AP® Calculus AB

Course Planning and Pacing Guide

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Welcome to the AP Calculus AB Course Planning and Pacing Guides

This guide is one of several course planning and pacing guides designed for AP Calculus AB teachers. Each provides an example of how to design instruction for the AP course based on the author's teaching context (e.g., demographics, schedule, school type, setting). These course planning and pacing guides highlight how the components of the *AP Calculus AB and BC Curriculum Framework*, which uses an Understanding by Design approach, are addressed in the course. Each guide also provides valuable suggestions for teaching the course, including the selection of resources, instructional activities, and assessments. The authors have offered insight into the *why* and *how* behind their instructional choices — displayed along the right side of the individual unit plans — to aid in course planning for AP Calculus teachers.

The primary purpose of these comprehensive guides is to model approaches for planning and pacing curriculum throughout the school year. However, they can also help with syllabus development when used in conjunction with the resources created to support the AP Course Audit: the Syllabus Development Guide and the four Annotated Sample Syllabi. These resources include samples of evidence and illustrate a variety of strategies for meeting curricular requirements.

Contents

Instructional Setting

John Bapst Memorial High School ▸ Bangor, Maine

School

John Bapst is an independent college preparatory high school in Bangor, Maine.

Student population

John Bapst has approximately 500 students including 55 international boarding students. Local students come from over 30 towns around the Bangor, Maine, area that are too small to have a high school. Ninety-nine percent of our students go on to college.

Instructional time

John Bapst typically starts school during the last week of August. We have 180 scheduled school days and we teach using a seven-period school day. Over the course of seven school days we meet the students for 43 minutes on five days and one long 75-minute block. For our AP Calculus classes we add in one 43-minute lab period each seven-day rotation.

Student preparation

Most of our students who take our AP Calculus AB class are either juniors or seniors. These students have typically completed Honors Precalculus, but because some of our students transfer into our school as upperclassmen we do allow students to test into this course.

Primary planning resources

Larson, Ron, and Bruce Edwards. *Calculus*. 10th ed. Australia: Cengage Learning, 2014.

The following resources are just for reference and are optional:

Foerster, Paul A. *Calculus: Concepts and Applications, Solutions Manual*. 2nd ed. Berkeley, CA: Key Curriculum, 2005.

Hopper, Clarence. *Visualizing Calculus: Powerful Programs for the TI-81, TI-82, TI-83, and TI-85 and CASIO Fx-7700GE, Fx-9700GE, and CFX-9800G.* Orangeburg, NY: Dale Seymour Publications, 1998.

Umbarger, Dan. *Twenty Key Ideas in Beginning Calculus*. Math Logarithms, 2012.

Overview of the Course

The focus of this course is conceptual understanding and real-world applications of the big ideas in AP Calculus AB, not rote memorization of different rules and algorithms. Knowing that my students have different learning styles and prior knowledge I use the Rule of Four throughout the course to try and effectively reach all students in my classroom.

In my class we first study Big Idea 1: Limits as the foundational building block for all calculus. For many students limits are theoretical and, although many students can correctly work/solve these problems, some do not understand why they need to learn about limits. It is my responsibility throughout the course to show students how limits can be used for both derivatives (limit definition of the derivative as h approaches zero) and with Riemann sums and the definite integral as we accumulate an infinite number of rectangles (limit as n approaches infinity). As we proceed through the various concepts, I make it a point to talk about the importance of limits and how they relate to the current topic.

In differential calculus (Big Idea 2) we introduce the concept of the instantaneous rate of change and the slope of a curve at a specific point. Building on students' knowledge from algebra, we learn how to find the slope of a curve at any specific x value using the tangent line and the limit definition of the derivative. Of course students need to learn the mechanics and various rules for taking derivatives (both of which are important), but I emphasize the practical use of derivatives (e.g., the rate of oil leaking out of a tank) and how we can use the first and second derivatives to learn about a function's behavior. As we continue our study of derivatives, we can move on to related rate and optimization problems, which are applicable to a wide variety of real-world scenarios.

As I introduce Big Idea 3 and the concepts of antiderivatives and integration, I want my students to see how these concepts are related to both limits and differentiation that we have previously studied. Using Riemann sums and the concept of accumulating the area of an infinite number of small rectangles, we can now find the area under a curve using the definite integral. One of my favorite parts of the course is the development of the Fundamental Theorem of Calculus (FTC), where students see how differential calculus and integral calculus are related.

