AP® Calculus BC

Course Planning and Pacing Guide

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Welcome to the AP Calculus BC Course Planning and Pacing Guides

This guide is one of several course planning and pacing guides designed for AP Calculus BC teachers. Each provides an example of how to design instruction for the AP course based on the author's teaching context (e.g., demographics, schedule, school type, setting). These course planning and pacing guides highlight how the components of the *AP Calculus AB and BC Curriculum Framework*, which uses an Understanding by Design approach, are addressed in the course. Each guide also provides valuable suggestions for teaching the course, including the selection of resources, instructional activities, and assessments. The authors have offered insight into the *why* and *how* behind their instructional choices — displayed along the right side of the individual unit plans — to aid in course planning for AP Calculus teachers.

The primary purpose of these comprehensive guides is to model approaches for planning and pacing curriculum throughout the school year. However, they can also help with syllabus development when used in conjunction with the resources created to support the AP Course Audit: the Syllabus Development Guide and the four Annotated Sample Syllabi. These resources include samples of evidence and illustrate a variety of strategies for meeting curricular requirements.

Contents

Instructional Setting

Scarsdale High School ▸ Scarsdale, NY

School Scarsdale High School is a traditional, college preparatory four-year high school located in Westchester County, a suburban county of New York City.

Student population The total enrollment is 1,443 students (722 male, 721 female) with the following racial/ethnic demographics: 76 percent Caucasian, 12 percent Asian, 7 percent Hispanic, 2 percent African American, and 3 percent multiracial. No students are eligible for free or reduced-price lunch. Nearly all of the students go to college.

Instructional time The school year starts right after Labor Day. There are 180 instructional days. There are seven periods in the school day, each lasting 49 minutes. Typically, a course meets four periods per week. However, Calculus BC students have class six periods per week.

Approximately 35 percent of our seniors take either Calculus AB or Calculus BC. Virtually all our students take four years of math. We usually have five to 10 students who are double-accelerated who will take Multivariable Calculus as seniors.

Student preparation In our school, Calculus BC is offered to students, typically seniors, who have completed our high honors or low honors precalculus course. Consequently, students at Scarsdale High School entering Calculus BC have a good background in intermediate algebra and trigonometry. Students coming into Calculus BC have a thorough knowledge of the function concept, parametric equations, polar curves, and have had a precalculus unit on sequences and series.

Primary planning resources

Foerster, Paul. *Calculus Concepts and Applications.* Emeryville, CA: Key Curriculum Press, 2005.

Stewart, James. *Single Variable Calculus.* 7th ed. Belmont, CA: Brooks/Cole, Cengage Learning, 2012.

Overview of the Course

This AP Calculus BC course is designed to be the college equivalent of two semesters of introductory calculus. The course is packed with content, so it is important that students come in with a strong precalculus background. In our school, when students enter the course, they already have a solid working knowledge of functions and have had exposure to sequences and series, parametric equations, and polar coordinates.

Although the course is filled with many topics, students are well served if a good portion of the lessons involve group work, helping one another to develop ideas and reinforce concepts. Group work permits students to learn from one another and encourages them to take responsibility for their own learning. The primary role of the teacher is to provide the structure in which this exchange among students can take place effectively, and to facilitate, guide, and summarize. As packed as the course may be, high school Calculus BC courses still have a far greater number of hours to teach the same topics than the equivalent college courses.

Throughout the course, a balance needs to be maintained between traditional and reform (post-Harvard consortium) calculus. There must also be a balance between technology and the grind-it-out mechanical differentiation and integration skills. Many of the assessments and exercises stipulate that calculators are not permitted, ensuring that students have the mechanical differentiation and integration skills and processes. Other exercises and assessments allowing calculators have more robust problems requiring deeper mastery of the underlying concepts and high-level problem-solving skills.

Spiraling of topics is a key element of this course. Many of the activities in this guide are identified as formative assessments because they require students to use skills from earlier lessons. Circulating the room while students work in groups during formative assessments allows me to gauge how well students have learned and retained calculus skills from previous lessons as they apply them to the concepts of the current unit. Toward the end of these formative assessments, I collect the students' work and scan it into the classroom sheet-fed scanner. I look at the students' scanned-in work the same evening that they complete their exercise sets and correct their own work to guide my instruction for subsequent lessons. Here I get to see where there are trouble spots or places where the class is ready to move on.

Many of the formative assessments are conducive to differentiated instruction in that they have two sets of exercises: one basic and one more challenging. Collecting and scanning student work allows me to return the papers so that they can continue working on those problems using their notes and the texts, yet this affords me the opportunity to see who may need a "call back" for some one-on-one conferencing. Some of the exercise sets in the homework assignments must also allow for scaffolding; some are intended as basic skill building and practice, and others lead up to thought-provoking exercises. The summative assessments are also designed in a tiered fashion so that everyone who has made it this far into a math program and puts in the effort can find some level of success, yet there may still be a challenge component for those who are in a better position to excel.

Of course, I want my students to do well on the AP Calculus BC Exam, but that takes care of itself if my primary focus is met: to keep the students engaged in every lesson, and to provide opportunities for them to see the themes and connections among the topics throughout the course. In AP Calculus BC, such opportunities are abundant.

Mathematical Thinking Practices for AP Calculus (MPACs)

The Mathematical Practices for AP Calculus (MPACs) capture important aspects of the work that mathematicians engage in, at the level of competence expected of AP Calculus students. They are drawn from the rich work in the National Council of Teachers of Mathematics (NCTM) Process Standards and the Association of American Colleges and Universities (AAC&U) Quantitative Literacy VALUE Rubric. Embedding these practices in the study of calculus enables students to establish mathematical lines of reasoning and use them to apply mathematical concepts and tools to solve problems. The Mathematical Practices for AP Calculus are not intended to be viewed as discrete items that can be checked off a list; rather, they are highly interrelated tools that should be utilized frequently and in diverse contexts.

MPAC 1: Reasoning with definitions and theorems

Students can:

a. use definitions and theorems to build arguments, to justify conclusions or answers, and to prove results;

b. confirm that hypotheses have been satisfied in order to apply the conclusion of a theorem;

c. apply definitions and theorems in the process of solving a problem;

d. interpret quantifiers in definitions and theorems (e.g., "for all," "there exists");

e. develop conjectures based on exploration with technology; and

f. produce examples and counterexamples to clarify understanding of definitions, to investigate whether converses of theorems are true or false, or to test conjectures.

MPAC 2: Connecting concepts

Students can:

a. relate the concept of a limit to all aspects of calculus;

b. use the connection between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, antidifferentiation) to solve problems;

c. connect concepts to their visual representations with and without technology; and

d. identify a common underlying structure in problems involving different contextual situations.

MPAC 3: Implementing algebraic/computational processes

Students can:

a. select appropriate mathematical strategies;

b. sequence algebraic/computational procedures logically;

c. complete algebraic/computational processes correctly;

d. apply technology strategically to solve problems;

e. attend to precision graphically, numerically, analytically, and verbally and specify units of measure; and

f. connect the results of algebraic/computational processes to the question asked.

Mathematical Thinking Practices for AP Calculus (MPACs)

MPAC 4: Connecting multiple representations

Students can:

a. associate tables, graphs, and symbolic representations of functions;

b. develop concepts using graphical, symbolical, or numerical representations with and without technology;

c. identify how mathematical characteristics of functions are related in different representations;

d. extract and interpret mathematical content from any presentation of a function (e.g., utilize information from a table of values);

e. construct one representational form from another (e.g., a table from a graph or a graph from given information); and

f. consider multiple representations of a function to select or construct a useful representation for solving a problem.

MPAC 5: Building notational fluency

Students can:

a. know and use a variety of notations (e.g., $f'(x)$, y', $\frac{dy}{dx}$);

b. connect notation to definitions (e.g., relating the notation for the definite integral to that of the limit of a Riemann sum);

c. connect notation to different representations (graphical, numerical, analytical, and verbal); and

d. assign meaning to notation, accurately interpreting the notation in a given problem and across different contexts.

MPAC 6: Communicating

Students can:

a. clearly present methods, reasoning, justifications, and conclusions;

b. use accurate and precise language and notation;

c. explain the meaning of expressions, notation, and results in terms of a context (including units);

d. explain the connections among concepts;

e. critically interpret and accurately report information provided by technology; and

f. analyze, evaluate, and compare the reasoning of others.

Pacing Overview

Unit	Hours of Instruction	Unit Summary
1: Limits and Continuity	10	In this unit we analyze rational functions that have deleted points in order to introduce the concept of the limit of a function, including limits involving infinity. We then consider limits involving trigonometric functions. By the end of the unit, students should come to realize that limits could fail to exist for different reasons. The definition of continuity is established in this unit as well.
2: Derivative Definition and Derivative Rules	12	The tangent line problem is used to motivate the definition of derivative. In this unit, we establish the power rule, as well as the sum, product, quotient, and chain rules for derivatives, plus the Inverse Function Theorem. We also establish the results for the derivatives of the trigonometric functions and their inverses. The process of implicit differentiation is likewise introduced.
3: Derivative Interpretations and Applications	13	In this unit we deal with derivative applications, such as linear approximations, rectilinear motion, and related rates. We connect average and instantaneous rates of change with the Mean Value Theorem and use it to develop curve-sketching techniques as we analyze function behavior. We close with optimization problems and discuss how to give a valid argument for guaranteeing minimum and maximum values of a function.
4: Integrals and the Fundamental Theorem of Calculus	14	In this unit we introduce the three types of integrals: definite integrals, indefinite integrals, and functions defined by an integral. We establish the notion of the average value of a function and prove the Fundamental Theorem of Calculus.
5: Techniques of Integration	12	Now that students have some theory of integration and some reasons to find antiderivatives, they are primed to develop some techniques. In this unit, we cover *u*-substitutions and integration by parts. Calculus of the log and exponential functions are introduced, providing an opportunity to spiral back to all the applications from Unit 3.
6: Applications of Integration	14	This unit covers applications to integration such as areas between curves, volumes of solids with known cross-sections, and arc length. This unit also provides an opportunity for spiraling of topics as particle motion is revisited in an integration context. Additionally, we cover L'Hospital's Rule and then use it as we develop improper integrals.
7: Parametric Equations and Polar Curves	13	In this unit, we begin by working with parametrically defined curves. That lends itself well to introducing curvilinear motion, affording an opportunity to extend all the previously learned particle motion ideas to two dimensions. We also cover the calculus of polar coordinate curves, including area and tangent line problems.

Pacing Overview (continued)

Unit	Hours of Instruction	Unit Summary
8: Differential Equations	13	In this unit students learn how to solve separable differential equations. Once again, we are able to spiral all the techniques of integration learned in previous units. Students learn how to set up differential equations (DEs) in contextual problems. When the DEs are not separable, students learn that they can still predict the behavior of such DEs with their slope fields and determine approximations for particular solutions using Euler's method. Population growth, including logistic growth, is studied in this unit.
9: Series	25	This unit begins with infinite series of constants as we develop numerous tests to determine whether or not such a given series converges. We introduce the concept of a power series very early in the unit. We establish the use of the ratio test to determine the interval of convergence of a power series, and the endpoint analysis reinforces the tests for convergence of infinite series of constants. We likewise introduce Maclaurin and Taylor expansions, and we use Taylor polynomials to approximate functions at points near their centers. We also study error analysis, establishing methods to guarantee certain levels of accuracy when only finitely many terms of an infinite series are used to approximate its sum.

UNIT 1: LIMITS AND CONTINUITY

BIG IDEA 1
Limits

Enduring Understandings:
- EU 1.1, EU 1.2

Estimated Time:
10 instructional hours

Guiding Questions:

▸ Even though a function may be undefined at a point, how can we describe the behavior of the function near that point? ▸ How can asymptotic behavior of a function be expressed mathematically? ▸ What are the different types of limits and what are the graphical manifestations of each type of limit? ▸ Why might a limit fail to exist? ▸ What is continuity, and what are the different types of discontinuities?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 1.1A(a): Express limits symbolically using correct notation. **LO 1.1B:** Estimate limits of functions. **LO 1.1C:** Determine limits of functions.	**Print** Foerster, section 2-2 Stewart, section 1.5	**Instructional Activity: Introduction to Limits** In this small-group activity, students first discuss the domain of the function $f(x)=\frac{x^2-9}{x-3}$. They complete a table of values and use those values to sketch a graph of the function *f*. Then I ask them to find another function, *g*, which is identical to *f* over the domain of *f*, but for which $g(3)$ is defined. I introduce the notation $\lim_{x\to a} f(x)=L$. Students use the graph to associate an image in their minds to correspond to the limit statement.	*I interrupt the groups frequently and there is teacher-directed Q&A, showing many "what ifs" on the board. Part of the activity involves providing graphs of functions for which no equation is given, forcing students to approximate limits of functions based on their graphs.*
LO 1.1A(a): Express limits symbolically using correct notation. **L.O 1.1A(b):** Interpret limits expressed symbolically. **LO 1.1C:** Determine limits of functions.	**Print** Stewart, section 1.5	**Instructional Activity: Introduction to Limits Involving Infinity** Working again in small groups, I have students discuss the domain of the function $f(x)=\frac{3x^2-13x+10}{x^2-5x+4}$. They complete a table of values and use those values to sketch a graph of the function *f*. As in the previous activity, students find another function, *g*, which is identical to *f* over the domain of *f*, but for which $g(1)$ is defined. I use the modified function *g* to reinforce the limit notation and concept of the previous activity, but then I use the graph to motivate the notation and concepts for limit statements of the type $\lim_{x\to\infty} f(x)=L$, $\lim_{x\to-\infty} f(x)=L$ as well as $\lim_{x\to a+} f(x)=\infty$, $\lim_{x\to a-} f(x)=-\infty$. I interrupt the groups for a guided class discussion and notational summary. As before, we make connections between the graph and the limit statements.	*These first two activities provide the students with limit notation. This sets up future lessons in which the students do more investigation. For these future investigations, students have the tools of limit notation to communicate their ideas. I usually consider additional, similar activities to reinforce instruction.*

UNIT 1: LIMITS AND CONTINUITY

BIG IDEA 1 Limits	Enduring Understandings:	Estimated Time:
	▸ EU 1.1, EU 1.2	10 instructional hours

Guiding Questions:

▸ Even though a function may be undefined at a point, how can we describe the behavior of the function near that point? ▸ How can asymptotic behavior of a function be expressed mathematically? ▸ What are the different types of limits and what are the graphical manifestations of each type of limit? ▸ Why might a limit fail to exist? ▸ What is continuity, and what are the different types of discontinuities?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 1.1A(a): Express limits symbolically using correct notation. **L.O 1.1A(b):** Interpret limits expressed symbolically. **LO 1.1B:** Estimate limits of functions. **LO 1.1C:** Determine limits of functions.		**Instructional Activity: Graphing Rational Functions Using Analysis of the Function's Limit Behavior** First I have students work in groups to analyze the limit behavior of $f(x)=\frac{x^3-x^2-2x}{x^3-2x^2+x}$. I circulate and guide the groups to use their limit analysis to help them sketch the graph. This analysis allows students an additional opportunity to connect the limit behavior referenced in the previous instructional activities with the graphs of the functions with respect to vertical and horizontal asymptotes, as well as (removable) points of discontinuity, which we reference as deleted points until the definition of continuity is introduced a little later on. I have one of the groups present their analysis on the board as we summarize as a class the connection between the limit statement and their graphs.
LO 1.1C: Determine limits of functions. **LO 1.1D:** Deduce and interpret behavior of functions using limits.	**Print** Stewart, section 1.6	**Formative Assessment: Evaluating Limits Without Having to Graph** Students work independently on evaluating limits involving algebraic manipulation, such as $\lim\limits_{x\to 1}\frac{2x^5+x-3}{x^2-1}$, $\lim\limits_{x\to 2}\frac{\sqrt{x^2+5}-3}{x-2}$, and $\lim\limits_{x\to\infty}\frac{2x^2+x-3}{x^2-4x+3}$, as well as limits for which $f(x)$ approaches ∞ or $-\infty$. They do so with notebooks open for reference, and without the burden of having to sketch. Students write their answers to each problem as a numerical limit value, ∞ or $-\infty$. I defer limits such as $\lim\limits_{x\to 0}\frac{1}{x}$ until later in the unit when we might be forced to write "DNE" (does not exist). There are two sets of problems: one basic and one more challenging. All students are expected to complete the basic problem set, but one purpose of the more challenging set is to provide differentiated instruction.

While students are working, I circulate and provide feedback. Toward the end, I collect the sheets and scan them in the classroom sheet-fed scanner for examination later that day. I immediately return the worksheets to the students and provide a solution key as the students exit. I use the scans to determine areas to work on for individual students. Students must correct their worksheets with the key and prepare requests for the next day.

UNIT 1: LIMITS AND CONTINUITY

BIG IDEA 1
Limits

Enduring Understandings:
- EU 1.1, EU 1.2

Estimated Time:
10 instructional hours

Guiding Questions:

▸ Even though a function may be undefined at a point, how can we describe the behavior of the function near that point? ▸ How can asymptotic behavior of a function be expressed mathematically? ▸ What are the different types of limits and what are the graphical manifestations of each type of limit? ▸ Why might a limit fail to exist? ▸ What is continuity, and what are the different types of discontinuities?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 1.1A(a): Express limits symbolically using correct notation. **L.O 1.1A(b):** Interpret limits expressed symbolically.	**Print** Stewart, section 1.7	**Instructional Activity: The Official Definition of *Limit*** In groups, students engage in a worksheet activity to try to devise a rigorous definition for statements of the type $\lim_{x \to a} f(x) = L$. Then they challenge each other to see if their definition stands up to various limit statements on the worksheet. I interrupt the activity for a guided class discussion leading to the epsilon-delta limit definition. The worksheet activity resumes in which students are tested on the definition in the context of different functions. We use graphing calculators to explore whether or not specific δ values are suitable for a given $\in$ challenge.
LO 1.1A(a): Express limits symbolically using correct notation. **LO 1.1D:** Deduce and interpret behavior of functions using limits.		**Instructional Activity: Limits That Don't Exist** In this activity I have students work in groups on limits that fail to exist for reasons other than the fact that they approach positive or negative infinity. They use their graphing calculators to help them examine the behavior of functions that oscillate infinitely often such as $f(x) = \sin\left(\frac{1}{x}\right)$ and piecewise-defined functions such as $f(x) = \begin{cases} x^2, & x > 2 \\ x+5, & x \le 2 \end{cases}$, which approach different values from each side. Then as a whole class, we discuss examples for which the absolute value of a function increases without bound, such as $\lim_{x \to 0} \frac{1}{x}$ versus $\lim_{x \to 0} \frac{1}{x^2}$. We discuss that even though neither limit exists, it is still appropriate to write $\lim_{x \to 0} \frac{1}{x^2} = \infty$, while in the case of $\lim_{x \to 0} \frac{1}{x}$, we may only write "does not exist."

Even though the $\in - \delta$ definition is not mandated by the Curriculum Framework, teachers who have the luxury of time will see a huge benefit in devoting some class time to this definition to overall student understanding of future units. The definition can be referenced as subtle questions arise later on since the derivative is itself a limit.

This is the perfect opportunity to provide clarification to students that for limits corresponding to vertical asymptotes, it may be appropriate to write $\lim_{x \to a} f(x) = \infty$ to indicate that the function increases without bound even though the limit does not exist.

