have an expected or theoretical value that you want to compare to a measured value, this is often expressed as a **percent error**, defined as the absolute value of the difference divided by the expected value:

$$\text{Percent error} = \left| \frac{\text{measured value} - \text{expected value}}{\text{expected value}} \right| \times 100$$

[Equation 9]

Note that when the expected value is very small, approaching zero, the percent error gets very large; and the percent error is undefined when the expected value is zero. Percent error may not be a very useful quantity in these cases.

The Null Hypothesis

Using what you know about combining uncertainties, you can compare a measured value with an accepted value, or two measured values with each other, to determine whether they are similar enough to be considered equal.

Suppose you measured the speed of a cart at an instant as 33.4 ± 0.4 cm / s. Suppose also that someone else measured it and told you that the speed was 34.0 ± 0.3 cm / s. Can you say that there is a significant difference between these two speeds? Another way to phrase this question would be: Is the difference between these two figures due to random effects only?

At first glance you might suspect that the two figures do not differ significantly, since the difference between them is 0.6 cm/s and this difference is less than the sum of the confidence intervals, which is 0.7 cm/s. This simple conclusion is incorrect because of statistics; the odds don't favor your being too low on one measurement while simultaneously being too high on the other. As seen when combining errors above, statistical theory tells us that the difference between the two figures, if due to random effects only, can be expected to be no larger than the square root of the sum of the squares of the individual uncertainties:

$$\sqrt{(0.3 \text{ cm/s})^2 + (0.4 \text{ cm/s})^2} = 0.5 \text{ cm/s}.$$

This is an example of using statistical methods to test a **null hypothesis**. In the present case, your hypothesis might be stated in the form of a question: is the difference between the two values equal to zero (is $v_1 - v_2 = 0$) ?

When you subtract these two numbers, the presence of random error makes it unlikely that you would get exactly zero, but statistically, they might still be measuring the same value. Since both numbers have an uncertainty associated with them, in order to answer this question you must first determine whether the number 0 is contained in the appropriate interval when you subtract the two quantities. In this example $34.0 \pm 0.3 \text{ cm/s} - 33.4 \pm 0.4 \text{ cm/s} = 0.6 \pm 0.5 \text{ cm/s}$ does not include 0. In the present case, since the two values for the speed differ by more than can be accounted for due to random effects only, the conclusion that the two speeds are the same is probably false.

Suppose that your two numbers for the speeds agreed with one another within the random uncertainties or even exactly. This would not “prove” the hypothesis that the speeds are equal, as the agreement might be a statistical anomaly. Even if experiments agree with one another, it is possible that continued improvement in the precision of the experiment might ultimately lead to a detection of a statistically significant difference.

This illustrates an important philosophical principle concerning experimental results. Although you can be reasonably confident that a given hypothesis is false (i.e., two measurements or measurement and theory differ by more than you can account for due to random effects only), you can never prove with equal assurance that it is true that two values are equal.

Chapter 3: Graphs

Graphs are often an excellent way to present or to analyze data. When making graphs, there are a few guidelines you should follow to make them as clear and understandable as possible:

- ▶ Graphs should be labeled completely and appropriately
 - › Each axis should be labeled with the quantity plotted including units
 - › Each axis should include a reasonable number of tick marks at even intervals, and include a scale
 - › Typically, graphs should be labeled with a meaningful title or caption
 - › If a legend is needed, the legend should be meaningful (e.g. Excel automatically includes a legend that often does not add any information unless the legend is edited)
- ▶ A typical problem with graphs is fitting them into a space that makes it difficult to see the trends in the data, so the graph should be of such a design that trends can be observed in the data
 - › Generally, you want to fill more than one-half of space vertically and horizontally (which means that the scale will not always start at zero). A particular exception to this is a set of data that indicates a horizontal line, or a line with zero slope. Because you expect random error in real data, if you make the y-axis scale such that the data fills the page vertically, it will not look like a horizontal line. In the case where the slope is near zero, analysis of the data will be improved by looking at the data with a significantly larger scale to determine if it actually looks like a horizontal line.
 - › The graph should be big enough to see, typically at least one-eighth of a page.