I have formative assessments for each chapter/unit and I use these to help gauge how well students are grasping the new concepts. These are quizzes and consist of short-answer and/or multiple-choice questions. These quizzes contain four problems and I shoot for the majority of my students to get at least three of these four questions correct. If this doesn't happen I revise my weekly syllabus to go back and revisit the topics/problems the student struggled with. I'm also lucky at my school to have an extra 43-minute period for a calculus lab each seven-day rotation. These labs have students work on self-discovery of concepts and I act as a facilitator. My Hershey's Kiss activity in Unit 8 is usually the most fun for the students, as they get to eat their "lab" when completed! As they work on "their lab" I will walk around the room and ask questions, listening to their discussions to help gauge their level of understanding.

Mathematical Thinking Practices for AP Calculus (MPACs)

The Mathematical Practices for AP Calculus (MPACs) capture important aspects of the work that mathematicians engage in, at the level of competence expected of AP Calculus students. They are drawn from the rich work in the National Council of Teachers of Mathematics (NCTM) Process Standards and the Association of American Colleges and Universities (AAC&U) Quantitative Literacy VALUE Rubric. Embedding these practices in the study of calculus enables students to establish mathematical lines of reasoning and use them to apply mathematical concepts and tools to solve problems. The Mathematical Practices for AP Calculus are not intended to be viewed as discrete items that can be checked off a list; rather, they are highly interrelated tools that should be utilized frequently and in diverse contexts.

MPAC 1: Reasoning with definitions and theorems

Students can:

a. use definitions and theorems to build arguments, to justify conclusions or answers, and to prove results;

b. confirm that hypotheses have been satisfied in order to apply the conclusion of a theorem;

c. apply definitions and theorems in the process of solving a problem;

d. interpret quantifiers in definitions and theorems (e.g., "for all," "there exists");

e. develop conjectures based on exploration with technology; and

f. produce examples and counterexamples to clarify understanding of definitions, to investigate whether converses of theorems are true or false, or to test conjectures.

MPAC 2: Connecting concepts

Students can:

a. relate the concept of a limit to all aspects of calculus;

b. use the connection between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, antidifferentiation) to solve problems;

c. connect concepts to their visual representations with and without technology; and

d. identify a common underlying structure in problems involving different contextual situations.

MPAC 3: Implementing algebraic/computational processes

Students can:

a. select appropriate mathematical strategies;

b. sequence algebraic/computational procedures logically;

c. complete algebraic/computational processes correctly;

d. apply technology strategically to solve problems;

e. attend to precision graphically, numerically, analytically, and verbally and specify units of measure; and

f. connect the results of algebraic/computational processes to the question asked.

Mathematical Thinking Practices for AP Calculus (MPACs)

MPAC 4: Connecting multiple representations

Students can:

a. associate tables, graphs, and symbolic representations of functions;
b. develop concepts using graphical, symbolical, or numerical representations with and without technology;
c. identify how mathematical characteristics of functions are related in different representations;
d. extract and interpret mathematical content from any presentation of a function (e.g., utilize information from a table of values);
e. construct one representational form from another (e.g., a table from a graph or a graph from given information); and
f. consider multiple representations of a function to select or construct a useful representation for solving a problem.

MPAC 5: Building notational fluency

Students can:

a. know and use a variety of notations (e.g., $f'(x), y', \frac{dy}{dx}$);
b. connect notation to definitions (e.g., relating the notation for the definite integral to that of the limit of a Riemann sum);
c. connect notation to different representations (graphical, numerical, analytical, and verbal); and
d. assign meaning to notation, accurately interpreting the notation in a given problem and across different contexts.

MPAC 6: Communicating

Students can:

a. clearly present methods, reasoning, justifications, and conclusions;
b. use accurate and precise language and notation;
c. explain the meaning of expressions, notation, and results in terms of a context (including units);
d. explain the connections among concepts;
e. critically interpret and accurately report information provided by technology; and
f. analyze, evaluate, and compare the reasoning of others.