UNIT 1: LIMITS AND CONTINUITY

BIG IDEA 1
Limits

Enduring Understandings:
- EU 1.1, EU 1.2

Estimated Time:
10 instructional hours

Guiding Questions:

▸ Even though a function may be undefined at a point, how can we describe the behavior of the function near that point? ▸ How can asymptotic behavior of a function be expressed mathematically? ▸ What are the different types of limits and what are the graphical manifestations of each type of limit? ▸ Why might a limit fail to exist? ▸ What is continuity, and what are the different types of discontinuities?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 1.2A: Analyze functions for intervals of continuity or points of discontinuity. **LO 1.2B:** Determine the applicability of important calculus theorems using continuity.	**Print** Foerster, section 2-4 Stewart, section 1.8	**Instructional Activity: The Definition of *Continuity*** Students collaborate in groups on a worksheet that directs them to explore various functions such as $f(x)=\begin{cases}\dfrac{3x^2-13x+10}{x^2-5x+4}, & x\neq 1,\ 4\\ 6, & x=1\\ 8, & x=4\end{cases}$, using their graphing calculators. As they see functions on their screen and on their hand-drawn sketches, which are discontinuous for different reasons, they are challenged to develop an official definition for a function to be continuous at a point $x=a$, as well as over an entire interval. I provoke students to produce a function defined on a closed interval that fails the conclusion of the Intermediate Value Theorem. In so doing, this activity encourages students to appreciate the requirement of continuity in order to be able to invoke the Intermediate Value Theorem.
All of the learning objectives in this unit are addressed.		**Summative Assessment: Limits and Continuity** Students are given 10 fairly short free-response questions directly targeting single learning objectives, and two longer free-response questions that require assimilating the concepts learned in the unit. This is followed by six multiple-choice questions.

This summative assessment addresses all of the guiding questions for the unit.

Mathematical Practices for AP Calculus in Unit 1

The following activities and techniques in Unit 1 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: The instructional activity "The Official Definition of *Limit*" requires students to reason with the definition of a limit in order to argue why certain limits exist and what their values are. In "The Definition of *Continuity*" students give verbal arguments to each other why certain functions are continuous or fail to be continuous based on the definition of continuity and critique each other's arguments. At least two or three items in the summative assessment require students to construct and articulate arguments using these definitions.

MPAC 2 — Connecting concepts: Many of the instructional activities in this unit, such as "Graphing Rational Functions Using Analysis of the Function's Limit Behavior" and "The Definition of *Continuity,*" are organized around connecting the concepts of graphs of functions and the corresponding limit statements. There are a number of problems in the assignments that have students graph functions that satisfy certain limit constraints.

MPAC 3 — Implementing algebraic/computational processes: In the "Evaluating Limits Without Having to Graph" instructional activity students must decide on algebraic techniques and manipulations that will produce functions that are nearly identical to the functions in their limit evaluation task, but for which the value of the limit is apparent. In many of the activities, the graphing calculator is used to explore what the limit might be, and the reasonableness of their conclusions.

MPAC 4 — Connecting multiple representations: In the "Introduction to Limits" instructional activity students are challenged to predict limit values based on tables provided by their graphing calculators. Many of the instructional activities throughout the unit, such as "Graphing Rational Functions Using Analysis of the Function's Limit Behavior" and "Introduction to Limits" challenge students to explore the graphical manifestations of limit statements. In several problem sets, students must graph a function that satisfies a collection of limit conditions.

MPAC 5 — Building notational fluency: Throughout the unit, notation involving left and right limits is used to communicate characteristics of functions and their graphs. In the instructional activity "The Official Definition of *Limit*" students must use quantifier notation to communicate the limit definition.

MPAC 6 — Communicating: In "The Definition of *Continuity*" students must communicate with each other in verbal and in written form the justification for why certain functions are continuous/discontinuous and whether or not their discontinuities are removable. In the "Graphing of Rational Functions Using Analysis of the Function's Limit Behavior" instructional activity students must communicate accurately with each other regarding the end behavior of various functions using the language of limits and communicate the behavior for which the function increases or decreases without bound.

UNIT 2: DERIVATIVE DEFINITION AND DERIVATIVE RULES

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 1.1, EU 2.1, EU 2.2, EU 2.3

Estimated Time:
12 instructional hours

Guiding Questions:

▸ **What is the derivative of a function?** ▸ **How do you find derivatives of products, quotients, composite, and inverse functions?** ▸ **How can we find the slope of a curve even when it is difficult to solve its equation for y in terms of x?** ▸ **When might a derivative fail to exist?** ▸ **What is the relationship between differentiability and continuity?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.1A: Identify the derivative of a function as the limit of a difference quotient. **LO 2.1B:** Estimate derivatives. **LO 2.3B:** Solve problems involving the slope of a tangent line.	**Print** Foerster, sections 3-1 and 3-2	**Instructional Activity: Definition of *Derivative*** We begin with the graphs of two functions on the board, $g(x)=\frac{3}{2}x+4$, and $f(x)=x^2-4x+5$, each on its own set of axes. I question students about the rate at which each function is changing, and a class discussion of slope ensues. Students come to appreciate the difference between constant slope and instantaneous slope at a specific point. I then have student groups find the slopes of various secant lines (identified as average rates of change over an interval) on the graph of f in order to determine the slopes of the tangent line at various points. Students present their findings, and I guide them through the definition $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. Student groups find derivatives of other polynomial functions, and then eventually $f(x)=\frac{1}{x}$ and $g(x)=\sqrt{x}$, followed by questions about lines tangent to those curves.	*As we progress through this activity, the concept of the derivative definition gets reinforced as the algebra intensifies. We go from polynomial functions to square root functions to reciprocal functions, providing the opportunity for algebra review.*
LO 2.1A: Identify the derivative of a function as the limit of a difference quotient. **LO 2.1C:** Calculate derivatives. **LO 2.3B:** Solve problems involving the slope of a tangent line.	**Print** Foerster, section 3-4	**Instructional Activity: The Power Rule** When students enter the room, there is already a list on the board of power functions paired with their derivatives that had been considered in previous lessons and homework. Students are asked to conjecture a general rule for the derivative of $f(x)=x^n$. After a teacher-guided class discussion leading to the result that $\frac{d}{dx}x^n=nx^{n-1}$, students work independently (except for a couple of volunteers who help each other work out the derivation on the board) in trying to prove the power rule for natural numbers n. We discuss the constant rule and sum rule for derivatives, and I say that a guided activity will be included in their homework to prove those results. Students will apply their new shortcut to problems involving lines tangent to polynomial functions.	*Prior to this lesson, it's a good idea to have provided the students practice with the binomial theorem. If they have found derivatives of functions involving cubic and quartic terms coming into this lesson, the students might need very little guidance in proving the power rule.*

UNIT 2: DERIVATIVE DEFINITION AND DERIVATIVE RULES

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 1.1, EU 2.1, EU 2.2, EU 2.3

Estimated Time:
12 instructional hours

Guiding Questions:

▸ What is the derivative of a function? ▸ How do you find derivatives of products, quotients, composite, and inverse functions? ▸ How can we find the slope of a curve even when it is difficult to solve its equation for y in terms of x? ▸ When might a derivative fail to exist? ▸ What is the relationship between differentiability and continuity?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 1.1C: Determine limit of functions.	**Print** Stewart, sections 1.5 (example 3) and 2.4	**Instructional Activity: A Special Trigonometric Limit** Students enter the class with two tangent line problems on the board. The first problem they can solve easily using previously learned ideas. The second problem involves the curve $y = \sin x$ and students come to appreciate that the power rule has only so much power. A class discussion foreshadowing the derivation of $\frac{d}{dx}\sin x$ ensues, and students come to realize that a prerequisite for that task will require knowing the value of $\lim_{\theta \to 0}\frac{\sin\theta}{\theta}$. In groups, students follow a guided worksheet enabling them to derive the result, and will appreciate that using radian measure yields the simple answer of 1, whereas using degrees yields the aesthetically less preferable answer $\frac{\pi}{180}$. One of the groups works at the board so they can present their derivation of that result.
LO 2.1A: Identify the derivative of a function as the limit of a difference quotient. **LO 2.1C:** Calculate derivatives. **LO 2.3B:** Solve problems involving the slope of a tangent line.	**Print** Stewart, section 2.4	**Instructional Activity: Some Trigonometric Derivatives** As in the previous activity, I present student groups with a tangent line problem involving the curve $y = \sin x$, only this time they recognize that they are equipped to derive the result for $\frac{d}{dx}\sin x$. In homework assignments leading up to this lesson, I review the $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$ formulas, so the student groups are pretty successful with their tasks of deriving the results for $\frac{d}{dx}\sin x$ and $\frac{d}{dx}\cos x$. As usual, I select one group to work at the board so they can present the derivation. Students reconvene in groups to work out some tangent line exercises for their newly found derivative rules.

Prior to this lesson, the students have already had homework practice involving trigonometric limits requiring the $\lim_{\theta \to 0}\frac{\sin\theta}{\theta}$ *result from the previous activity. I always seize this opportunity to emphasize that throughout our AP Calculus BC course they need to have their calculators in radian mode, or else we'd have* $\frac{d}{dx}\sin x = \frac{\pi}{180}\cos x$ *instead of the preferred* $\frac{d}{dx}\sin x = 1 \cdot \cos x$.

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 1.1, EU 2.1, EU 2.2, EU 2.3

Estimated Time:
12 instructional hours

Guiding Questions:

▶ **What is the derivative of a function?** ▶ **How do you find derivatives of products, quotients, composite, and inverse functions?** ▶ **How can we find the slope of a curve even when it is difficult to solve its equation for y in terms of x?** ▶ **When might a derivative fail to exist?** ▶ **What is the relationship between differentiability and continuity?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.1C: Calculate derivatives.	**Print** Foerster, section 4-2	**Instructional Activity: The Product Rule** Students work in groups in search of a rule for finding the derivative of the product of two functions. Following the development in Foerster section 4-2, students determine that $\frac{d}{dx}\left(x^5 \cdot x^8\right) \neq \left(\frac{d}{dx}x^5\right)\cdot\left(\frac{d}{dx}x^8\right)$ and conclude that the derivative of a product is not equal to the product of the derivatives. The worksheet guides the student groups through the derivation of the product rule, which is outlined in this textbook section. One of the groups presents their findings. I summarize the rule and present a guided example. Students resume working in groups to practice with the new rule. Having the derivatives of $\sin x$ and $\cos x$ on the books prior to this activity really helps a calculus teacher provide meaningful illustrations of the product rule.
LO 2.1C: Calculate derivatives.	**Print** Foerster, section 4-3	**Instructional Activity: The Quotient Rule** In a lesson structure similar to the previous activity, student groups work out examples to discover that the derivative of a quotient is not equal to the quotient of the derivatives. The worksheets guide students to develop the quotient rule by rearranging the equation $Q(x)=\frac{N(x)}{D(x)}$ into the form $N(x)=D(x)\cdot Q(x)$, and then applying the product rule. Students solve the resulting equation for $Q'(x)$, establishing the rule. As usual, one of the groups presents their findings and summarizes the rule. Students resume their group formations to work out examples. This is an excellent time to use the quotient rule to have the class work in groups and derive the result for the derivatives of the other trigonometric function using the quotient rule.

In introducing the product, quotient, and chain rules, it really helps that we already have access to the results $\frac{d}{dx}\sin x$ *and* $\frac{d}{dx}\cos x$.

UNIT 2: DERIVATIVE DEFINITION AND DERIVATIVE RULES

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 1.1, EU 2.1, EU 2.2, EU 2.3

Estimated Time:
12 instructional hours

Guiding Questions:

▶ **What is the derivative of a function?** ▶ **How do you find derivatives of products, quotients, composite, and inverse functions?** ▶ **How can we find the slope of a curve even when it is difficult to solve its equation for y in terms of x?** ▶ **When might a derivative fail to exist?** ▶ **What is the relationship between differentiability and continuity?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.1A: Identify the derivative of a function as the limit of a difference quotient. **LO 2.1C:** Calculate derivatives.	**Print** Foerster, Special Problem Set 3-6 and section 3-7 Stewart, section 2.5	**Instructional Activity: The Chain Rule** Students work in groups on a worksheet that is a modified version of the special group activity in Foerster section 3-6. The purpose of the problem set is to motivate the statement of the chain rule. As usual, I have one group work at the board while I circulate and help to guide the students through the problem set. The group working at the board presents their findings, then I take over for a summary of the rule, using both notational forms $\frac{d}{dx}f(g(x))=f'(g(x))\cdot g'(x)$ and $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$. I defer the proof of the chain rule until Unit 3, using the Harvard Consortium justification by virtue of the principle of local linearity. The remainder of this activity involves practice examples, including problems involving tabular data.	*Throughout this unit, I assign "hidden derivative" homework problems corresponding to the rules taught for the most recent instructional activity. For example,* $\lim_{h\to 0}\frac{\sec^2 \frac{\pi}{4}+h -2}{h}$ *after the chain rule has been established, and* $\lim_{x\to\frac{\pi}{3}}\frac{\tan x-\sqrt{3}}{x-\frac{\pi}{3}}$ *after the previous activity. This way, the different definitions and derivative rules are all in play.*
LO 2.1A: Identify the derivative of a function as the limit of a difference quotient. **LO 2.1C:** Calculate derivatives. **LO 2.3B:** Solve problems involving the slope of a tangent line.		**Formative Assessment: Practice with Derivatives and Tangent Lines** Students work independently on finding derivatives of functions. The functions are combinations of algebraic and trigonometric, and they involve products, quotients, and compositions. Students find equations of tangent and normal lines at specified points and find points on the graph where the tangent line is horizontal. Students also evaluate limits that are really "hidden derivatives," such as $\lim_{x\to\frac{\pi}{4}}\frac{\tan^2(x)-1}{x-\frac{\pi}{4}}$. Students work on these problems with notebooks open. I circulate around the room and if I see good opportunities for instruction or warnings about common errors, I interrupt the class for instruction. There are two sets of problems: one basic and one more challenging. All students are expected to complete the basic problem set, but one purpose of the more challenging set is to provide differentiated instruction.	*Toward the end of this formative assessment, I collect the sheets and scan them in the classroom sheet-fed scanner for examination later that day. The worksheets are immediately returned to the students and a solution key is provided as the students exit. I use the scans to determine areas to work on for individual students. Students are required to correct their worksheet with the key and prepare requests for the next day.*

UNIT 2: DERIVATIVE DEFINITION AND DERIVATIVE RULES

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 1.1, EU 2.1, EU 2.2, EU 2.3

Estimated Time:
12 instructional hours

Guiding Questions:

▸ What is the derivative of a function? ▸ How do you find derivatives of products, quotients, composite, and inverse functions? ▸ How can we find the slope of a curve even when it is difficult to solve its equation for y in terms of x? ▸ When might a derivative fail to exist? ▸ What is the relationship between differentiability and continuity?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.1C: Calculate derivatives.	**Print** Stewart, section 6.1	**Instructional Activity: The Inverse Function Theorem** I give students a worksheet that reviews some basics from their precalculus course about inverse functions. Students work independently to try to find $g'(5)$, where g is the inverse of $f(x)=x^3+4x$. Students soon realize that it is difficult to find f^{-1} as an explicit function of x. I then interrupt the class and guide them through a derivation of the inverse function rule, following the Stewart text. I give them a similar problem to reinforce their understanding of the rule, and after a student-presented solution we summarize the rule. I challenge the class to find the rules for $\frac{d}{dx}\arctan x$ and $\frac{d}{dx}\arcsin x$. Using the new rules for the derivatives of these inverse trigonometric functions we spiral back to the product, quotient, and chain rules for the rest of the activity.
LO 2.1C: Calculate derivatives. **LO 2.3B:** Solve problems involving the slope of a tangent line.	**Print** Stewart, section 2.6	**Instructional Activity: Implicit Differentiation** I write two tangent line problems on the board: the first is to find the slope of the curve $x^2+y^2=25$ at the point $(4,3)$, and the second is to find the slope of the curve $x^2+3xy-4y^3=6$ at the point $(2,1)$. Students realize that explicit differentiation has its limitations. I model the implicit differentiation method to the first problem. Having already solved the first problem using explicit differentiation, students have a good chance to understand what it means to say that y is an implied function of x. Then students work in groups to retry the second example now that they have the new derivative rule. As usual, I choose one group to work on the board, so they can explain their findings.

This is a good opportunity to show new derivations for $\frac{d}{dx}\arctan x$ and $\frac{d}{dx}\arcsin x$, but this time using implicit differentiation.

UNIT 2: DERIVATIVE DEFINITION AND DERIVATIVE RULES

BIG IDEA 2
Derivatives

Enduring Understandings:
▸ EU 1.1, EU 2.1, EU 2.2, EU 2.3

Estimated Time:
12 instructional hours

Guiding Questions:

▸ What is the derivative of a function? ▸ How do you find derivatives of products, quotients, composite, and inverse functions? ▸ How can we find the slope of a curve even when it is difficult to solve its equation for y in terms of x? ▸ When might a derivative fail to exist? ▸ What is the relationship between differentiability and continuity?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.2B: Recognize the connection between differentiability and continuity.	**Print** Foerster, section 4-6 Stewart, section 2.2	**Instructional Activity: The Relationship Between Continuity and Differentiability** Student groups debate the existence of $f'(2)$ where $f(x)=\lvert x-2\rvert$. Students consider the graphical interpretation of derivative. They express the function f in piecewise form and use the definition of derivative to decide. I guide the class through an analysis of the left and right derivatives, and since they are unequal, we conclude that $f'(2)$ does not exist. I write the official definition for differentiability on the board and pose the question, "Well, it seems that a function may be continuous at a point, yet it may not be differentiable there. If a function is differentiable, must it be continuous?" I allow students to debate the issue in groups. After a few minutes, I interrupt them and guide them through section 4-6 from Foerster.
All of the learning objectives in this unit are addressed.		**Summative Assessment: Derivative Definition and Rules** Students are given 10 fairly short free-response questions directly targeting single learning objectives, and two longer free-response questions that require assimilating the concepts learned in the unit. This is followed by six multiple-choice questions.

This summative assessment addresses all of the guiding questions for the unit.

Mathematical Practices for AP Calculus in Unit 2

The following activities and techniques in Unit 2 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: The derivations of all the derivative shortcuts, such as the power rule, the product rule, the quotient rule, and the inverse function rule require this mathematical practice. At least two or three items in the summative assessment require students to construct and articulate arguments using these definitions.

MPAC 2 — Connecting concepts: In the instructional activity "The Relationship Between Continuity and Differentiability," we must work with both the definitions of derivative and continuity to establish this foundational result.

MPAC 3 — Implementing algebraic/computational processes: In the instructional activities—which include all the derivative shortcuts, such as the power rule, the product rule, the quotient rule, the chain rule, and implicit differentiation—the students are engaged in a tremendous degree of algebraic and computational processes as they invoke their newly learned rules.

MPAC 4 — Connecting multiple representations: In the instructional activity "The Relationship Between Continuity and Differentiability," students use a graphical interpretation of derivative to help them reason that certain functions are not differentiable. Students are challenged to predict limit values based on tables provided by their graphing calculators. Throughout the unit, assignments include examples of functions where only tabular data is provided.