Linearizing Data

Most often when you take data in an experiment, you intend to do one of three things:

1. You want to verify that the data has a certain relationship. For example, you want to determine if a body is moving with a constant acceleration, so you plot position versus time data for the body to see if it is a parabola.
2. You expect that your data has a certain relationship and want to determine some parameter. For example, you have position and time data for constant-acceleration motion of a body, and you want to determine the acceleration.
3. You are trying to find an unknown relationship between two variables. For example, you take position versus time data for a body, but you do not know what kind of motion it is, and you may ask, is position related to time with a linear, square, exponential, power law, or some other relationship?

The Equation of a Straight Line

When you make a plot on x - y axes, a straight line (line with constant slope) is the simplest relationship that data can have. Representing a straight line with a function on x - y axes only requires two arbitrary parameters, m and b , such that

$$y = m * x + b$$

[Equation 10]

Because there are only two parameters in the linear function, it is the easiest function to use as a model, the meaning of the parameters are most clear (slope and y -intercept), and the parameters can always be worked out with a best fit line with very little manipulation compared to a higher-order function. Be cautious, though, to only model something with a linear function if indeed it is linear.

The slope is a measure of how the y variable changes with changes in x , $m = \Delta y / \Delta x$. Be careful when estimating slopes from best-fit lines: the slope should be determined from the best-fit line, *not* by taking two of the data points and subtracting their y and x values.

The y -intercept is where the line crosses the y -axis (where $x = 0$). It is often interpreted as the initial value of the function, assuming the function starts when $x = 0$.

Linearizing data so that you can do a straight-line fit is an important data analysis technique. Even if the data you take do not have a linear relationship, if you have a model for it, you can often figure out what to do to linearize it.

For example, in physics we often start with the quadratic relationship between position in one dimension, y , and time, t , for constant acceleration: $y = \frac{1}{2}at^2$. When y vs. t is graphed, it yields a parabola. If instead you set the x -axis variable equal to t^2 , so y is graphed vs. t^2 , the model is $y = \frac{1}{2}ax$, and should yield a straight line with slope $m = \frac{1}{2}a$.

A more general technique of linearizing data is to do a “log-log” plot of data. If the data is exponential, $y = Ae^{bx}$ or a power law, $y = ax^n$, taking the log of both sides of the relationships will linearize them. So if you take the log (typically base 10 or natural log) of both of your x and y data sets, you can determine the unknown parameters:

Exponential: $\ln(y) = \ln(Ae^{bx}) = \ln(A) + bx$. If you plot $\ln(y)$ vs. x , the data will approximate a line with y -intercept $\ln(A)$ and slope b .

Power Law: $\log(y) = \log(ax^n) = \log(a) + n * \log(x)$. If you plot $\log(y)$ vs. $\log(x)$, the data will approximate a line with y -intercept $\log(a)$ and slope n .

Note that not all functions can be linearized. If you simply take the first example of the quadratic time dependence of position with time, and add an unknown initial speed, $y = \frac{1}{2}at^2 + v_0t$, there are now too many unknown parameters multiplied by different combinations of variables, and the equation cannot be linearized without knowing one parameter ahead of time, such as the initial speed.

Graphical Analysis of Linear Data

Now that you have linearized your data, a straight-line fit to the data will yield parameter values and allow you to determine unknown quantities. For example, in the first example in the previous section, a determination of the slope allows you to determine the acceleration of the object.

The Least Squares Fit of a Straight Line

As discussed above, in an experiment there is often a linear relationship between the measured variables, or you can linearize them. For example, the velocity of an object in free fall changes linearly with time, in the absence of air resistance. After you plot a set of data and find that it approximates a straight line, the next question is how to find the slope and the intercept of the line that seems to provide the best fit. If you were to actually measure speed versus time for a falling object that is not impacted significantly by any forces other than gravity, the data would be scattered around a straight line.

You should not expect the data to look exactly like a straight line because you know that the presence of random error causes some scatter away from the ideal line. You can find approximate values for slope and intercept by using a straight edge to draw a line that appears to split the difference between the scattered points; the line should have as many points above it as below. A more exact answer is given by a statistical analysis, which is described here. You will often be able to use a computer or graphing calculator to do this analysis.

The process of finding the best-fit line proceeds as follows: Suppose you do an experiment where you have your two variables y and x that depend on each other (e.g. measuring speed of a freely falling body as a function of time), or perhaps some more complicated combination of variables as described in linearizing data, above. If there were no random errors present, all of your experimental results would fall exactly on a line given by Equation 10: $y = m * x + b$.

Equation 9 is your model for the data. If you expect your data to have this relationship, then you must have a theoretical equation that takes this form, where y and x are variables and m and b are undetermined constants. Note, though, that if your data does not look linear, perhaps it is not!