MPAC 5 — Building notational fluency: Throughout the unit, the Leibnitz and functional forms of derivatives are prominent. Notation involving left and right derivatives is needed in "The Relationship Between Continuity and Differentiability" instructional activity.

MPAC 6 — Communicating: Throughout this unit, students are put into groups where they analyze, evaluate, and compare the reasoning of others. For example, in the instructional activity "The Product Rule," students need to come to the conclusion and articulate that the derivative of a product is not equal to the product of the derivatives. In "The Inverse Function Theorem" students communicate at a high level using precise language and notation when they justify their results using the theorem $\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))}$.

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 1.2, EU 2.1, EU 2.2, EU 2.3, EU 2.4

Estimated Time:
13 instructional hours

Guiding Questions:

- **How does the principle of local linearity allow us to analyze a function's behavior near a given point?**
- **Where do instantaneous rates of change arise contextually?** ▸ **How can we interpret higher-order derivatives with regard to rectilinear motion, curve sketching, and contextually?** ▸ **How can the Mean Value Theorem (MVT) be applied to describe the behavior of a differentiable function over an interval?** ▸ **What information do the first and second derivatives of a function reveal about its graph?** ▸ **What are the different ways to determine the relative and absolute extrema of a function?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.3B: Solve problems involving the slope of a tangent line.	**Print** Foerster, section 5-2	**Instructional Activity: The Principle of Local Linearity** Students independently work on two problems: first, to find the equation of the tangent line to the graph of $f(x)=\sqrt{x}$ at the point where $x=6$, and second, to approximate $\sqrt{36.2}$ without a calculator. One student works at the board. After a few minutes, the student at the board explains his or her solution to the first problem. Using the graphing calculator connection to the interactive whiteboard, I demonstrate with multiple ZOOMs that the tangent line and the function f are nearly indistinguishable "near" $x=6$. After giving similar exercises we develop the principal, "For small h and a differentiable function f, $f(x+h)\approx f(x)+f'(x)\cdot h$. With some questioning, students realize that this underlined statement is algebraically equivalent to $f'(x)\approx\frac{f(x+h)-f(x)}{h}$. We summarize by writing just below the underlined equation, "$new\ y = old\ y + \Delta y$" or "$new\ y = old\ y + slope \cdot \Delta x$".

Since the principle of local linearity is such a recurring theme in the Calculus BC course, this evening's homework assignment contains many similar problems, and forces students to write the different versions of the principle mentioned in the activity.

UNIT 3: DERIVATIVE INTERPRETATIONS AND APPLICATIONS

BIG IDEA 3
Derivatives

Enduring Understandings:
- EU 1.2, EU 2.1, EU 2.2, EU 2.3, EU 2.4

Estimated Time:
13 instructional hours

Guiding Questions:

- **How does the principle of local linearity allow us to analyze a function's behavior near a given point?**
- **Where do instantaneous rates of change arise contextually?** ▸ **How can we interpret higher-order derivatives with regard to rectilinear motion, curve sketching, and contextually?** ▸ **How can the Mean Value Theorem (MVT) be applied to describe the behavior of a differentiable function over an interval?** ▸ **What information do the first and second derivatives of a function reveal about its graph?** ▸ **What are the different ways to determine the relative and absolute extrema of a function?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.1D: Determine higher order derivatives. **LO 2.2A:** Use derivatives to analyze properties of a function. **LO 2.3C:** Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion. **LO 2.3D:** Solve problems involving rates of change in applied contexts.	**Print** Stewart, section 2.7	**Instructional Activity: The Calculus of Rectilinear Motion** Students groups do a worksheet activity (modified from the first example in Stewart section 2.7). One group works at the board. Students are given a particle's position function, and work through the notions of instantaneous versus average velocity and displacement versus total distance traveled. Acceleration as the rate of change of velocity with respect to time is introduced. Students understand how the signs of acceleration and velocity are both necessary to determine whether the speed of the particle is increasing or decreasing. The group at the board explains their findings to the class, and I show an animation of the particle's motion, setting $\begin{cases} x1(t)=s(t) \\ y1(t)=3 \end{cases}$ in parametric mode. I use a viewing window $1 \le y \le 5$ so the particle travels linearly right in the middle of the screen.

UNIT 3: DERIVATIVE INTERPRETATIONS AND APPLICATIONS

BIG IDEA 3
Derivatives

Enduring Understandings:
- EU 1.2, EU 2.1, EU 2.2, EU 2.3, EU 2.4

Estimated Time:
13 instructional hours

Guiding Questions:

▸ How does the principle of local linearity allow us to analyze a function's behavior near a given point? ▸ Where do instantaneous rates of change arise contextually? ▸ How can we interpret higher-order derivatives with regard to rectilinear motion, curve sketching, and contextually? ▸ How can the Mean Value Theorem (MVT) be applied to describe the behavior of a differentiable function over an interval? ▸ What information do the first and second derivatives of a function reveal about its graph? ▸ What are the different ways to determine the relative and absolute extrema of a function?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.3A: Interpret the meaning of derivative within a problem. **LO 2.3C:** Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion. **LO 2.3D:** Solve problems involving rates of change in applied contexts.	**Print** Foerster, section 4-9	**Instructional Activity: Related Rates** Student groups work on a guided worksheet based on classic related rates problems in Foerster section 4-9. One group presents their solution. After some Q&A, I elaborate that when multiple variable quantities each vary with time, they can be thought of as an implied function of time. For example, in the formula $A=\frac{1}{2}bh$, the rates of change of the area, base, and height are related. Students come to understand that while they never get to see the functions of time corresponding to A, b, and h explicitly, the chain rule still applies as we differentiate with respect to time. Groups resume working on the second problem in the section, and another group presents their solution. The class ends with the summary in the Foerster section.

BIG IDEA 3
Derivatives

Enduring Understandings:
- EU 1.2, EU 2.1, EU 2.2, EU 2.3, EU 2.4

Estimated Time:
13 instructional hours

Guiding Questions:

- How does the principle of local linearity allow us to analyze a function's behavior near a given point?
- Where do instantaneous rates of change arise contextually?
- How can we interpret higher-order derivatives with regard to rectilinear motion, curve sketching, and contextually?
- How can the Mean Value Theorem (MVT) be applied to describe the behavior of a differentiable function over an interval?
- What information do the first and second derivatives of a function reveal about its graph?
- What are the different ways to determine the relative and absolute extrema of a function?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.1D: Determine higher order derivatives. **LO 2.2A:** Use derivatives to analyze properties of a function. **LO 2.3C:** Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion. **LO 2.3D:** Solve problems involving rates of change in applied contexts.	**Print** Foerster, section 4-9 **Web** AP Calculus AB 1999 Free-Response Question 6	**Formative Assessment: Practice with Local Linearity, Rectilinear Motion, and Related Rates** Students work independently on problems involving the applications to derivatives taught so far in this unit. They make linear approximations for given functions near known points. There are problems on particle motion. I include position functions that spiral back to the derivative rules from Unit 2, such as $s(t)=\cos^2 t+\cos t$. I include one related rates example similar to a problem we've already done in recent homework assignments, such as the classic liquid leaking from a cone-type problem from Foerster, section 4-9. Then I include something less traditional, such as the AP Calculus AB 1999 Free-Response Question 6 from AP Central. I circulate, occasionally interrupting the class to provide light instruction.

Toward the end of this formative assessment, I collect the sheets and scan them in the classroom sheet-fed scanner for examination later that day. The worksheets are immediately returned to the students and a solution key is provided as the students exit. I use the scans to determine areas to work on for individual students. Students are required to correct their worksheet with the key and prepare requests for the in-class Q&A the next day.

UNIT 3: DERIVATIVE INTERPRETATIONS AND APPLICATIONS

BIG IDEA 3
Derivatives

Enduring Understandings:
- EU 1.2, EU 2.1, EU 2.2, EU 2.3, EU 2.4

Estimated Time:
13 instructional hours

Guiding Questions:

- How does the principle of local linearity allow us to analyze a function's behavior near a given point? ▸ Where do instantaneous rates of change arise contextually? ▸ How can we interpret higher-order derivatives with regard to rectilinear motion, curve sketching, and contextually? ▸ How can the Mean Value Theorem (MVT) be applied to describe the behavior of a differentiable function over an interval? ▸ What information do the first and second derivatives of a function reveal about its graph? ▸ What are the different ways to determine the relative and absolute extrema of a function?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 1.2B: Determine the applicability of important calculus theorems using continuity. **LO 2.4A**: Apply the Mean Value Theorem to describe the behavior of a function over an interval.	**Print** Foerster, section 4-2 Stewart, section 3.2	**Instructional Activity: The Mean Value Theorem (MVT)** Students work independently to find the average rate of change of the function $f(x)=\sqrt{4x+1}$ over the interval $2 \le x \le 6$. After a student-presented solution, I ask the class to find a value in $2 \le x \le 6$ for which the instantaneous rate of change of f equals the average rate of change for $2 \le x \le 6$. The problem sheet includes a diagram showing the curve, as well as the associated secant and tangent lines. We discuss the geometric significance of the MVT, and a student solution is presented. Student groups consider the question, "Given a function f and a closed interval $[a,b]$, will there always be a point in the interval for which the instantaneous rate of change equals the average rate of change over the interval?" The ensuing discussion elicits the statement of the MVT.
LO 2.2A: Use derivatives to analyze properties of a function. **LO 2.4A**: Apply the Mean Value Theorem to describe the behavior of a function over an interval.	**Print** Stewart, section 3.3	**Instructional Activity: Increasing and Decreasing Functions** Students work independently, using their graphing calculators to sketch $f(x)=3x^4-4x^3-12x^2+5$ and determine the intervals on which f is increasing/decreasing, which refreshes work done in the prior year's precalculus course. The challenge is posed: "Could we determine this same information with calculus and without technology?" Students work in groups while I circulate, guiding students as necessary. One group presents their solution, and there is an intuitive discussion about positive derivatives corresponding to increasing functions, and negative derivatives corresponding to decreasing functions. With the Mean Value Theorem fresh in their minds from the previous lesson, we discuss the proof given in the Stewart section.

Students exit with the Mean Value Theorem conclusion, but are challenged to complete an assignment to prove the theorem. Naturally, the proof is too much to expect BC students to pull off without a lot of guidance. A worksheet-style homework assignment guides the student to follow the development of the Foerster section referenced. The subsequent lesson involves going over this homework assignment in detail, and then the examples in the referenced Stewart section.

UNIT 3: DERIVATIVE INTERPRETATIONS AND APPLICATIONS

BIG IDEA 3
Derivatives

Enduring Understandings:
- EU 1.2, EU 2.1, EU 2.2, EU 2.3, EU 2.4

Estimated Time:
13 instructional hours

Guiding Questions:

▶ How does the principle of local linearity allow us to analyze a function's behavior near a given point? ▶ Where do instantaneous rates of change arise contextually? ▶ How can we interpret higher-order derivatives with regard to rectilinear motion, curve sketching, and contextually? ▶ How can the Mean Value Theorem (MVT) be applied to describe the behavior of a differentiable function over an interval? ▶ What information do the first and second derivatives of a function reveal about its graph? ▶ What are the different ways to determine the relative and absolute extrema of a function?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.1D: Determine higher order derivatives. **LO 2.2A:** Use derivatives to analyze properties of a function.	**Print** Stewart, sections 3.3 and 3.5	**Instructional Activity: Concavity and Curve Sketching** The task of the day is to sketch $f(x)=x^{\frac{4}{3}}+x^{\frac{1}{3}}$ on the interval $-8 \le x \le 8$. Students decide that it will be useful to find intercepts, and from the prior activity, students recognize that it will be valuable to analyze f'. This is the perfect moment to display an interactive whiteboard file with the diagram in the referenced Stewart section motivating the concept of concavity. I follow the subsection in Stewart section 3.3 titled "What Does f'' Say About f?" in my discussion with the class. Students then break off into groups where they use the ideas in the discussion to perform the calculus analysis on the function f and to produce its graph. After a group presents a solution, we write a summary of the steps for curve sketching.	*The assignment that students exit this activity with includes sketching curves that contain trigonometric functions, such as $y=\cos^2 x+2\cos x$. The subsequent lesson involves reinforcing these ideas, and much of the next lesson is spent going over this assignment.*
LO 2.2A: Use derivatives to analyze properties of a function.	**Web** AP Calculus BC 1980 Free-Response Question 7 (see Appendix)	**Formative Assessment: Using a Given Graph of f' to Formulate Conclusions About f and f''** As students enter the room I give them worksheets to complete independently. The sheet has a graph of f' on a given interval. I use the graph from the 1980 AP Calculus BC Exam Free-Response Question 7. I modify the original AP problem by putting some instruction and guidance on the worksheet, along with little "recall" boxes highlighting recently learned concepts (see Appendix). I circulate around the room and give some direction and instruction. If I notice misconceptions, I interrupt the class for quick discussions and reminders. In this formative assessment, there is also a question that reverts back to tangent line approximations. Students are challenged to investigate and eventually see that the concavity of the function determines whether the approximations are underestimates or overestimates.	*Toward the end of this formative assessment, I collect the sheets and scan them in the classroom sheet-fed scanner for examination later that day. The worksheets are immediately returned to the students and a solution key is provided as the students exit. I use the scans to determine areas to work on for individual students. Students are required to correct their worksheet with the key and prepare requests for the next day.*

UNIT 3: DERIVATIVE INTERPRETATIONS AND APPLICATIONS

BIG IDEA 3
Derivatives

Enduring Understandings:
- EU 1.2, EU 2.1, EU 2.2, EU 2.3, EU 2.4

Estimated Time:
13 instructional hours

Guiding Questions:

▶ How does the principle of local linearity allow us to analyze a function's behavior near a given point? ▶ Where do instantaneous rates of change arise contextually? ▶ How can we interpret higher-order derivatives with regard to rectilinear motion, curve sketching, and contextually? ▶ How can the Mean Value Theorem (MVT) be applied to describe the behavior of a differentiable function over an interval? ▶ What information do the first and second derivatives of a function reveal about its graph? ▶ What are the different ways to determine the relative and absolute extrema of a function?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 1.2B: Determine the applicability of important calculus theorems using continuity. **LO 2.1D**: Determine higher order derivatives. **LO 2.2A**: Use derivatives to analyze properties of a function.	**Print** Stewart, section 3.1	**Instructional Activity: Absolute Versus Relative Extrema** Students work independently to sketch $f(x)=x^3-6x^2+9x+7, x\in[0,5]$ and identify the maximum and minimum values of the function f on the interval. We have a class discussion about the meaning of absolute versus relative extrema. I guide the class as we develop definitions of these important terms. At this point, I distribute worksheets and the class breaks off into groups. The worksheet guides students through the examples in this Stewart section, instructing them on the Extreme Value Theorem, the first derivative test, the closed interval test, and the second derivative test. The length and gravity of the activity requires two lessons. As groups present their solutions, the main emphasis is on communicating the justification. I then summarize the tests and theorems referenced above on the board.

UNIT 3: DERIVATIVE INTERPRETATIONS AND APPLICATIONS

BIG IDEA 3
Derivatives

Enduring Understandings:
- EU 1.2, EU 2.1, EU 2.2, EU 2.3, EU 2.4

Estimated Time:
13 instructional hours

Guiding Questions:

▶ How does the principle of local linearity allow us to analyze a function's behavior near a given point? ▶ Where do instantaneous rates of change arise contextually? ▶ How can we interpret higher-order derivatives with regard to rectilinear motion, curve sketching, and contextually? ▶ How can the Mean Value Theorem (MVT) be applied to describe the behavior of a differentiable function over an interval? ▶ What information do the first and second derivatives of a function reveal about its graph? ▶ What are the different ways to determine the relative and absolute extrema of a function?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.1C: Calculate derivatives. **LO 2.2A:** Use derivatives to analyze properties of a function. **LO 2.3A:** Interpret the meaning of derivative within a problem. **LO 2.3C:** Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion. **LO 2.3D:** Solve problems involving rates of change in applied contexts.	**Print** Stewart, section 3.7	**Instructional Activity: Optimizing Functions in Context** At the start of class, I hand out a "Do Now" problem sheet, which is example 4 in this Stewart section. We solve this problem as a class, emphasizing the justification beyond just finding the critical value. We summarize the steps of solving a contextual optimization problem, as outlined in this Stewart section. Students then break off into groups and solve examples 2 and 3 from this same section, written on the flip side of the Do Now sheet. Several groups then present their solutions to these two problems. I lead the class through a summary and use this opportunity to review the different ways to justify the absolute extrema of a function from the previous activity, but this time, it is in context.	
All of the learning objectives in this unit are addressed.		**Summative Assessment: Derivative Applications** I give students 10 fairly short free-response questions directly targeting single learning objectives, and two longer free-response questions that require assimilating the concepts learned in the unit. This is followed by six multiple-choice questions.	*This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 3

The following activities and techniques in Unit 3 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: In the "Increasing and Decreasing Functions" instructional activity we use the Mean Value Theorem to justify our results. In the "Absolute Versus Relative Extrema" instructional activity we define those key terms and use them to formulate methods by which we can find those extrema. In the "Optimizing Functions in Context" instructional activity we use theorems and definitions in our justifications.

MPAC 2 — Connecting concepts: In the "Concavity and Curve Sketching" instructional activity we must connect the new idea of concavity of functions to the earlier principle of local linearity. At least one item in the summative assessment requires students to connect curve sketching ideas to rectilinear motion.

MPAC 3 — Implementing algebraic/computational processes: In the "Concavity and Curve Sketching" and "Optimizing Functions in Context" instructional activities there is a tremendous amount of algebraic manipulations required to analyze the signs of the first and second derivatives to determine their signs.

MPAC 4 — Connecting multiple representations: Once the "Concavity and Curve Sketching" instructional activity is completed, it opens the door for assignments in which tabular information about a function f and its first and second derivatives is given. Students use the information to make conclusions about the graphs of this function f. Once this activity has been covered, the assignments include exercises in which the derivative of a function is presented symbolically, and information about the graph of the original function must be determined.

MPAC 5 — Building notational fluency: In the instructional activities "The Calculus of Rectilinear Motion" and "Concavity and Curve Sketching" students use the notation of higher-order derivatives. Through most of the activities in the second half of the unit, students are required to connect the notation to the graphical properties of the functions. In the "Related Rates" instructional activity students must connect verbal descriptions to the corresponding derivative notations.

MPAC 6 — Communicating: In the "Absolute Versus Relative Extrema" and "Optimizing Functions in Context" instructional activities students must communicate with one another in verbal and in written form the justification for how they can be certain that they have found the relative or absolute extrema. In the "Related Rates" instructional activity students must interpret their derivative results in the context of the specific problem given.

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
14 instructional hours

Guiding Questions:

▸ How does an indefinite integral relate to a derivative? ▸ What is the difference between the true variable and the dummy variable in a function defined by an integral? ▸ How does a definite integral measure net change? ▸ How can the Fundamental Theorem of Calculus (FTC) be applied to evaluate a definite integral or find the derivative of a function defined by an integral?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.2B: Approximate a definite integral.	**Print** Foerster, section 5-4 Stewart, section 4.1 **Technology** C++ compiler or graphing calculator	**Instructional Activity: Riemann Sums** Students receive two problems as they enter the room. The first problem asks students to approximate the area under the curve $y=4x^2+2$ between $x=3$ and $x=5$. The second problem provides tabular data representing the velocity of a particle traveling along a line for selected times, and students must approximate the distance traveled. Students come to appreciate the idea of summing up products to solve these problems. This all motivates the definition of the Riemann sum: left, right, and midpoint. Students realize that the smaller the subintervals, the better the approximation. We write up a computer program to calculate a right Riemann with 1,000 subintervals. Although I do this using a C++ compiler on the interactive whiteboard, this can be done using the programming capabilities of a graphing calculator.	*Students entering Calculus BC in our school already have basic programming skills from their precalculus course. Writing the Riemann sum programs really gets students to understand the definition. If a school has a computer science teacher, this is a great opportunity for interdisciplinary teaching.*
LO 3.2A(a): Interpret the definite integral as the limit of a Riemann sum. **LO 3.2A(b)**: Express the limit of a Riemann sum in integral notation. **LO 3.2B**: Approximate a definite integral. **LO 3.3B(b)**: Evaluate definite integrals.	**Print** Foerster, section 5-4 Stewart, section 4.2	**Instructional Activity: A Race Against the Computer and the Definition of the Definite Integral** We begin with an area problem corresponding to $\int_1^3 f(x)dx$ for $f(x)=2x^3+1$. Students are tasked to write a program for a right Riemann sum approximation, this time with 60,000 subintervals. We run the program. I ask, "How can we improve upon this approximation?" We decide to sum up infinitely many infinitesimally small products. The definition of definite integral as the limit of Riemann sums is achieved. We write $\int_1^3 f(x)dx=\lim_{n\to\infty}\sum_{k=1}^{n}\left(f(x_k)\cdot\Delta x\right)$, where $x_k=a+k\cdot\Delta x$, $a=1,b=3$, and $\Delta x=\frac{b-a}{n}$. We explicitly take the limit of these right Riemann sums while the program is running. Students already know the necessary summation formulas from their precalculus course. I guide the Q&A so that the computer generates the approximation around the same time as it takes for us to determine the exact answer.	*The choice of 60,000 subintervals isn't totally arbitrary. For the slow processor on the computer we use, I have determined over the years that number of subintervals allows for around 30 minutes. That is just the right amount of time to develop the definition and explicitly take the limit of the Riemann sums.*

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
14 instructional hours

Guiding Questions:

▸ How does an indefinite integral relate to a derivative? ▸ What is the difference between the true variable and the dummy variable in a function defined by an integral? ▸ How does a definite integral measure net change? ▸ How can the Fundamental Theorem of Calculus (FTC) be applied to evaluate a definite integral or find the derivative of a function defined by an integral?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.2C: Calculate a definite integral using areas and properties of definite integrals.	**Web** AP Calculus AB 2004 Free-Response Question 5	**Formative Assessment: Evaluating Definite Integrals Using Signed Area Interpretations** Students work, at first independently, on a two-page worksheet with a problem set that involves definite integrals that can be evaluated using area interpretations, such as $\int_{-2}^{5}(2x-3)\,dx$, $\int_{1}^{5}\lvert x-2\rvert\,dx$, and $\int_{-4}^{4}\sqrt{16-x^2}\,dx$. Students must express the definite integrals as the limit of Riemann sums, but they should find the values of these definite integrals by considering the signed area to which they correspond. The worksheet also consists of problems involving definite integrals of functions whose equations are not given, but whose graphs consist of lines and semicircles (such as AP Calculus AB 2004 Free-Response Question 5). I circulate and give individual guidance and occasionally interrupt for instruction as necessary. Students break off into groups for the second page while I scan their work on the first.
LO 3.3A: Analyze functions defined by an integral.	**Print** Stewart, Discovery Project following section 4.2	**Instructional Activity: Functions Defined by an Integral** The Discovery Project following section 4.2 in Stewart is the perfect mechanism to introduce functions defined by an integral. There are four problems, each with multiple parts in this exercise. Students work in groups. As groups work on this exercise, I circulate and provide guidance. I choose four groups, one for each of the four problems, which are structured in parts. This provides the opportunity for a collaborative presentation among the members of each group. This activity not only serves to make students feel comfortable with the subtleties of functions defined by an integral, but it foreshadows the Fundamental Theorem of Calculus (FTC), form 1, that is, $\frac{d}{dx}\int_{a}^{x} f(t)\,dt = f(x)$.

About two-thirds of the way through the period, I collect each student's first page and run them through the sheet-fed scanner for examination later that day. While I am scanning, students reconfigure into groups, and help each other on the second page. I return the first page. Their assignment that evening is to use the knowledge gained from their groups to correct their first page and complete their second page.

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
14 instructional hours

Guiding Questions:

- **How does an indefinite integral relate to a derivative?** ▸ **What is the difference between the true variable and the dummy variable in a function defined by an integral?** ▸ **How does a definite integral measure net change?** ▸ **How can the Fundamental Theorem of Calculus (FTC) be applied to evaluate a definite integral or find the derivative of a function defined by an integral?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.2A(b): Express the limit of a Riemann sum in integral notation. **LO 3.2B:** Approximate a definite integral. **LO 3.4B:** Apply definite integrals to problems involving the average value of a function.	**Print** Stewart, section 5.5	**Instructional Activity: Average Value of a Function** The Do Now exercise on the board tells students that the number of bacteria in a culture at time t is given by some specific function. I ask students for the average number of bacteria in the culture over a given time interval (it might be strategic to use a function and interval from one of the earlier activities so that students have already calculated the value of the corresponding definite integral). Unsuspecting that this problem involves definite integrals, students suggest determining the population at sample points within the interval and finding the average of those populations. With that contextual motivation, I guide the students through the development in this Stewart section, including the Mean Value Theorem for Integrals, used in the following instructional activity.
LO 3.3A: Analyze functions defined by an integral.	**Print** Stewart, section 4.3 **Web** Khan Academy, "Proof of Fundamental Theorem of Calculus"	**Instructional Activity: The Fundamental Theorem of Calculus, Form 1** Students work in groups on a worksheet that essentially guides the students through the proof of the FTC, form 1, that is, $\frac{d}{dx}\int_a^x f(t)dt = f(x)$. Students have already conjectured this result from the "Functions Defined by an Integral" instructional activity above. The proof I use is one that can be found in many textbooks, also used by Khan Academy. This proof utilizes the Mean Value Theorem for integrals established in the prior activity. This worksheet guides students through the steps of the proof by asking key questions, and the students within each group help one another through the exercise. Because of the gravity of the theorem and the subtleties involved, I personally present the solutions to the exercises via a PowerPoint presentation.

UNIT 4: INTEGRALS AND THE FUNDAMENTAL THEOREM OF CALCULUS

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
14 instructional hours

Guiding Questions:

▸ How does an indefinite integral relate to a derivative? ▸ What is the difference between the true variable and the dummy variable in a function defined by an integral? ▸ How does a definite integral measure net change? ▸ How can the Fundamental Theorem of Calculus (FTC) be applied to evaluate a definite integral or find the derivative of a function defined by an integral?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.3A: Analyze functions defined by an integral.	**Print** Stewart, section 4.3	**Formative Assessment: Using The Fundamental Theorem of Calculus, Form 1** Students work independently on a problem set inspired by problems 1–18, and especially 59 and 60 in this section of Stewart. Students need to use the chain rule in conjunction with the FTC. In other cases, the graph of a function f is provided, and students must determine information about the graph of $F(x)=\int_a^x f(t)dt$. I circulate and give light instruction and guidance. I intentionally do not use the exact Stewart problems so that those problems can be used in subsequent assignments, reinforcing the learning that occurs in this formative assessment. There are two sets of problems: one basic and one more challenging. All students must complete the basic problem set, but one purpose of the more challenging set is to provide differentiated instruction.	*As in the other formative assessments, I scan and return the worksheets along with solution keys. I use the scans to determine areas to work on for individual students. For example, students often confuse the x values corresponding to the relative extrema of the function in the integrand with the extrema of the function defined by the integral. Students must correct their worksheet with the key and prepare requests for the next day.*
LO 3.1A: Recognize antiderivatives of basic functions. **LO 3.3B(b):** Evaluate definite integrals.	**Print** Stewart, section 4.3	**Instructional Activity: The Fundamental Theorem of Calculus, Form 2** Student groups work on a problem set involving particle motion. Students are given the velocity function corresponding to the motion $v(t)=3t^2+1$. Students use ideas from previous activities to set up the appropriate definite integral corresponding to the distance traveled for $1\le t\le 3$. As done in previous lessons, students evaluate that definite integral by explicitly taking the limit of the Riemann sums. On the subsequent part to this worksheet students find the distance in a totally different manner— by finding the position function $s(t)$, (their first exposure to an antiderivative). When we go over the exercise as a class, students appreciate that $\int_1^3 v(t)dt=s(3)-s(1)$, and the table is set for that evening's assignment, the proof of the FTC, form 2 (a worksheet based on this Stewart section).	*At this point, students understand the definite integral as the limit of the Riemann Sums. They have spent more than a few lessons working with this idea. In this activity, for the first time, students should be mesmerized that there is a connection between this concept and the process of antidifferentiation.*

UNIT 4: INTEGRALS AND THE FUNDAMENTAL THEOREM OF CALCULUS

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
14 instructional hours

Guiding Questions:

▶ How does an indefinite integral relate to a derivative? ▶ What is the difference between the true variable and the dummy variable in a function defined by an integral? ▶ How does a definite integral measure net change? ▶ How can the Fundamental Theorem of Calculus (FTC) be applied to evaluate a definite integral or find the derivative of a function defined by an integral?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.1A: Recognize antiderivatives of basic functions. **LO 3.2C:** Approximate a definite integral. **LO 3.3B(a):** Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals. **LO 3.4A:** Interpret the meaning of a definite integral within a problem. **LO 3.4E:** Use the definite integral to solve problems in various contexts.	**Print** Stewart, section 4.4	**Instructional Activity: Indefinite Integrals and the Net Change Theorem** The lessons in between the previous activity and this one all serve to reinforce the FTC, form 2, that is, $\int_a^b f'(x)dx = f(b) - f(a)$. Assignments and lessons in between the prior activity and this one have students well-rehearsed on this result. In this activity, groups work on today's worksheet with an exercise set bringing them back to a point similar to the previous activity, namely, $\int_a^b v(t)dt = s(b) - s(a)$. The problem set guides students to the more general statement $s(b) = s(a) + \int_a^b v(t)dt$. The worksheet has a modified version of example 7 in this Stewart section regarding the rate of energy consumption in San Francisco on a given day in September. At the end of the activity, students fully appreciate $f(x) \quad = \quad f(a) \quad + \quad \int_a^x f'(t)dt$ End Amt = Start Amt + Net Change.
All of the learning objectives in this unit are addressed.		**Summative Assessment: Integrals and the Fundamental Theorem of Calculus** Students are given 10 fairly short free-response questions directly targeting single learning objectives, and two longer free-response questions that require assimilating the concepts learned in the unit. This is followed by six multiple-choice questions.

This summative assessment addresses all of the guiding questions for the unit.

Mathematical Practices for AP Calculus in Unit 4

The following activities and techniques in Unit 4 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: In the "A Race Against the Computer and the Definition of the Definite Integral" instructional activity students work as a group to develop the definition of the definite integral, building on their understanding of the definition of Riemann sums. Students work with both forms of the Fundamental Theorem of Calculus throughout the second half of this unit.

MPAC 2 — Connecting concepts: In the "Using the Fundamental Theorem of Calculus, Form 1" formative assessment students connect ideas from previous units, such as the chain rule, with the new concepts of functions defined by an integral. In the "Indefinite Integrals and the Net Change Theorem" instructional activity students connect the rectilinear motion ideas from Unit 3 with the new ideas of definite integrals. In using the Fundamental Theorem of Calculus, form 2, students must connect the derivative ideas from Unit 2 with the concept of the definite integral.

MPAC 3 — Implementing algebraic/computational processes: In the "A Race Against the Computer and the Definition of the Definite Integral" instructional activity there is a tremendous amount of algebraic manipulation required to explicitly calculate the limit of the Riemann sums. In the "Evaluating Definite Integrals Using Signed Area Interpretations" formative assessment computational skills of finding the areas of compound regions are being developed.

MPAC 4 — Connecting multiple representations: In the "Riemann Sums" and "Indefinite Integrals and the Net Change Theorems" instructional activities students must connect information given in tabular data form with the definition of definite integrals of functions given in symbolic form. In the "Evaluating Definite Integrals Using Signed Area Interpretations" formative assessment students must connect information obtained from a graph with the analytic definition of definite integrals as the limit of Riemann sums.

MPAC 5 — Building notational fluency: In the "Riemann Sums" and "A Race Against the Computer and the Definition of the Definite Integral" instructional activities there is a tremendous level of notational fluency being developed within these definitions. Through much of the unit, when dealing with the Fundamental Theorem of Calculus, form 1, students must appreciate the distinction between the true variable and the dummy variable in applying $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

MPAC 6 — Communicating: In all the group activities, students must communicate within their groups, and also present to the class using precise mathematical language. This is especially true in the "Functions Defined by an Integral" instructional activity.

UNIT 5: TECHNIQUES OF INTEGRATION

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.1, EU 2.3, EU 3.1, EU 3.3

Estimated Time:
12 instructional hours

Guiding Questions:

- **When trying to find an antiderivative, how does the chain rule work in reverse?** ▸ **In the rule $\int u^n du = \frac{u^{n+1}}{n+1} + c$, what should we do when $n = -1$?** ▸ **How does integrating exponential functions differ from integrating power functions?** ▸ **When trying to find an antiderivative, how can the product rule be used in reverse?** ▸ **How can we integrate rational functions?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.1A: Recognize antiderivatives of basic functions. **LO 3.3B(a):** Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals.	**Print** Foerster, section 5-3 Stewart, section 4.5	**Instructional Activity: Change of Variable *U*-Substitutions and the Chain Rule in Reverse** Students independently work on a worksheet consisting of pairs of problems, such as "differentiate $\sin(x^2)$," and then to find a similar antiderivative, such as $\int x^3 \cos(x^4)dx$. After a student-presented solution, I guide the class through the process of u-Substitutions, as done in the referenced Foerster and Stewart sections, showing the students that if $u = x^4$, then $dx = \frac{1}{4}x^3 du$, resulting in $\frac{1}{4}\int \cos(u)du$. Then I distribute a worksheet of indefinite integral problems similar to the exercises in these two sections. Students work in groups and for each problem, one group works at the board. Each group explains their solution. I summarize, and students are set up nicely for their homework assignment, which has a broad variety of integration problems requiring use of the u-forms they know from the derivative units.

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.1, EU 2.3, EU 3.1, EU 3.3

Estimated Time:
12 instructional hours

Guiding Questions:

- **When trying to find an antiderivative, how does the chain rule work in reverse?** - **In the rule $\int u^n du = \frac{u^{n+1}}{n+1} + c$, what should we do when $n = -1$?** - **How does integrating exponential functions differ from integrating power functions?** - **When trying to find an antiderivative, how can the product rule be used in reverse?** - **How can we integrate rational functions?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.1C: Calculate derivatives. **LO 2.3B:** Solve problems involving the slope of a tangent line. **LO 3.1A:** Recognize antiderivatives of basic functions. **LO 3.3B(a):** Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals.	**Print** Foerster, section 6-2 Stewart, section 6.3	**Instructional Activity: The Integral of the Reciprocal Function and the Derivative of the Logarithmic Function** When students enter the room, the problems on the board are to find $\int x^{-3}dx$, $\int x^{-2}dx$ and $\int x^{-1}dx$. Of course, the students realize the inability of the power rule to find the last integral. We discuss the situation and agree that we need to determine a function whose derivative is $\frac{1}{x}$. Our discussion parallels the Foerster presentation. We follow that up by evaluating $\lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} = \frac{1}{x}$ using log properties and the definition of e. We place the rules of the day, $\frac{d}{dx}\ln x = \frac{1}{x}$ and $\int \frac{1}{u}du = \ln\lvert u\rvert + c$, on the board. Students then collaborate in groups on a worksheet activity with derivative and antiderivative practice using those rules. These exercises utilize the chain rule in both directions and set up the formative assessment for the next lesson. Students present solutions to these problems.

Students are primed for this lesson days in advance. Prior to this lesson, I give assignments reviewing the definition $e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ from their precalculus course, along with basic log and exponent properties.

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.1, EU 2.3, EU 3.1, EU 3.3

Estimated Time:
12 instructional hours

Guiding Questions:

▶ When trying to find an antiderivative, how does the chain rule work in reverse? ▶ In the rule $\int u^n du = \frac{u^{n+1}}{n+1} + c$, what should we do when $n=-1$? ▶ How does integrating exponential functions differ from integrating power functions? ▶ When trying to find an antiderivative, how can the product rule be used in reverse? ▶ How can we integrate rational functions?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.2A: Use derivatives to analyze properties of a function. **LO 2.3B:** Solve problems involving the slope of a tangent line. **LO 2.1C:** Calculate derivatives. **LO 2.3C:** Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion. **LO 3.1A:** Recognize antiderivatives of basic functions. **LO 3.3B(a):** Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals.	**Print** Foerster, section 6-2 Stewart, section 6.3	**Formative Assessment: Applications** Students work in groups on a worksheet involving examples that revisit the referenced derivative applications from Unit 3, this time using the new result $\frac{d}{dx}\ln x = \frac{1}{x}$. For example, students analyze the function $f(x) = \frac{\ln x}{x}$ for extrema and concavity. There is a broad selection of such problems to choose from in the sections referenced from Foerster and Stewart. The worksheet also has problems incorporating the result $\int \frac{1}{u} du = \ln\lvert u\rvert + C$ to area problems from Unit 4. I circulate and provide guidance. Students try to complete the worksheet independently for homework. The next day, groups reconvene and assist one another in comparing and critiquing their work while I circulate and have discussions with the groups. On the second day, after some additional collaboration, each of the problems is presented by a different group.	*This two-day formative assessment allows me to circulate and have discussions with individual students to determine their level of understanding not only of the new rules, but as they apply to previous units. Not only does it provide an opportunity to spiral topics, I also get a good sense of which concepts need to be reinforced in future assignments.*

UNIT 5: TECHNIQUES OF INTEGRATION

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.1, EU 2.3, EU 3.1, EU 3.3

Estimated Time:
12 instructional hours

Guiding Questions:

▶ **When trying to find an antiderivative, how does the chain rule work in reverse?** ▶ **In the rule** $\int u^n du = \frac{u^{n+1}}{n+1} + c$ **, what should we do when** $n = -1$**?** ▶ **How does integrating exponential functions differ from integrating power functions?** ▶ **When trying to find an antiderivative, how can the product rule be used in reverse?** ▶ **How can we integrate rational functions?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.3B: Solve problems involving the slope of a tangent ine. **LO 3.1A:** Recognize antiderivatives of basic functions. **LO 3.3B(a):** Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals.	**Print** Foerster, section 6-4 Stewart, section 6.4	**Instructional Activity: Exponential Functions** Students work independently on three problems. The first is a review problem from Unit 2; any basic problems to review the inverse function theorem, namely, that $\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}$, will do. The second problem asks them to explain why, even though $\frac{d}{dx}x^5 = 5 \cdot x^4$, one should *not* expect that $\frac{d}{dx}5^x = x \cdot 5^{x-1}$. A student presents the solution to the first problem, and then I lead a class conversation about the second. We then solve the third problem, which is to use the Inverse Function Theorem to derive $\frac{d}{dx}e^x = e^x$ establishing $\int e^u du = e^u + C$. Students then collaborate in groups on a worksheet activity with derivative and antiderivative practice using those rules. These exercises utilize the chain rule in both directions and set up the formative assessment for the following lesson. Students present solutions to these problems.