As an example, take the expected physical relationship for the velocity versus time for an object undergoing constant acceleration:

$$v = v_0 + at$$

[Equation 11]

If you are measuring velocity at varying times, the relationship between v and t should be a linear one: this equation looks a lot like the equation for the straight line above. If you have N pairs of data points, and your data consists of time values, t , on the x -axis, with each data point denoted as x_i ; and velocity values, v , on the y -axis, with each value denoted as y_i , then you would expect it to look like a straight line. You need to determine the best-fit line from these values in order to determine the initial speed v_0 (the y -intercept) and acceleration, a (the slope).

Assuming only random errors are present, based on the mathematics of the normal distribution discussed earlier, in order to maximize the probability that you have the correct fitted line, you have to minimize the sum of the squares of the deviations of the measured points from the fitted line. For data points that each have approximately equal absolute random error (equally *weighted*), this minimum value occurs when the slope, m , and y -intercept, b , are given by:

$$m = \frac{N \sum (x_i y_i) - \sum x_i \sum y_i}{N \sum (x_i^2) - (\sum x_i)^2}$$

[Equation 12]

$$b = \frac{\sum y_i - m \sum x_i}{N}$$

[Equation 13]

The uncertainties in these values, Δm and Δb , are given by:

$$\Delta m = \sqrt{\frac{N \sum (y_i - mx_i - b)^2}{(N - 2)[N \sum (x_i^2) - (\sum x_i)^2]}}$$

[Equation 14]

$$\Delta b = \sqrt{\frac{\sum (x_i^2) \sum (y_i - mx_i - b)^2}{(N - 2)[N \sum (x_i^2) - (\sum x_i)^2]}}$$

[Equation 15]

The procedure can be carried out systematically by a computer or graphing calculator using regression or best-fit line analysis. In order to use a computer program intelligently, keep in mind the following points:

1. You must enter at least three data points or the least squares procedure will not work.
2. You need at least five data points to determine approximate uncertainties for the slope and y -intercept.

3. No matter how wildly scattered your data may be, or even if the variables are not linearly related, the computer can always come up with a slope and an intercept that is a best-fit in the least squares sense. It's a good idea to make at least a rough plot of your data to be sure that your chosen method of plotting does yield something close to a straight line. An example you may want to show your students is the set of data and graphs known as the Anscombe quartet of graphs which all have the same linear "best fit" line, but have plots that are wildly different. See https://en.wikipedia.org/wiki/Anscombe%27s_quartet.

You can use a computer to make calculations of the above equations, or use built-in functions to determine slope, y-intercept, and uncertainty in each. As well as determining the unknown parameters and their errors, you should also be sure to plot the data and the best fit line to be sure it looks as expected.

Weighted Least Squares

Instead of minimizing the sum of the squares of the differences between each data point and the fitted line, you could minimize the weighted sum of the squares. This can be done when data quality varies and you know some data with more confidence than others.

Each data point should be weighted by the inverse square of its standard error. If each data point y_i has individual standard errors, e_i , then in order to minimize the weighted sum of the squares of the distance from each data point to the fitted line, the slope and y-intercept become:

$$m = \frac{\sum \frac{1}{e_i^2} \sum \frac{x_i y_i}{e_i^2} - \sum \frac{x_i}{e_i^2} \sum \frac{y_i}{e_i^2}}{\sum \frac{1}{e_i^2} \sum \frac{x_i^2}{e_i^2} - \left(\sum \frac{x_i}{e_i^2} \right)^2}$$

[Equation 16]

$$b = \frac{\sum \frac{x_i y_i}{e_i^2} - m \sum \frac{x_i^2}{e_i^2}}{\sum \frac{x_i}{e_i^2}}$$

[Equation 17]

While these expressions might look daunting, they only involve a few different sums, and can be implemented in a spreadsheet without too much difficulty.

Resources

“AP Physics 1 and 2 Lab Investigations: Student Guide to Data Analysis.” AP Physics 1 and AP Physics 2 Essential Course Resources. College Board. Accessed February 10, 2015. <https://media.collegeboard.com/digitalServices/pdf/ap/physics-1-2-data-analysis-student-guide.pdf>

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“Useful Excel Commands for Lab.” Department of Physics. Randolph College. Accessed January 6, 2015. <http://physics.randolphcollege.edu/lab/IntroLab/Reference/exchint.html>.