UNIT 5: TECHNIQUES OF INTEGRATION

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.1, EU 2.3, EU 3.1, EU 3.3

Estimated Time:
12 instructional hours

Guiding Questions:

▸ When trying to find an antiderivative, how does the chain rule work in reverse? ▸ In the rule $\int u^n du = \frac{u^{n+1}}{n+1} + c$, what should we do when $n = -1$? ▸ How does integrating exponential functions differ from integrating power functions? ▸ When trying to find an antiderivative, how can the product rule be used in reverse? ▸ How can we integrate rational functions?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.2A: Use derivatives to analyze properties of a function. **LO 2.3B:** Solve problems involving the slope of a tangent line. **LO 2.3C:** Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion. **LO 3.1A:** Recognize antiderivatives of basic functions. **LO 3.3B(a):** Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals.	**Print** Foerster, section 6-4 Stewart, section 6.4	**Formative Assessment: Applications of Exponential Functions** Students work in groups on a worksheet involving examples that revisit the derivative applications from Unit 3, this time using the new result $\frac{d}{dx}e^x = e^x$. For example, students analyze the function $f(x) = \frac{x}{e^x}$ for extrema and concavity. There is a broad selection of such problems from which to choose in the sections referenced from Foerster and Stewart. The worksheet also has problems incorporating the result $\int e^u du = e^u + C$ to area problems from Unit 4. I circulate and provide guidance. Students try to complete the worksheet independently for homework. The following day, groups reconvene and assist one another in comparing work. I circulate and have discussions with the groups. On the second day, after some additional collaboration, each of the problems is presented by a different group.

This two-day formative assessment allows me to circulate and have discussions with individual students to determine their level of understanding, not only on the new rules, but as they apply to previous units. Not only does it provide an opportunity to spiral topics, I also get a good sense of which concepts need to be reinforced in future assignments.

UNIT 5: TECHNIQUES OF INTEGRATION

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.1, EU 2.3, EU 3.1, EU 3.3

Estimated Time:
12 instructional hours

Guiding Questions:

▸ When trying to find an antiderivative, how does the chain rule work in reverse? ▸ In the rule $\int u^n du = \frac{u^{n+1}}{n+1} + c$, what should we do when $n=-1$? ▸ How does integrating exponential functions differ from integrating power functions? ▸ When trying to find an antiderivative, how can the product rule be used in reverse? ▸ How can we integrate rational functions?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.1A: Recognize antiderivatives of basic functions. **LO 3.3B(a):** Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals.	**Print** Stewart, section 7.1	**Instructional Activity: Integration by Parts** The Do Now problem to find $\frac{d}{dx}x^3\cos x$ is on the board, which provides students with a review of the product rule. After we go over the problem, we discuss that, since our derivative rules so far have led to indefinite integral rule counterparts, there should be an integration technique corresponding to the product rule. Then student groups collaborate on a worksheet that guides them through an introduction to the technique of integration by parts. The worksheet is a modified version of the Stewart section referenced. After leading students through $\int x\sin x\,dx$ and $\int \ln x\,dx$, the worksheet asks students to find $\int xe^{2x}\,dx$. After one of the groups presents their solution on the board, the class independently tries one more problem, and a student presents the solution to the class.
LO 3.1A: Recognize antiderivatives of basic functions. **LO 3.3B(a):** Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals.	**Print** Foerster, section 9-7 Stewart, section 7.4	**Instructional Activity: Partial Fractions** There are two Do Now problems on the board. The first is to express $\frac{1}{2x-1}+\frac{3}{x+1}$ as a single fraction, and the second is an integration problem, $\int \frac{7x-2}{2x^2+x-1}dx$. After doing the first problem, students realize that the second problem is "rigged" to be solved using knowledge from the first. Then students groups collaborate on a worksheet that guides them through an introduction to the technique of integration by partial fractions. The worksheet is a modified version of the Foerster section referenced. After leading students through the process and model problem, the worksheet asks students to find $\int \frac{4x+41}{x^2+3x-10}dx$. After one of the groups presents their solution on the board, the class independently tries one more problem, and one student presents the solution to that problem to the class.

UNIT 5: TECHNIQUES OF INTEGRATION

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.1, EU 2.3, EU 3.1, EU 3.3

Estimated Time:
12 instructional hours

Guiding Questions:

- **When trying to find an antiderivative, how does the chain rule work in reverse?** ▸ **In the rule $\int u^n du = \frac{u^{n+1}}{n+1} + c$, what should we do when $n = -1$?** ▸ **How does integrating exponential functions differ from integrating power functions?** ▸ **When trying to find an antiderivative, how can the product rule be used in reverse?** ▸ **How can we integrate rational functions?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.1A: Recognize antiderivatives of basic functions. **LO 3.3B(a):** Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals.		**Formative Assessment: Mixing It Up with Integration** Students collaborate on a worksheet that consists of sets of integrals. In each set, the integrands appear to be similar, but the techniques of integration are different. For example, one set is $\int \frac{e^x}{e^x+1}dx$, $\int \frac{e^{2x}+1}{e^x}dx$, and $\int \frac{e^x}{e^{2x}+1}dx$. Another set is $\int \frac{2x}{x^2+x-2}dx$, $\int \frac{2x+1}{x^2+x+2}dx$, and $\int \frac{1}{x^2+2x+2}dx$. I circulate to look for misconceptions and provide guidance.	*Toward the end of this formative assessment, I collect the sheets and scan them in the classroom sheet-fed scanner for examination later that day. The worksheets are immediately returned to the students and a solution key is provided as the students exit. I use the scans to determine areas to work on for individual students. Students are required to correct their worksheet with the key and prepare requests for the next day.*
All of the learning objectives in this unit are addressed.		**Summative Assessment: Techniques of Integration** Students are given 10 fairly short free-response questions directly targeting single learning objectives and two longer free-response questions that require assimilating the concepts learned in the unit. This is followed by six multiple-choice questions.	*This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 5

The following activities and techniques in Unit 5 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: In the "Change of Variable *U*-Substitutions and the Chain Rule in Reverse" instructional activity students must have mastery of the chain rule to the extent that they can apply it in reverse. The same is true in the "Integration by Parts" instructional activity. In "The Integral of the Reciprocal Function and the Derivative of the Logarithmic Function" instructional activity we use the definition of derivative to develop our new derivative result. In the "Exponential Functions" instructional activity we use the Inverse Function Theorem and the definition of inverse functions to determine the derivative of the exponential function.

MPAC 2 — Connecting concepts: All of the activities following the first reinforce u-substitutions with the newer integration techniques. The first two formative assessments in this unit spiral back to curve-sketching concepts and other derivative applications from Unit 3 in the context of the newly learned derivative rules.

MPAC 3 — Implementing algebraic/computational processes: In the "Partial Fractions" instructional activity students review and strengthen their algebraic skills involving rational functions. Throughout the unit, students use their algebra skills to manipulate the integrands into familiar u-forms.

MPAC 4 — Connecting multiple representations: Once the rules for the logarithmic and exponential functions have been established, assignments can contain problems that revisit ideas from all the previous units that use graphical and tabular stems, except this time the corresponding functions are logarithmic or exponential. There are many such problems in the Foerster sections referenced in this unit.

MPAC 5 — Building notational fluency: Beginning with the first instructional activity in this unit, "Change of Variable *U*-Substitutions and the Chain Rule in Reverse," students increase their fluency in differential notation with the u-substitutions. Starting with the second instructional activity, "The Integral of the Reciprocal Function and the Derivative of the Logarithmic Function," and in the subsequent problem sets, students continue to learn the significance of the different variables in Fundamental Theorem of Calculus in appreciating that $\ln x = \int_a^x \frac{1}{t}\,dt$.

MPAC 6 — Communicating: Once the derivative results for the logarithmic and exponential functions have been established, homework assignments can contain problems involving rates of change of such functions and interpreting these derivatives in a variety of contexts, except this time the functions are exponential and logarithmic. Students must articulate their interpretations in such problems, which are abundant in the referenced sections of Stewart and Foerster.

UNIT 6: APPLICATIONS OF INTEGRATION

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 1.1, EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
14 instructional hours

Guiding Questions:

▸ How can integrals be used to measure accumulation? ▸ How can the definite integral be used to calculate volumes and arc lengths? ▸ How can L'Hospital's Rule help us to evaluate limits involving indeterminate forms? ▸ What are improper integrals, and how can they be evaluated?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.3B(a): Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals. **LO 3.4D:** Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.	**Print** Stewart, section 5.1	**Instructional Activity: Areas Between Curves** Students work independently to find the area bounded by the graphs of $f(x)=4-x^2$ and $g(x)=-5$. At this point in their development, students may still think of definite integrals as providing area "under a curve," so they are often inclined to unnecessarily split this region into sub regions. I walk around and provide guidance. A student who solved the problem with the single definite integral $\int_{-3}^{3}(f(x)-g(x))dx$ presents the solution at the board. The idea of summing up infinitely many infinitesimally small products is reinforced. Students are given a worksheet with additional problems, including regions where they have the option to produce a single definite integral with respect to y or integrate with respect to x, requiring two integrals, such as the region R, bounded by the graphs $y=\frac{1}{x^2}$, $y=4$, and $y=x$.	*The last example referenced in this activity allows for a rich discussion about how definite integrals really sum up products, and so they are indeed the limit of Riemann sums. Seeing the same problem from both a Δx perspective (two definite integrals) and then a Δy perspective (one definite integral) is a revealing experience for students.*
LO 3.3B(a): Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals. **LO 3.4D:** Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.	**Print** Stewart, section 5.2 **Web** Manipula Math with Java, "Volume of a Solid"	**Instructional Activity: Volumes of Solids Whose Cross Sections Are Known** The classic problem on the board describes $x^2+y^2=1$ as the base of a solid whose cross-sections perpendicular to the x-axis are equilateral triangles. Many students have difficulty visualizing the solid, so we show the picture from the Stewart section as well as the Java applet referenced. We develop the Riemann sum approximation with four slices, and then take the limit of the general Riemann sum. Students work in groups on a similar problem from Stewart. This is a two-period activity — in the follow-up, students find volumes of solids of revolution in a similarly formatted lesson. In the lesson summary we reemphasize that a definite integral is the tool for summing up products. The volume of these solids equals $\int_a^b A(x)dx$, where $A(x)$ represents the cross-sectional area for $a \le x \le b$.	*The referenced applet, "Volume of a Solid," requires the most up-to-date version of Java plug-in installed on your browser.*

UNIT 6: APPLICATIONS OF INTEGRATION

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 1.1, EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
14 instructional hours

Guiding Questions:

▸ How can integrals be used to measure accumulation? ▸ How can the definite integral be used to calculate volumes and arc lengths? ▸ How can L'Hospital's Rule help us to evaluate limits involving indeterminate forms? ▸ What are improper integrals, and how can they be evaluated?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.3B(a): Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals. **LO 3.4D:** Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.	**Print** Stewart, section 8.1	**Instructional Activity: Arc Length** Student groups complete a worksheet that provides a diagram showing the graph of $y=x^{1.5}$ for $1\le x\le 4$. Students approximate the length of this curve by partitioning the interval $1\le x\le 4$ into three subintervals and creating three line segments. They use the Pythagorean theorem to obtain their approximation. A group presents their solution and I ask how we can improve the approximation and, eventually, how we can find the actual arc length. Following the development in the referenced Stewart section, we develop the general Riemann sum and corresponding integral. Groups then work on a similar problem from this section, and one group presents a solution. We summarize and point out that we are summing up products as the arc length is the limit of the Riemann sums.
LO 3.3A: Analyze functions defined by an integral. **LO 3.3B(a):** Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals. **LO 3.4C:** Apply definite integrals to problems involving motion.	**Print** Foerster, section 10-1 Stewart, section 4.4	**Instructional Activity: Particle Motion Revisited** Student groups work on a sheet of problems revisiting the particle motion ideas from Unit 3. I circulate and provide reminders as necessary. Among the problems, students must find the total distance traveled by a particle over an interval when the velocity function is given. One of the groups presents a solution (by finding the position function and using ideas from Unit 3) as we review the concepts. I pose the question, "Is it possible to determine the total distance traveled without actually finding the position function?" We follow the developments of the referenced Foerster and Stewart sections to conclude that $\int_a^b v(t)\,dt$ represents displacement, while $\int_a^b \lvert v(t)\rvert\,dt$ represents total distance.

This is a perfect opportunity to reinforce the particle motion idea statement $s(b)=s(a)+\int_a^b v(t)\,dt$ *and more generally,*
$f(x)=f(a)+\int_a^x f'(t)\,dt$

End Amt = Start Amt + Net Change.

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 1.1, EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
14 instructional hours

Guiding Questions:

▸ How can integrals be used to measure accumulation? ▸ How can the definite integral be used to calculate volumes and arc lengths? ▸ How can L'Hospital's Rule help us to evaluate limits involving indeterminate forms? ▸ What are improper integrals, and how can they be evaluated?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.3A: Analyze functions defined by an integral. **LO 3.3B(a):** Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals. **LO 3.4A:** Interpret the meaning of a definite integral within a problem. **LO 3.4E:** Use definite integrals to solve problems in various contexts.	**Print** Stewart, section 4.3 **Web** AP Calculus AB 2002 Free-Response Question 2	**Instructional Activity: The Integral as an Accumulator** The Do Now exercise is AP Calculus AB 2002 Free-Response Question 2. Students work independently while I circulate and provide guidance whenever students are confused about applying the net change idea in context. One student presents the solution. As we review the solution, we reinforce the net change idea and the idea that a definite integral is a tool to sum up products. Now, however, we can see contextually that a function of the type $f(x)=f(a)+\int_a^x f'(t)dt$ accumulates change because the definite integral $\int_a^x f'(t)dt$ sums up the appropriate products of rate*time. Groups work on similar problems from the referenced Stewart section. Some groups work at the board and present their solutions. Then I summarize that this is an example of one of our main themes: $f(x)=f(a)+\int_a^x f'(t)dt$ End Amt = Start Amt + Net Change.

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 1.1, EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
14 instructional hours

Guiding Questions:
▸ How can integrals be used to measure accumulation? ▸ How can the definite integral be used to calculate volumes and arc lengths? ▸ How can L'Hospital's Rule help us to evaluate limits involving indeterminate forms? ▸ What are improper integrals, and how can they be evaluated?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.3A: Analyze functions defined by an integral. **LO 3.3B(a):** Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals. **LO 3.4A:** Interpret the meaning of a definite integral within a problem. **LO 3.4C:** Apply definite integrals to problems involving motion. **LO 3.4D:** Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve. **LO 3.4E:** Use definite integrals to solve problems in various contexts.	**Print** Foerster, section 10.1 Stewart, section 4.3, 5.2, and 8.1	**Formative Assessment: Summing up Infinitely Many Infinitesimally Small Products** Students work in groups on an exercise set taken from the referenced textbook sections with applications from each of the prior activities of this unit. The theme of the worksheet is summing up infinitely many infinitesimally small products. Some of the problems permit the use of a calculator, and some require that the students evaluate their definite integrals by hand using the FTC and reinforcing the integration techniques. I circulate around the room and provide light guidance, but mainly I listen for student misconceptions and try to watch their interaction to assess understanding. I have conversations with individual students within each group to help me with this assessment of their understanding.

Toward the end of this formative assessment, I collect the worksheets and scan them in the classroom sheet-fed scanner for examination later that day. The worksheets are immediately returned to the students and a solution key is provided as the students exit. I use the scans to determine areas to work on for individual students. Students are required to correct their worksheet with the key and prepare requests for the next day.

UNIT 6: APPLICATIONS OF INTEGRATION

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 1.1, EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
14 instructional hours

Guiding Questions:

▶ How can integrals be used to measure accumulation? ▶ How can the definite integral be used to calculate volumes and arc lengths? ▶ How can L'Hospital's Rule help us to evaluate limits involving indeterminate forms? ▶ What are improper integrals, and how can they be evaluated?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 1.1C: Determine limits of functions.	**Print** Stewart, section 6.8	**Instructional Activity: Indeterminate Forms and L'Hospital's Rule** Student groups collaborate on a worksheet whose first problem is to use the principle of local linearity to approximate $\sqrt{36.2}$. The second problem asks them to evaluate limits involving the indeterminate forms $\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty-\infty$. Students use techniques from Unit 1 involving algebraic manipulations to evaluate the limits. I circulate and at times interrupt the class to provide review and guidance. Students present their solutions, and then I present the limit problem $\lim\limits_{x\to 0}\frac{2e^x-4x-2}{\sin x}$, which they are unable to solve using the old techniques. I project the diagram from this Stewart section onto the board and use the principal of local linearity reviewed in the first Do Now problem. I then lead the class through the development and necessary conditions in Stewart for the technique of L'Hospital's Rule.
LO 1.1C: Determine limits of functions. **LO 3.3B(a):** Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals. **L0 3.2D: (BC)** Evaluate an improper integral or show that an improper integral diverges.	**Print** Stewart, section 7.8	**Instructional Activity: Improper Integrals** Students work independently on a Do Now problem that has them fill in the chart of values (no calculator) for $\int_0^m \frac{1}{1+x^2}dx$, where $m=0,\frac{\sqrt{3}}{3},1,\sqrt{3}$. After a student presents a solution, I lead the class into suggesting that there may be a limit that the value of the definite integral approaches as the value of m increases without bound. I lead the class through questions that lead to $\int_0^\infty \frac{1}{1+x^2}dx = \lim\limits_{m\to\infty}\int_0^m \frac{1}{1+x^2}dx$. Student groups then complete a worksheet guiding them through the examples in the referenced Stewart section.

This evening's homework assignment not only has problems similar to the ones encountered in class, but other problems from the referenced Stewart section that require manipulating the expressions into the $\frac{0}{0}$ and $\frac{\infty}{\infty}$ forms from the other indeterminate forms. I encourage students to try the problems and then look at the model problems provided in this Stewart section.

During this activity, students are always intrigued when our motivating example leads to the fact that $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx=\pi$.

UNIT 6: APPLICATIONS OF INTEGRATION

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 1.1, EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
14 instructional hours

Guiding Questions:
▸ How can integrals be used to measure accumulation? ▸ How can the definite integral be used to calculate volumes and arc lengths? ▸ How can L'Hospital's Rule help us to evaluate limits involving indeterminate forms? ▸ What are improper integrals, and how can they be evaluated?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 1.1C: Determine limits of functions. **LO 3.1A:** Recognize antiderivatives of basic functions. **LO 3.3B(b):** Calculate definite integrals. **LO 3.2D: (BC)** Evaluate an improper integral or show that an improper integral diverges.		**Formative Assessment: Techniques of Integration Applied to Improper Integrals** Students work in groups on an exercise set consisting of improper integrals corresponding to both horizontal and vertical asymptotes. The idea is to reinforce the improper integral ideas from the previous lessons, while spiraling back to integration techniques from Unit 5. The problems require techniques of integration from Unit 5, such as *u*-substitutions and integration by parts, as well as partial fractions. Examples include $\int_1^\infty \frac{x}{e^{x^2}}dx$, $\int_1^\infty \frac{x}{e^x}dx$, $\int_4^\infty \frac{1}{x^2-x-6}dx$, and $\int_0^1 \frac{\ln x}{x}dx$. I circulate around the room and provide light guidance, but mainly I listen for student misconceptions and try to watch their interaction to assess understanding. I have conversations with individual students within each group to help me with this assessment of their understanding.	*Toward the end of this formative assessment, I collect the worksheets and scan them in the classroom sheet-fed scanner for examination later that day. The worksheets are immediately returned to the students and a solution key is provided as the students exit. I use the scans to determine areas to work on for individual students. Students are required to correct their worksheet with the key and prepare requests for the next day.*
All of the learning objectives in this unit are addressed.		**Summative Assessment: Applications of Integration** Students are given 10 fairly short free-response questions directly targeting single learning objectives and two longer free-response questions that require assimilating the concepts learned in the unit. This is followed by six multiple-choice questions.	*This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 6

The following activities and techniques in Unit 6 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: In nearly all of the activities, students work with the definition of the definite integral as a limit of Riemann sums. In the "Improper Integrals" instructional activity students devise and apply the definition of an improper integral as the limit of the definite integral. At the very start of that activity, as through much of the unit, students use the Fundamental Theorem of Calculus to evaluate definite integrals.

MPAC 2 — Connecting concepts: In the "Particle Motion Revisited" instructional activity students connect rectilinear motion from a derivative perspective with the new integration concepts, including that of net change.

MPAC 3 — Implementing algebraic/computational processes: In nearly all the applications of the definite integral, students need to use algebraic and computational skills in finding antiderivatives and to perform the appropriate calculations.

MPAC 4 — Connecting multiple representations: After the "Particle Motion Revisited" instructional activity we are able to give homework assignments that consist of problems in which the motion of the particle is given in tabular and graphical forms.

MPAC 5 — Building notational fluency: In working with the improper integral definition $\int_0^{\infty} f(x)dx = \lim_{m\to\infty}\int_0^m f(x)dx$, students deal with intricate notation involving the limit of a definite integral, which is itself a limit. In dealing with accumulation functions during activities and assessments, there is a great deal of fluency to be gained in the abstract equation $f(x) = f(a) + \int_a^x f'(t)dt$.

MPAC 6 — Communicating: In all the group activities, students must communicate within their groups as well as present to the class using precise mathematical language. This is especially true in the "Summing up Infinitely Many Infinitesimally Small Products" formative assessment as well as in the instructional activity "The Integral as an Accumulator."

UNIT 7: PARAMETRIC EQUATIONS AND POLAR CURVES

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.1, EU 2.2, EU 2.3, EU 3.3, EU 3.4

Estimated Time:
13 instructional hours

Guiding Questions:

- **How can we find the slope of a parametrically defined curve or a polar curve?** ▶ **How can we extend the ideas of rectilinear motion to describe behavior of motion along a planar curve?** ▶ **How can we use definite integrals to find the length of a parametrically defined curve?** ▶ **How can we find areas in polar coordinates?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.1C: Calculate derivatives.	**Print** Foerster, section 4-7 Stewart, section 10.1	**Instructional Activity: Parametric Equations** Students fill out a chart for t, x, and y values for the equations $\begin{cases} x = t^2 - 2t \\ y = t+1 \end{cases}$, and then plot the corresponding points. When students are done we present a PowerPoint presentation of the referenced Stewart section. The presentation goes quickly because parametric equations is a precalculus topic in our school. After the presentation, student groups collaborate on a worksheet with 15 parametrically defined curves. Students need to eliminate the parameter and sketch the curves. These curves involve trigonometric, exponential, and logarithmic relationships. Examples include curves with restrictions such as $\begin{cases} x = \sin t \\ y = \cos 2t \end{cases}$ and $\begin{cases} x = 2\sec t \\ y = \tan t \end{cases}$, as well as $\begin{cases} x = 2^t \\ y = 2^{-t} \end{cases}$. I circulate and provide guidance. Students complete the worksheet as their homework assignment and we go over it in the next lesson.
LO 2.1C: Calculate derivatives. **LO 2.3C:** Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion.	**Print** Foerster, section 10-6 Stewart, section 10.2	**Instructional Activity: Introduction to Curvilinear Motion** I write a problem on the board that describes the planar motion of a particle. Its position at time t is given by $\begin{cases} x = 1+3t \\ y = 2+4t \end{cases}$. Students find the position at time $t = 0,1,2,3,4$. They sketch the path. This example motivates the idea of the velocity vector and the notion of speed as its magnitude. We introduce the concept of the acceleration vector. Then student groups collaborate on a worksheet with particles moving along parametric curves described in the previous activity. They find velocity and acceleration vectors as well as speed at specified times. I ask students to interpret their results in the context of the motion. For each problem, one group works at the board and presents solutions when everyone is finished.

I usually emphasize to my students that restrictions on t do not impact the graph, but restrictions on x and y do. For example, I would mention to the class that in the second parametrically defined curve, t cannot equal $\frac{\pi}{2}$, yet that has no impact on the curve.

Prior to this lesson, assignments include rectilinear motions problems. By the time students get to this lesson, students should be ready to extend these ideas learned in previous units to motion along a curve.

UNIT 7: PARAMETRIC EQUATIONS AND POLAR CURVES

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.1, EU 2.2, EU 2.3, EU 3.3, EU 3.4

Estimated Time:
13 instructional hours

Guiding Questions:

▸ How can we find the slope of a parametrically defined curve or a polar curve? ▸ How can we extend the ideas of rectilinear motion to describe behavior of motion along a planar curve? ▸ How can we use definite integrals to find the length of a parametrically defined curve? ▸ How can we find areas in polar coordinates?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.1C: Calculate derivatives. **LO 2.3B**: Solve problems involving the slope of a tangent line. **LO 2.3C**: Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion. **LO 3.4C**: Apply definite integrals to problems involving motion. **LO 3.4D**: Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.	**Print** Foerster, section 10-6 Stewart, section 10.2	**Instructional Activity: Arc Length for Parametrically Defined Curves** Student groups collaborate on a worksheet that provides yet another exercise working with position, velocity and acceleration vectors, and the notion of speed as the magnitude of velocity. This time the particle travels along the curve from example 1 in the referenced Stewart section. The worksheet guides students through the examples of finding first and second derivatives of the parametrically defined curve. In a second exercise, the particle travels along the parametrically defined curve from example 4 from the referenced Stewart section. This time, the worksheet guides students to find the length of the path traveled following the development from this Stewart section. I circulate and clarify, and after a group presents a solution, I summarize the lesson.

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.1, EU 2.2, EU 2.3, EU 3.3, EU 3.4

Estimated Time:
13 instructional hours

Guiding Questions:

- **How can we find the slope of a parametrically defined curve or a polar curve?** ▸ **How can we extend the ideas of rectilinear motion to describe behavior of motion along a planar curve?** ▸ **How can we use definite integrals to find the length of a parametrically defined curve?** ▸ **How can we find areas in polar coordinates?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.1C: Calculate derivatives. **LO 2.3C:** Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion. **LO 3.3B(b):** Evaluate definite integrals. **LO 3.4C:** Apply definite integrals to problems involving motion. **LO 3.4D:** Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.	**Web** AP Calculus BC 2003 Free-Response Question 3	**Formative Assessment: Applications with Curvilinear Motion** Students work independently on a worksheet. The first problem is from the AP Calculus BC Exam Free-Response Question 3, which emphasizes calculator-active applications. Students must interpret derivatives, and this exercise spirals back to some old ideas such as net change. The flip side of the worksheet involves a few "grind-it-out," traditional problems where the integration must be done by hand, such as finding the length of the arc of the curve $\begin{cases} x = \cos^3 t \\ y = \sin^3 t \end{cases}$. I circulate, occasionally interrupting the class to provide light instruction.

Toward the end of this formative assessment, I collect the worksheets and scan them in the classroom sheet-fed scanner for examination later that day. The worksheets are immediately returned to the students and a solution key is provided as the students exit. I use the scans to determine areas to work on for individual students. Students are required to correct their worksheet with the key and prepare requests for the next day.

UNIT 7: PARAMETRIC EQUATIONS AND POLAR CURVES

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.1, EU 2.2, EU 2.3, EU 3.3, EU 3.4

Estimated Time:
13 instructional hours

Guiding Questions:

▶ How can we find the slope of a parametrically defined curve or a polar curve? ▶ How can we extend the ideas of rectilinear motion to describe behavior of motion along a planar curve? ▶ How can we use definite integrals to find the length of a parametrically defined curve? ▶ How can we find areas in polar coordinates?

Learning Objectives	Materials	Instructional Activities and Assessments	
(See notes to the right of this instructional activity.)	**Print** Stewart, section 10.3	**Instructional Activity: Polar Coordinate Curves** Students fill out a chart for the equation $r = 2\sin 3\theta$ for specified values of θ. I lead the class through a review of graphing in polar coordinates from their precalculus course. We graph $r = 2\sin 3\theta$ in Cartesian coordinates, with r as the vertical axis and θ as the horizontal axis. We analyze that graph and use it to draw the polar version. Students then collaborate on a worksheet to sketch $r = 3\sin 2\theta$, $r = 2 + 4\sin\theta$, and $r = 6\sec\theta$. Of course, the latter can't be drawn using the method we reviewed, but after students present the first two, we review the conversion formulas $x = r\cos\theta$, $y = r\sin\theta$, and $r^2 = x^2 + y^2$. This way, students review that another method to sketch curves is to convert the equations into Cartesian form. We check by graphing the curves on the graphing calculator.	*This activity supports the learning objectives involving the calculus of polar curves in future activities. We are setting the stage toward finding the range of θ values that generate portions of polar graphs for which we will soon be finding areas. Working with the conversion formulas sets the stage for finding slopes of tangent lines in polar form.*
LO 3.3B(b): Evaluate definite integrals. **LO 3.4D:** Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.	**Print** Stewart, section 10.4	**Instructional Activity: Areas in Polar Coordinates** Students work on a Do Now worksheet independently. The first problem goes back to Unit 4 asking for a Riemann sum approximation for the area of the region under $y = x^2$ between $x = 1$ and $x = 3$. The second problem is to approximate the area of $r = \theta^2$ between $\theta = 1$ and $\theta = 3$ by dividing the regions into four sectors. Going over this problem as a class leads to an interesting discussion about why $\int_1^3 x^2\,dx$ corresponds to the first area, while $\int_1^3 \theta^2\,d\theta$ does *not* correspond to the second area. I lead the class through the derivation of the formula $\frac{1}{2}\int_a^b r^2\,d\theta$, following the development in the Stewart section. Students then work in groups on a problem sheet that requires them to find areas of regions sketched in the prior activity, including the area from the Do Now worksheet.	*This lesson gives us yet another opportunity to emphasize that a definite integral sums up infinitely many infinitesimally small products.*

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Derivatives
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Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.1, EU 2.2, EU 2.3, EU 3.3, EU 3.4

Estimated Time:
13 instructional hours

Guiding Questions:

- How can we find the slope of a parametrically defined curve or a polar curve? ▸ How can we extend the ideas of rectilinear motion to describe behavior of motion along a planar curve? ▸ How can we use definite integrals to find the length of a parametrically defined curve? ▸ How can we find areas in polar coordinates?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.3B(b): Evaluate definite integrals. **LO 3.4D:** Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.	**Print** Stewart, section 10.4	**Instructional Activity: Areas Between Curves** Student groups collaborate on a worksheet with two problems. One is to find the area of the region inside the circle $r = 3\sin\theta$ and outside the limacon $r = 2 - \sin\theta$. The second problem is to find the area of the region common to the graphs of the circles $r = 2\sin\theta$ and $r = 2\cos\theta$. I select one group to present each solution, and as they present we have an interesting class discussion about why we are subtracting integrals for the first problem, yet we are adding integrals for the second. I allow students to use the graphing calculators to evaluate the definite integrals so we can emphasize the concepts. The following activity forces them to practice their integration skills learned in the previous units.
LO 3.3B(b): Evaluate definite integrals. **LO 3.4D:** Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.	**Print** Stewart, section 10.4 **Web** AP Calculus BC 2007 Free-Response Question 3	**Instructional Activity: Mixing It Up with Polar Coordinates, Curvilinear Motion, and Related Rates** Student groups collaborate on a worksheet. The first side is a problem based on AP Calculus BC 2007 Free-Response Question 3. The problem deals with polar curves in a particle motion context and incorporates related rates ideas. Students are allowed to use their graphing calculators for this portion of the worksheet. On the flip side, students have to calculate areas of regions bounded by polar curves, except, unlike the previous activity, students need to integrate by hand. I circulate and provide guidance and have several students work at the board to present their solutions.

This is a great opportunity to reinforce the meaning of the definite integral as the limit of Riemann sums. I emphasize to students that when finding areas in rectangular coordinates, we evaluate $\int_a^b f(x)dx$ because the Δx values in the corresponding Riemann sums represent widths of rectangles. In the integral $\frac{1}{2}\int_a^b (f(\theta))^2 d\theta$, the $\Delta\theta$ values in the corresponding Riemann sums represent angles of triangles, or sectors.

At this point in the school year, it is early February. I've found that it's beneficial to give some of these "put-it-all-together" worksheets well before the "official" review for the AP exam. There are so many great old AP problems that serve to help instruct the student that it would be a waste of a fantastic resource if we didn't find opportunities to use some of them well before that official review.

UNIT 7: PARAMETRIC EQUATIONS AND POLAR CURVES

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.1, EU 2.2, EU 2.3, EU 3.3, EU 3.4

Estimated Time:
13 instructional hours

Guiding Questions:

▶ How can we find the slope of a parametrically defined curve or a polar curve? ▶ How can we extend the ideas of rectilinear motion to describe behavior of motion along a planar curve? ▶ How can we use definite integrals to find the length of a parametrically defined curve? ▶ How can we find areas in polar coordinates?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.1C: Calculate derivatives. **LO 2.2A:** Use derivatives to analyze the properties of a function.	**Print** Stewart, section 10.3	**Instructional Activity: Tangent Lines to Polar Curves** Student groups collaborate on a worksheet that guides them through a procedure to find the slope of lines tangent to a polar curve at a given point. The sheet follows the development of the referenced Stewart section. I interrupt to show a PowerPoint presentation of this section. Student groups resume working on the flip side of this worksheet, which has two problems. One is to find the slope of the line tangent to the curve. The other is to find the points on the curve $r = 1 + \sin\theta$ where the tangent line is either horizontal or vertical. I have two of the groups work at the board, and they present their solutions. After some dialogue with the class during the group presentation, I summarize the day's results.	
All of the learning objectives in this unit are addressed.		**Summative Assessment: Parametric Equations and Polar Curves** Students are given 10 fairly short free-response questions directly targeting single learning objectives and two longer free-response questions that require assimilating the concepts learned in the unit. This is followed by six multiple-choice questions.	*This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 7

The following activities and techniques in Unit 7 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: In the "Arc Length for Parametrically Defined Curves" and "Areas in Polar Coordinates" instructional activities students must reason with the definition of the definite integral at the limit of the Riemann sums to develop the appropriate formulas. In the "Tangent Lines to Polar Curves" instructional activity students must reason with the chain rule to develop the appropriate slopes.

MPAC 2 — Connecting concepts: In the instructional activities involving polar area and in those involving arc length, we reinforce u-substitutions and other integration techniques. In the "Mixing It Up with Polar Coordinates, Curvilinear Motion, and Related Rates" instructional activity students combine all these ideas from different sections.

MPAC 3 — Implementing algebraic/computational processes: Throughout the unit, student use their algebra skills to manipulate the integrands into familiar U-forms. In the "Tangent Lines to Polar Curves" instructional activity students determine the points at which the tangent lines are vertical while they engage in some intense algebra, trigonometry, and computation.

MPAC 4 — Connecting multiple representations: Throughout the unit, students use the parametric and polar graphs in coordination with the definite integrals. Once students know how to determine the slopes of polar curves, that opens the door for problems in which they are given information in tabular form about θ , r, and $\frac{dr}{d\theta}$.

MPAC 5 — Building notational fluency: Students need to work well with Riemann sums as they develop formulas for arc length and area in polar coordinates.

MPAC 6 — Communicating: In the "Applications with Curvilinear Motion" instructional activity students must communicate their interpretation of the derivatives in the context of motion. After polar coordinates have been introduced, the homework assignment can contain problems in which students determine the maximum distance from a point on the curve to the origin. Such optimization problems require careful communication to justify conclusions.

UNIT 8: DIFFERENTIAL EQUATIONS

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.1, EU 2.3, EU 3.1, EU 3.3, EU 3.5

Estimated Time:
13 instructional hours

Guiding Questions:

▶ How do we solve separable differential equations? ▶ How does a slope field help us to analyze solution curves to differential equations? ▶ How does Euler's method help us to find approximations for solutions to differential equations? ▶ How does the logistic growth model differ from the exponential growth model?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.3E: Verify solutions to differential equations. **LO 3.1A:** Recognize antiderivatives of basic functions. **LO 3.5A:** Analyze differential equations to obtain general and specific solutions.	**Print** Stewart, sections 9.1 and 9.2	**Instructional Activity: What Is a Differential Equation?** When students enter the room, there is a true/false question on the board: "If $y=\sqrt[3]{x^3+8}$, then $\frac{dy}{dx}=\frac{x^2}{y^2}$." After I lead the class though verification, we define the term "differential equation" (DE). We follow the development in Stewart section 9.2 regarding solving such DEs, and discuss the notions of "general solution" and "particular solution." We solve and check some of the other DEs in the Stewart sections, setting the class up for the subsequent activity.
LO 3.1A: Recognize antiderivatives of basic functions. **LO 3.3B(b):** Evaluate definite integrals. **LO 3.5A:** Analyze differential equations to obtain general and specific solutions.	**Print** Stewart, section 9.2	**Formative Assessment: Solving Separable Differential Equations** Students work in groups on a sheet that contains a dozen separable DEs. The first is to solve the DE $x^2 \cdot f'(x)=f(x)$, subject to the initial condition $f(1)=\frac{2}{e}$. I ask students to check that their solution indeed satisfies the DE. The subsequent problems provide practice in various u-substitutions, integration by part, and partial fractions. I circulate and provide guidance. Students try to complete the worksheet independently for homework. The following day, groups reconvene and assist one another in comparing and critiquing their work while I circulate and have discussions with the groups. On the second day, after some additional collaboration, a different group presents each of the problems.

This two-day formative assessment allows me to circulate and have discussions with individual students to determine their level of understanding of the integration techniques covered in our course, not only on the new rules, but as they apply to previous units. Not only does it provide an opportunity to spiral topics, but I also get a good sense of which concepts need to be reinforced in future assignments.

UNIT 8: DIFFERENTIAL EQUATIONS

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.1, EU 2.3, EU 3.1, EU 3.3, EU 3.5

Estimated Time:
13 instructional hours

Guiding Questions:

▶ How do we solve separable differential equations? ▶ How does a slope field help us to analyze solution curves to differential equations? ▶ How does Euler's method help us to find approximations for solutions to differential equations? ▶ How does the logistic growth model differ from the exponential growth model?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.1C: Calculate derivatives. **LO 2.3F:** Estimate solutions to differential equations. **LO 2.3C:** Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion. **LO 3.3B(b):** Evaluate definite integrals.	**Print** Foerster, section 7-4 Stewart, section 9.2	**Instructional Activity: Slope Fields** There are two DEs on the board. The first is a separable differential equation in the spirit of the previous activity. The second is the DE $\frac{dy}{dx}=x+y$. Once students come to realize that the second equation is not separable, I lead the class though a PowerPoint presentation introducing slope fields. The presentation is a merge of the referenced Stewart and Foerster sections. Following this, student groups collaborate on a worksheet. The first task is to draw a slope field for $x\cdot\frac{dy}{dx}=(y-1)^2$ on the provided grid. The second is a matching exercise containing six DEs that have to be matched with their slope fields. We go over the problems and then I provide a summary.
LO 2.3F: Estimate solutions to differential equations.	**Print** Foerster, section 7-5	**Instructional Activity: Euler's Method** Students collaborate on a worksheet that revisits the DE $\frac{dy}{dx}=x+y$. The computer-generated slope field from the previous activity appears next to the equation, and students are asked to draw a solution curve to this DE satisfying initial condition $f(0)=1$. The worksheet walks the students through the development of Euler's method from this Foerster section, culminating in an approximation for $f(.5)$. We go over the worksheet and summarize the technique. The student groups collaborate on a problem from the Foerster resource guide, which is a modified version of problem 1 from the exercise set in this Foerster section. The problem reinforces both the slope field concept as well as Euler's method. After a group-presented solution, we write a TI-89 program for Euler's method.

Although the AP Exam is constructed so as to avoid an advantage for students who already have a Euler's method program on their calculators, the exercise of writing the program reinforces the process and is conducive to a deeper student understanding of the technique.

UNIT 8: DIFFERENTIAL EQUATIONS

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.1, EU 2.3, EU 3.1, EU 3.3, EU 3.5

Estimated Time:
13 instructional hours

Guiding Questions:

▸ **How do we solve separable differential equations?** ▸ **How does a slope field help us to analyze solution curves to differential equations?** ▸ **How does Euler's method help us to find approximations for solutions to differential equations?** ▸ **How does the logistic growth model differ from the exponential growth model?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.5A: Analyze differential equations to obtain general and specific solutions. **LO 3.5B**: Interpret, create, and solve differential equations in context.	**Print** Foerster, section 7-2	**Instructional Activity: Exponential Growth and Decay** Student groups collaborate on a guided worksheet that is a modified version of this Foerster section. This is the students' first exposure to translating a phrase such as "The rate of change of the population is proportional to the population" into mathematical notation. Students are guided to solving example 1 from this Foerster section, and one of the groups presents their solution. The problem on the flip side of the worksheet is example 2 from this section. Students test themselves by working individually on the problem of setting up and solving this DE based on the verbal description given. There are follow-up questions for the students to make predictions based on their solution to the DE. After a student-presented solution, I summarize.
LO 3.3B(b): Evaluate definite integrals. **LO 3.5A**: Analyze differential equations to obtain general and specific solutions. **LO 3.5B**: Interpret, create, and solve differential equations in context.	**Print** Foerster, section 7-3	**Instructional Activity: Additional Modeling with Differential Equations** Student groups collaborate on a worksheet that is a modified version of this Foerster section. This activity is the natural follow-up to the prior activity; the perfect one, two punch for introducing the solution of DEs in context. In this exercise, students are guided through setting up and solving DEs based on a contextual verbal description. For each of the first two problems in this section, I have student groups present their solution. After going over the problems, students work individually on the third problem from this section. After a student-presented solution, I summarize how different phrases may be interpreted to help form a DE and how to solve DEs when the variables are separable.

After this lesson, the door is opened to give assignments consisting of problems in both the Stewart and Foerster chapters in which students solve DEs in context, such as problems involving Newton's law of cooling.

UNIT 8: DIFFERENTIAL EQUATIONS

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.1, EU 2.3, EU 3.1, EU 3.3, EU 3.5

Estimated Time:
13 instructional hours

Guiding Questions:

- **How do we solve separable differential equations?** - **How does a slope field help us to analyze solution curves to differential equations?** - **How does Euler's method help us to find approximations for solutions to differential equations?** - **How does the logistic growth model differ from the exponential growth model?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.3B(b): Evaluate definite integrals. **LO 3.5A:** Analyze differential equations to obtain general and specific solutions. **LO 3.5B:** Interpret, create, and solve differential equations in context.	**Print** Foerster, section 7-6	**Instructional Activity: The Logistic Model** Student groups collaborate on a worksheet that guides them through an introduction to the logistic growth model, which follows the development of this Foerster section. This multi-part problem culminates in the solution to a logistic DE utilizing partial fractions from Unit 5. One group presents their solution at the board. I summarize, contrasting the logistic growth model with the exponential growth model. Then I present a new problem asking for the carrying capacity $\left(\lim_{t\to\infty} P(t)\right)$, as well as the time t at which the population is growing most rapidly for a population of rabbits that grows according to $\frac{dP}{dt} = 3P - 3P^2$, where P is units of a hundred thousand rabbits. Students must find these answers without attempting to solve the DE.	
All of the learning objectives in this unit are addressed.		**Summative Assessment: Differential Equations** Students are given 10 fairly short free-response questions directly targeting single learning objectives and two longer free-response questions that require assimilating the concepts learned in the unit. This is followed by six multiple-choice questions.	*This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 8

The following activities and techniques in Unit 8 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: In the “What Is a Differential Equation?” instructional activity students must use the definition to determine if a given function satisfies the required constraints of a specific DE.

MPAC 2 — Connecting concepts: In the “Solving Separable Differential Equations” formative assessment students must incorporate their knowledge from Unit 5 regarding various techniques of integration in order to solve the DEs. When approximating solutions using Euler’s method, students see the influence of the principle of local linearity from Unit 3.

MPAC 3 — Implementing algebraic/computational processes: In solving logistic differential equations, students implement their algebra skills in exercising the technique of partial fractions. Computational processes are implemented in the Euler’s method exercises.

MPAC 4 — Connecting multiple representations: Throughout the unit, students compare information about solutions to differential equations by solving analytically, as well as using graphical information provided by the slope fields and approximations in tabular form via Euler’s method. In the “Additional Modeling with Differential Equations” instructional activity students must work with verbal descriptions leading to DEs.

MPAC 5 — Building notational fluency: Throughout the unit, students’ good understanding of differential notation is reinforced as they encounter and solve the differential equations, and as they employ integration techniques such as u-substitutions.

MPAC 6 — Communicating: In the “Additional Modeling with Differential Equations” instructional activity students must communicate within their groups as well as present to the class interpreting solutions to the differential equations.

UNIT 9: SERIES

BIG IDEA 4
Series (BC)

Enduring Understandings:
- EU 4.1, EU 4.2

Estimated Time:
25 instructional hours

Guiding Questions:

▶ What does it mean for a series to converge versus diverge, and what are the criteria? ▶ How can we determine how many terms of a convergent series must be added before a desired level of accuracy is guaranteed? ▶ How can we find a power series expansion for a given function? ▶ How can we find the interval of convergence for a given power series? ▶ How can we find new power series expansions using known power series expansions? ▶ What are the applications of infinite series?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 4.1A: Determine whether a series converges or diverges. **LO 4.1B:** Determine or estimate the sum of a series.	**Print** Stewart, sections 11.1 and 11.2	**Instructional Activity: Reviewing Sequences and Series** Student groups collaborate on a worksheet that is a fusion of the two Stewart sections referenced. Our precalculus course has a substantial unit on sequences and series, so the first few lessons of this unit are mostly the review-and-extend type. The worksheet guides students through this review as they work out the examples while I circulate and provide guidance. Students complete the worksheet as their homework assignment, and we go over it in the next lesson. At that time, I take the opportunity to present a PowerPoint presentation of these sections. We discuss the ideas of the sequence of partial sums of a series, infinite geometric series, and telescoping series. Students exit with a similar worksheet for homework.
LO 4.1A: Determine whether a series converges or diverges.	**Print** Stewart, section 11.2	**Instructional Activity: The General Term Test and the Harmonic Series** The problem on the board describes an infinite series $\sum_{n=1}^{\infty} a_n$ with *sequence of partial sums* $\{S_n\}$ given by $S_n = \frac{5n+8}{n+2}$. Students must find the first two terms in the sequence of partial sums, then use them to find a_1, a_2, and the value to which $\sum_{n=1}^{\infty} a_n$ converges. This reinforces the definition of a convergent series and the meaning of its sequence of partial sums. After we review the exercise, student groups collaborate on a worksheet from Stewart section 11.2 . The first part introduces them to the harmonic series and asks whether the series converges. The second part asks them whether the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ converges or diverges. The worksheet guides them through the results that "$\sum_{n=1}^{\infty} a_n$ converges" implies that $\lim_{n\to\infty} a_n = 0$, but not conversely. We review the worksheet and summarize this crucial result.

UNIT 9: SERIES

BIG IDEA 4
Series (BC)

Enduring Understandings:
- EU 4.1, EU 4.2

Estimated Time:
25 instructional hours

Guiding Questions:

▶ What does it mean for a series to converge versus diverge, and what are the criteria? ▶ How can we determine how many terms of a convergent series must be added before a desired level of accuracy is guaranteed? ▶ How can we find a power series expansion for a given function? ▶ How can we find the interval of convergence for a given power series? ▶ How can we find new power series expansions using known power series expansions? ▶ What are the applications of infinite series?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 4.1A: Determine whether a series converges or diverges.	**Print** Stewart, section 11.4	**Instructional Activity: The Comparison Test** Today's Do Now problem asks students whether either of the series $\sum_{n=1}^{\infty}\frac{1}{2^n+3}$ or $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ converge. A class discussion ensues and we reach the conclusion of the comparison test. Then student groups collaborate on a worksheet that leads them through the official statement of the comparison test and guides them through the exercises in this section, illustrating how this test can be used to argue that some series converge while others diverge. A solution for each of the problems on the worksheet is presented by a different student group, and then I summarize.	*After this lesson, the students are primed for a hefty assignment asking them to decide if a given series converges or diverges. It is good to spend a period going over such a comprehensive assignment before making the leap from infinite series of constants to infinite series functions (rational functions expressed as infinite geometric series), which are introduced in the next activity.*
LO 4.2B: Write a power series representing a given function. **LO 4.2C:** Determine the radius and interval of convergence of a power series.	**Print** Foerster, section 12-1 Stewart, section 11.2	**Formative Assessment: Introduction to Functions Defined as Infinite Series** I present the function $S_1(x)=6x+6x^2+6x^3+6x^4+\cdots+6x^n+\cdots$ and student groups explore whether $S(3)$ or $S(.25)$ exist. They must find all the values of x for which the infinite series function converges. Students find a rational function f to which S converges over that domain. I circulate, provide guidance, and engage each group in a discussion. I ask groups similar questions about $S_2(x)=\sum_{n=0}^{\infty}12\cdot\left(\frac{1}{3}(x-2)\right)^n$. After student-presented solutions of these problems, I am able to use the infinite geometric series ideas to unofficially introduce the notions of power series, interval of convergence, and radius of convergence. Students see that there are infinite series representations for "finite" functions. I'm sure to promise students that we will not stop with rational functions!	*Circulating and interacting with students allows me to see where they are in their development of infinite series of constants, and how they may be poised to handle the transition to what I call at this point, "infinite series functions," until they eventually learn that these are special cases of power series. Introducing this concept early in the unit allows for the notion of "interval of convergence" to percolate before revisiting it in the context of more general power series.*

BIG IDEA 4
Series (BC)

Enduring Understandings:
- EU 4.1, EU 4.2

Estimated Time:
25 instructional hours

Guiding Questions:

- What does it mean for a series to converge versus diverge, and what are the criteria? ▸ How can we determine how many terms of a convergent series must be added before a desired level of accuracy is guaranteed? ▸ How can we find a power series expansion for a given function? ▸ How can we find the interval of convergence for a given power series? ▸ How can we find new power series expansions using known power series expansions? ▸ What are the applications of infinite series?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 4.2A: Construct and use Taylor polynomials. **LO 4.2B:** Write a power series representing a given function.		**Instructional Activity: Polynomial Approximators** This time, we go backward. The problem on the board asks students to find an infinite geometric series function S corresponding to the rational function $f(x)=\frac{x}{1+2x}$. After going over this problem, we introduce the idea of "polynomial approximators," priming the student for the official lesson on Maclaurin expansion later in the unit. We find the fourth-degree polynomial approximator $P_4(x)$ for this function, and then use $P_4(x)$ to approximate $f(.1)$. Then we introduce the error function, $E(x)=\lvert P_4(x)-f(x)\rvert$. Next, student groups collaborate on a worksheet in which they find the interval of convergence for f. They must then use their graphing calculators to explore the graphs of S, f, P_4, and E over the interval of convergence. We go over the exploration, and I summarize.
LO 4.1A: Determine whether a series converges or diverges.	**Print** Stewart, section 11.3	**Instructional Activity: The Integral Test** After we go over the day's Do Now problem showing that $\int_1^\infty \frac{1}{x^3}dx$ converges, student groups collaborate on a worksheet containing a diagram that shows inscribed rectangles corresponding to the right Riemann sum approximation for $\int_1^{100} \frac{1}{x^3}dx$ with 100 equal subintervals. Students use this diagram to argue that because the improper integral $\int_1^\infty \frac{1}{x^3}dx$ converges, so too does the infinite series $\sum_{n=1}^{\infty}\frac{1}{n^3}$. The worksheet guides student groups through the rationale of the integral test, as developed in this Stewart section. A group presents their findings, and I show a PowerPoint presentation of this section. Students turn to the flip side of their worksheet to find two exercises utilizing the integral test. After student-presented solutions to these two problems, I provide a summary.

This lesson provides an opportunity to introduce the notion that a polynomial can be used to approximate rational functions. At this point in their development, students are only able to find the interval of convergence of a power series if that power series corresponds to an infinite geometric series. This sets the students up well for lessons to follow.

BIG IDEA 4
Series (BC)

Enduring Understandings:
- EU 4.1, EU 4.2

Estimated Time:
25 instructional hours

Guiding Questions:
▸ What does it mean for a series to converge versus diverge, and what are the criteria? ▸ How can we determine how many terms of a convergent series must be added before a desired level of accuracy is guaranteed? ▸ How can we find a power series expansion for a given function? ▸ How can we find the interval of convergence for a given power series? ▸ How can we find new power series expansions using known power series expansions? ▸ What are the applications of infinite series?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 4.1A: Determine whether a series converges or diverges.	**Print** Stewart, section 11.3	**Instructional Activity: The p-Series Test** The first problem on the Do Now worksheet asks students to determine whether the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$, $\sum_{n=1}^{\infty}\frac{1}{n}$, and $\sum_{n=1}^{\infty}\frac{1}{n^{.5}}$ converge or diverge. The second problem asks students to conjecture a general rule about when the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges or diverges. After a student-presented solution to the first question and then a discussion about the second, I direct the students to turn to the flip side of the worksheet, which guides them through the proof of the p-series test. After a student-presented solution, I provide a summary.
LO 4.1A: Determine whether a series converges or diverges.	**Print** Stewart, section 11.4	**Instructional Activity: Limit Comparison Test** The problem on the board asks students to determine whether the series $\sum_{n=1}^{\infty}\frac{1}{n^2+3}$, $\sum_{n=1}^{\infty}\frac{1}{5n^2-3}$, and $\sum_{n=1}^{\infty}\frac{3n+5}{n^3+1}$ converge or diverge. We discuss these problems, and students easily justify that the first converges while they "suspect" the second and third also converge, but they are unable to justify this suspicion based on results we have established to date. At that point, I have students form groups and they collaborate on a worksheet that guides them through the development of the limit comparison test (LCT), as presented in this Stewart section. After some time we go over the worksheet at the board. I direct students to the flip side of the worksheet, which has two more problems from the Stewart exercises. After student-presented solutions, I provide a summary.

I've always thought it was best to hold off on presenting the LCT until the p-series test was covered since it provides the best vehicle for invoking the LCT.

UNIT 9: SERIES

BIG IDEA 4
Series (BC)

Enduring Understandings:
- EU 4.1, EU 4.2

Estimated Time:
25 instructional hours

Guiding Questions:

▸ What does it mean for a series to converge versus diverge, and what are the criteria? ▸ How can we determine how many terms of a convergent series must be added before a desired level of accuracy is guaranteed? ▸ How can we find a power series expansion for a given function? ▸ How can we find the interval of convergence for a given power series? ▸ How can we find new power series expansions using known power series expansions? ▸ What are the applications of infinite series?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 4.1A: Determine whether a series converges or diverges.	**Print** Stewart, section 11.5	**Instructional Activity: The Alternating Series Test for Series of Constants** Student groups collaborate on a worksheet that guides them through the development of the alternating series test as presented in the referenced Stewart section. I interrupt to show a PowerPoint presentation of this section. Student groups resume working on the flip side of this worksheet, which has several problems from this Stewart exercise set. I circulate and provide guidance. For each problem, I have one of the groups work at the board, and they present their solution. After some dialogue with the class during the group presentation, I summarize the day's results.
LO 4.1A: Determine whether a series converges or diverges.	**Print** Stewart, section 11.7	**Formative Assessment: Mixing It Up with Series of Constants** Students work independently on a worksheet that has 10 infinite series. In each case, students must decide if the given series converges or diverges. I construct this set of problems so that students are forced to use different convergence tests from the unit. Examples include $\sum_{n=3}^{\infty} \frac{1}{n \ln n}$, $\sum_{n=1}^{\infty} \frac{5n^2-3}{3n^4+1}$, $\sum_{n=1}^{\infty} \frac{5n^2-3}{3n^4+1}$, and $\sum_{n=1}^{\infty} \frac{.01n}{6n+7}$ so that students may have options for their choice of techniques on some series, but are forced to use all the tests from this unit. I circulate and provide light guidance, occasionally interrupting the class to clarify concepts. The following day, after we go over requested problems, I hand out a flowchart based on the summary at the end of Stewart section 11.7.

Toward the end of this formative assessment I collect the sheets and scan them in the classroom sheet-fed scanner for examination later that day. The worksheets are immediately returned to the students and a solution key is provided as the students exit. I use the scans to determine areas to work on for individual students. Students are required to correct their worksheet with the key and prepare requests for the next day

BIG IDEA 4
Series (BC)

Enduring Understandings:
- EU 4.1, EU 4.2

Estimated Time:
25 instructional hours

Guiding Questions:

▸ What does it mean for a series to converge versus diverge, and what are the criteria? ▸ How can we determine how many terms of a convergent series must be added before a desired level of accuracy is guaranteed? ▸ How can we find a power series expansion for a given function? ▸ How can we find the interval of convergence for a given power series? ▸ How can we find new power series expansions using known power series expansions? ▸ What are the applications of infinite series?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 4.2A: Construct and use Taylor polynomials.	**Print** Stewart, section 11.10	**Instructional Activity: Maclaurin Expansions** The problem on the board asks students to find the fourth-degree "polynomial approximator" for $f(x)=\frac{x}{1-2x}$. After we go over this problem as a class using ideas from previous activities, I ask students to do the same in groups for $\sin x$, $\cos x$, and e^x. Of course, students are perplexed, but they are intrigued. I then hand out a worksheet based on Stewart section 11.10, and groups proceed to collaborate on the worksheet while I circulate and provide guidance. One group works at the board and presents their solution. I provide a summary of the main idea in the section that many familiar transcendental functions may be converted to their infinite polynomial versions, namely $\sum_{n=0}^{\infty} a_n x^n$, where $a_n = \frac{f^{(n)}(0)}{n!}$ to conclude the activity.
LO 4.2A: Construct and use Taylor polynomials.	**Print** Stewart, section 11.10	**Instructional Activity: Taylor Expansions** Today's Do Now problem has student groups find the first four terms for the Maclaurin series for $\arctan x$ and then $\ln x$. After a student-presented solution at the board for the first function, we conclude that we need something new for the second. Student groups then collaborate on a worksheet (with a new group at the board) that guides them to develop the technique for finding Taylor series centered about a given value $x=a$. Student groups must then find the expansion for $f(x)=\ln x$ centered about $x=1$. The group at the board presents their solution. My summary links back to earlier activities in which we found infinite series functions corresponding to rational functions, and I use this opportunity to give the general definition of a power series.

Leading up to this activity, many of my assigned problem sets have exercises reinforcing the ideas from the "Introduction to Functions Defined as Infinite Series" formative assessment and the "Polynomial Approximators" instructional activity. In this way, students are primed for learning about Maclaurin expansions, Taylor polynomials, and power series in general.

UNIT 9: SERIES

BIG IDEA 4
Series (BC)

Enduring Understandings:
- EU 4.1, EU 4.2

Estimated Time:
25 instructional hours

Guiding Questions:

▸ What does it mean for a series to converge versus diverge, and what are the criteria? ▸ How can we determine how many terms of a convergent series must be added before a desired level of accuracy is guaranteed? ▸ How can we find a power series expansion for a given function? ▸ How can we find the interval of convergence for a given power series? ▸ How can we find new power series expansions using known power series expansions? ▸ What are the applications of infinite series?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 4.1A: Determine whether a series converges or diverges.	**Print** Stewart, section 11.6	**Instructional Activity: The Ratio Test for Infinite Series of Constants** Student groups collaborate on a worksheet that guides them through the development of the ratio test as presented in this Stewart section. Naturally, the worksheet begins with the definition of absolute versus conditional convergence, followed by some guided examples. I circulate and provide comments, and occasionally interrupt the class for some instruction and ultimately to summarize the ratio test result. After a class discussion, student groups resume working on the flip side of this worksheet, which has several problems from this Stewart exercise set. I circulate and provide guidance. One group works at the board and they present their solution. After some dialogue with the class during the group presentation, I summarize the day's results.
LO 4.2A: Construct and use Taylor polynomials. **LO 4.2C**: Determine the radius and interval of convergence of a power series.	**Print** Foerster, section 12-1 Stewart, sections 11.6 and 11.8	**Instructional Activity: Using the Ratio Test to Find the Interval of Convergence of a Power Series** The problem on the board asks students to use ideas from the infinite geometric series activities to find the interval of convergence of $S(x)=\sum_{n=0}^{\infty} 9\cdot\left(\frac{1}{3}(x-1)\right)^n$. This allows for a review discussion of the ideas of the center and radius of a power series developed in earlier activities. Our next task is to find the interval of convergence for the power series $\sum_{n=1}^{\infty}\frac{(x-1)^n}{n\cdot 3^n}$. I guide the class discussion and we develop the procedure of using the ratio test to determine the radius of convergence and then an endpoint analysis to determine the interval of convergence. Then students work individually on determining the interval of convergence for $\sum_{n=3}^{\infty}(-1)^{n+1}\frac{(x-2)^n}{n\ln n}$. After a student-presented solution, I provide a summary of the procedure outlined in the referenced Stewart section.

UNIT 9: SERIES

BIG IDEA 4
Series (BC)

Enduring Understandings:
- EU 4.1, EU 4.2

Estimated Time:
25 instructional hours

Guiding Questions:

▸ What does it mean for a series to converge versus diverge, and what are the criteria? ▸ How can we determine how many terms of a convergent series must be added before a desired level of accuracy is guaranteed? ▸ How can we find a power series expansion for a given function? ▸ How can we find the interval of convergence for a given power series? ▸ How can we find new power series expansions using known power series expansions? ▸ What are the applications of infinite series?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 4.1B: Determine or estimate the sum of a series. **LO 4.2A:** Construct and use Taylor polynomials.	**Print** Stewart, section 11.11	**Instructional Activity: The Lagrange Error Bound** Today's Do Now problem has students find an approximation for the number e by using the seventh-degree Taylor polynomial of e^x centered about $x=0$. After solving the problem, I guide the class through a discussion about how accurate such approximations would be, and how we could choose enough terms to guarantee certain levels of accuracy. Then student groups collaborate on a worksheet that guides them through the development of the Lagrange error bound, following the development in this referenced Stewart section. I circulate and provide guidance. Eventually, I interrupt and provide a PowerPoint presentation of this development. I then return to the Do Now problem and illustrate how the Lagrange error bound can guarantee that our approximation for e is correct within .0001.
LO 4.1B: Determine or estimate the sum of a series. **LO 4.2A:** Construct and use Taylor polynomials.	**Print** Stewart, section 11.5	**Instructional Activity: The Alternating Series Test Error Bound** This time, I ask students to construct the tenth-degree Taylor polynomial of $\ln(x+1)$ centered about $x=0$ and use it to approximate $\ln 2$. Students are fascinated that this leads to the alternating harmonic series. After solving the problem, I guide the class through a discussion and they come to realize that our level of accuracy can't easily be determined through our recently learned tool of the Lagrange error bound. I present the argument for the alternating series error bound as developed in this section of Stewart. I put up a PowerPoint presentation slide of two more problems from this Stewart section. After student-presented solutions, I summarize. My summary includes the slides from the Lagrange error bound activity, so I can emphasize the differences.

The complexity and subtleties of this error bound result force me to construct a slightly more teacher-dominated lesson than usual. It also requires staying on this topic a second day before moving on.

BIG IDEA 4
Series (BC)

Enduring Understandings:
- EU 4.1, EU 4.2

Estimated Time:
25 instructional hours

Guiding Questions:

▸ What does it mean for a series to converge versus diverge, and what are the criteria? ▸ How can we determine how many terms of a convergent series must be added before a desired level of accuracy is guaranteed? ▸ How can we find a power series expansion for a given function? ▸ How can we find the interval of convergence for a given power series? ▸ How can we find new power series expansions using known power series expansions? ▸ What are the applications of infinite series?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 4.1B: Determine or estimate the sum of a series. **LO 4.2A:** Construct and use Taylor polynomials.	**Web** AP Calculus BC 2008 Free-Response Question 3 AP Calculus BC 2012 Free-Response Question 6	**Formative Assessment: Putting It Together with Power Series and Errors** Student groups collaborate on a worksheet. The first side is AP Calculus BC 2008 Free-Response Question 3, dealing with tabular data for a function and its derivatives at various points. Students construct Taylor polynomials and use them to calculate approximations. Students use the Lagrange error bound to assess the quality of their approximation. On the flip side, students work on AP Calculus BC 2012 Free-Response Question 6. This problem utilizes the ratio test to determine the interval of convergence of a power series. Students are asked to provide an approximation and argue that it is small based on the alternating series error bound. I circulate and provide guidance. Several students work at the board and present solutions.

I've found that it's beneficial to give some of these "put-it-all-together" worksheets before the "official" review for the AP exam. There are so many great old AP problems that serve to help instruct the student, that it would be a waste of a fantastic resource if we didn't find opportunities to use some of them well before that official review.

UNIT 9: SERIES

BIG IDEA 4
Series (BC)

Enduring Understandings:
- EU 4.1, EU 4.2

Estimated Time:
25 instructional hours

Guiding Questions:

- What does it mean for a series to converge versus diverge, and what are the criteria? ▸ How can we determine how many terms of a convergent series must be added before a desired level of accuracy is guaranteed? ▸ How can we find a power series expansion for a given function? ▸ How can we find the interval of convergence for a given power series? ▸ How can we find new power series expansions using known power series expansions? ▸ What are the applications of infinite series?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 4.2A: Construct and use Taylor polynomials.	**Print** Stewart, section 11.11	**Instructional Activity: Manipulation with Series** The Do Now problem asks students to find the sum $4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots+\frac{1}{2n-1}+\cdots\right)$. Students are equally perplexed and intrigued. I present an introduction of the techniques of manipulation to construct new series from given series. I start with the expansion for $\frac{1}{1-x}$, previously established from the infinite geometric series activities. I use it to develop series for $\frac{1}{1+x}$ (composition) and $\ln(x+1)$ (term-by-term antidifferentiation). Students are amazed that the last time we developed the expansion for $\ln(x+1)$ we had to engage in the laborious task of using Taylor's formula for the coefficients, $a_n=\frac{f^{(n)}(a)}{n!}$, and it was required to find many derivatives. Next we develop the expansions for $\frac{1}{1+x^2}$ (composition) and $\arctan x$ (term-by-term antidifferentiation). We substitute $x=1$ into our expansion for $\arctan x$ and it immediately yields the result from the Do Now worksheet, and students are mesmerized that $\pi=4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots+(-1)^{n+1}\frac{1}{2n-1}+\cdots\right)$.	*This activity opens the door for assignments containing the rich choices in the referenced section of Stewart, as well as future worksheet-based activities from the robust AP problems such as AP Calculus BC 2011 Free-Response Question 6. I stay on this topic a few days as I lead up to a review for the upcoming summative assessment.*
All of the learning objectives in this unit are addressed.		**Summative Assessment: Series** Students are given 10 fairly short free-response questions directly targeting single learning objectives and two longer free-response questions that require assimilating the concepts learned in the unit. This is followed by six multiple-choice questions.	*This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 9

The following activities and techniques in Unit 9 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: In the very first activity, as well as throughout the unit, we are working with the definition of a convergent series as the limit of its sequence of partial sums. In every activity involving a test for convergence, as well as the activities involving the two error bound theorems, we construct arguments using the theorem du jour. Here, we are reasoning with the theorem.

MPAC 2 — Connecting concepts: Once the integral test has been established, students get to practice with integration techniques from the prior units, as well as the concept of improper integrals. Once the ratio test has been established, assignments contain ratio test exercises that require the use of L'Hospital's Rule.

MPAC 3 — Implementing algebraic/computational processes: Activities involving the ratio test, both for series of constants and in determining the radius of convergence of power series, students implement their algebra skills at an intensive level.

MPAC 4 — Connecting multiple representations: In the "Putting It Together with Power Series and Errors" formative assessment and subsequent assignments, students use tabular data to construct Taylor polynomials. After the instructional activity "The Lagrange Error Bound" the door is opened for assignments such as AP Calculus BC 2011 Free-Response Question 6, in which the bound is constructed using graphical information about a fifth derivative.

MPAC 5 — Building notational fluency: From the start, students are exposed to the definition of the sequence of partial sums of a sequence $\{a_n\}$, as $s_n = \sum_{k=1}^{n} a_k$. An even more intense dose of notation is served up in Taylor's formula $a_n = \frac{f^{(n)}(a)}{n!}$. The ultimate exercise in understanding notation occurs in "The Lagrange Error Bound" instructional activity.

MPAC 6 — Communicating: In arguing whether a series converges or diverges, students are required to articulate careful arguments using the convergence tests of the unit. They must also argue that the hypotheses of the convergence tests have been satisfied. In constructing error bounds using LaGrange, students have to craft and present arguments as to why the error bound expression must be smaller than specific tolerances.

Resources

General Resources

Foerster, Paul. *Calculus Concepts and Applications.* Emeryville, CA: Key Curriculum Press, 2005.

Stewart, James. *Single Variable Calculus*, 7th ed. Belmont, CA: Brooks/Cole, Cengage Learning, 2012.

Unit 3 (Derivative Interpretations and Applications) Resources

"1980 BC7." *AP Calculus AB/AP Calculus BC Free-Response Questions and Solutions 1979–1988,* 47-48. New York: The College Board, 2004.

"AB-6." *AP Calculus AB 1999 Scoring Guidelines,* 7. AP Central – The AP Calculus AB Exam. Accessed June 2, 2015. **http://apcentral.collegeboard.com/apc/members/repository/sg_calculus_ab_99.pdf**.

Unit 4 (Integrals and the Fundamental Theorem of Calculus) Resources

Khan, Sal. "Proof of Fundamental Theorem of Calculus." Khan Academy. Video, 14:00. Accessed June 2, 2015. **https://www.khanacademy.org/math/integral-calculus/indefinite-definite-integrals/fundamental-theorem-of-calculus/v/proof-of-fundamental-theorem-of-calculus**.

"Question 5." *AP Calculus AB 2004 Scoring Guidelines,* 6. AP Central – The AP Calculus AB Exam. Accessed June 2, 2015. **http://apcentral.collegeboard.com/apc/public/repository/ap04_sg_calculus_ab_36998.pdf**.

Unit 6 (Applications of Integration) Resources

"Question 2." *AP Calculus AB 2002 Scoring Guidelines,* 3. AP Central – The AP Calculus AB Exam. Accessed June 2, 2015. **http://apcentral.collegeboard.com/apc/public/repository/sg_calculus_ab_02_11386.pdf**.

"Volume of a Solid." Manipula Math with Java. Accessed June 3, 2015. **http://www.ies-math.com/math/java/calc/renshi/renshi.html**.

Unit 7 (Parametric Equations and Polar Curves) Resources

"Question 3." *AP Calculus BC 2003 Scoring Guidelines,* 3. AP Central – The AP Calculus BC Exam. Accessed June 2, 2015. **http://apcentral.collegeboard.com/apc/members/repository/ap03_sg_calculus_bc_26477.pdf**.

"Question 3." *AP Calculus BC 2007 Scoring Guidelines,* 4. AP Central – The AP Calculus BC Exam. Accessed June 2, 2015. **http://apcentral.collegeboard.com/apc/public/repository/_ap07_calculus_bc_sgs_final.pdf**.

Unit 9 (Series) Resources

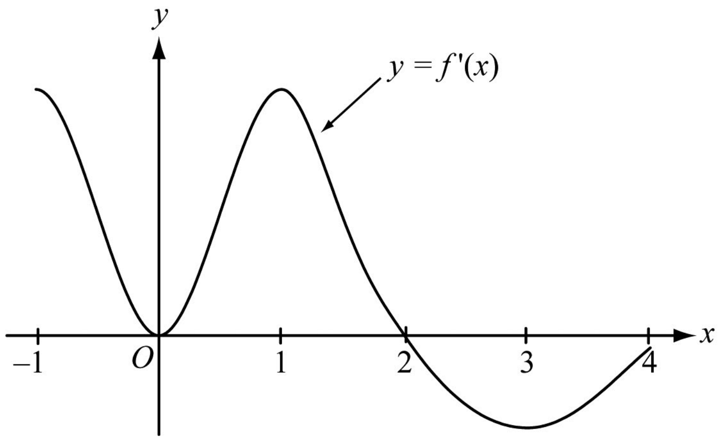
"Question 3." *AP Calculus BC 2008 Scoring Guidelines,* 4. AP Central – The AP Calculus BC Exam. Accessed June 2, 2015. **http://apcentral.collegeboard.com/apc/public/repository/ap08_calculus_bc_sgs.pdf.**

"Question 6." *AP Calculus BC 2011 Scoring Guidelines*, 7. AP Central – The AP Calculus BC Exam. Accessed June 2, 2015. **http://apcentral.collegeboard.com/apc/public/repository/ap11_calculus_bc_scoring_guidelines.pdf**.

"Question 6." *AP Calculus BC 2012 Scoring Guidelines,* 7. AP Central – The AP Calculus BC Exam. Accessed June 2, 2015. **http://apcentral.collegeboard.com/apc/public/repository/ap12_calculus_bc_scoring_guidelines.pdf**.

Appendix

AP Calculus BC 1980 Free-Response Question 7



Note: This is the graph of the *derivative* of f, NOT the graph of f.

1. Let f be a function that has domain the closed interval $[-1,4]$ and range the closed interval $[-1,2]$. Let $f(-1)=-1$, $f(0)=0$, and $f(4)=1$. Also let f have the derivative function f' that is continuous and that has the graph shown in the figure above.

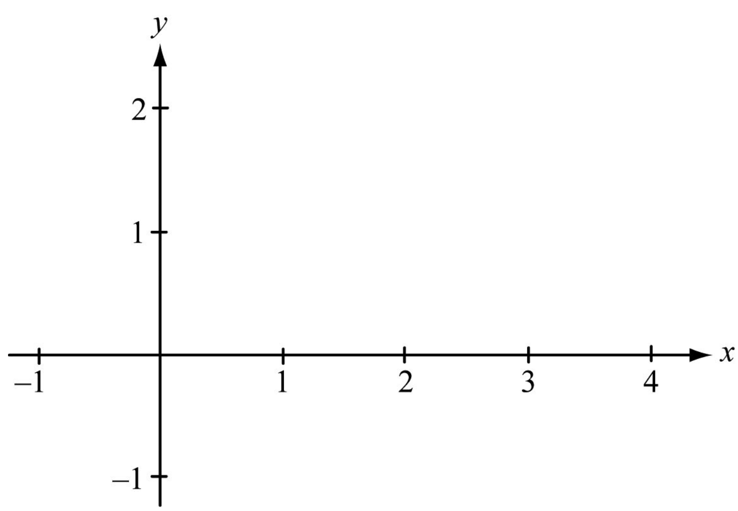
   (a) Find all values of x for which f assumes a relative maximum. Justify your answer.

   (b) Find all values of x for which f assumes its absolute minimum. Justify your answer.

   (c) Find the intervals on which f is concave downward.

   (d) Give all the values of x for which f has a point of inflection.

   (e) On the axes provided, sketch the graph of f.

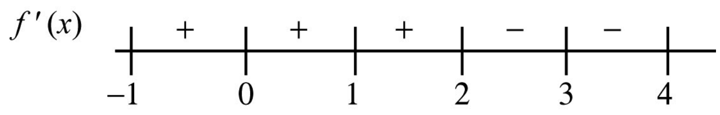


Note: The graph of f' has been slightly modified from the original on the 1980 exam to be consistent with the given values of f at $x=-1$, $x=0$, and $x=4$.

AP Calculus BC 1980 Free-Response Question 7

Solution

(a) $f'(x)=0$ at $x=0,2$



There is a relative maximum at $x=2$, since $f'(2)=0$ and $f'(x)$ changes from positive to negative at $x=2$.

(b) There is no minimum at $x=0$, since $f'(x)$ does not change sign there. So the absolute minimum must occur at an endpoint. Since $f(-1)<f(4)$, the absolute minimum occurs at $x=-1$.

(c) The graph of f is concave down on the intervals $(-1,0)$ and $(1,3)$ because f' is decreasing on those intervals.

(d) The graph of f has a point of inflection at $x=0$, 1, and 3 because f' changes from decreasing to increasing or from increasing to decreasing at each of those x values.

(e)