Pacing Overview

Unit	Hours of Instruction	Unit Summary
1: Limits	12	In this unit students develop an understanding of limits as the foundational building block for both derivatives and integration. One goal of this unit is to ensure that students are comfortable solving limit problems using the Rule of Four. The Rule of Four is a method where students can solve problems using: 1. A graphical approach 2. A numerical/tabular approach 3. An algebraic approach 4. A verbal or written approach, communicating effectively what their final answer means in the context of the problem
2: Derivatives (Late Transcendentals)	12	In this unit students use their understanding of limits to explore the meaning of a derivative and instantaneous rate of change. Building on the limit definition of the derivative, students will explore and begin to use the various rules for taking a derivative. One goal of this unit is for students to use the Rule of Four to solve for derivatives of many different types of functions.
3. Implicit Differentiation and Related Rates	10	In this unit students expand on their understanding of derivatives and their use in real-world related rates problems. Students explore how to take derivatives of equations that are not mathematical functions using implicit differentiation. One goal of this unit is for students to take derivatives of an expression with relation to any variable, typically time with related rates problems.
4. Applications of Differentiation/ Curve Sketching/ Optimization	10	In this unit students discover how we can use the first and second derivatives of functions to describe the function's behavior and sketch it accurately. One goal of this unit is for students to understand how to apply the Existence Theorems (which include the Intermediate Value Theorem, Extreme Value Theorem, Rolle's Theorem, and the Mean Value Theorem) to help problem solve and justify their conclusions.
5. Integration and Accumulation/ Fundamental Theorem of Calculus	20	In this unit students discover the relationship between differentiation and integration as inverse operations. Students learn how to integrate functions and then, using the definite integral, learn how to "accumulate" in various real-world settings. As the unit progresses they learn the importance of the Fundamental Theorem of Calculus and its many applications.

Pacing Overview

Unit	Hours of Instruction	Unit Summary
6. Transcendental Functions — Derivatives and Integration	10	In this unit students build upon their knowledge of taking derivatives and integrating using transcendental functions. The textbook that I use introduces these concepts later in the textbook, hence the term “later transcendentals.” Students see how powerful the chain rule can be and how to apply it to this unit when taking various derivatives.
7. Differential Equations/Slope Fields	10	In this unit students discover how to “read” a slope field to see how a function (or other equations that are not mathematical functions) behave. Slope fields are the graphical interpretation of a differential equation (DE) and tie in nicely to the Rule of Four. Students will also build upon their knowledge of integration, using separation of variables to solve more complicated DEs.
8. Area/Volume of Revolution — Applications of Integration	14	In this unit students discover the real power and beauty of calculus in a variety of integration problems. Building upon their knowledge of accumulation (and specifically area under a curve), students will be able to find the area between two curves given two functions. Students also learn to find the volume of a solid when a function (or two functions) is rotated around a horizontal line or vertical line. Using a variety of geometric shapes, students will also be able to find the volume of a 3-D solid using known cross-sectional areas.

UNIT 1: LIMITS

BIG IDEA 1
Limits

Enduring Understandings:
- EU 1.1, EU 1.2

Estimated Time:
12 instructional hours

Guiding Questions:

▶ How can we use limits as the building block for our study of calculus? ▶ How can a function have a limit at certain x values even though the function is undefined at that x value? ▶ What is the Rule of Four and how can we use this process to solve problems and understand calculus? ▶ How can we use limits to determine the continuity of a function at specific x values?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 1.1A(a): Express limits symbolically using correct notation. **LO 1.1A(b):** Interpret limits expressed symbolically.	**Print** Larson and Edwards, section 1.2	**Instructional Activity: The Mile Run** In this activity students gain an understanding of the concept of a limit and learn to express limits using appropriate notation. Each student collects data on the world record time for the mile run since 1900 in five-year increments, then graphs the data on a scatterplot using graphing calculators. As a class, we discuss the behavior of the data: for many years they are decreasing and reasonably linear. They tend to level off into a horizontal nature and although the times still decrease, the improvements are marginal. We then discuss what we think will happen in 50 years, whether at some point we won't see any improvements in time, and if we will reach a "limit" as to how fast humans can run the mile.
LO 1.1A(b): Interpret limits expressed symbolically. **LO 1.1B:** Estimate limits of functions. **LO 1.1C:** Determine limits of functions.	**Print** Larson and Edwards, section 1.2	**Instructional Activity: The Basics of Limits** In this activity students use their graphing calculators and the graphing/table features to determine the limits of various functions, starting with something as simple as $\lim_{x \to 2}(x+4)$. We solve this problem by substitution, graphing the function, and by looking at the table feature on our graphing calculators.

Many students quickly learn how to solve limit problems but may struggle with why they need to learn about limits. It is my responsibility as we move through the future calculus concepts to make the connection with limits and their importance in the study of calculus.

BIG IDEA 1
Limits

Enduring Understandings:
- EU 1.1, EU 1.2

Estimated Time:
12 instructional hours

Guiding Questions:

▸ How can we use limits as the building block for our study of calculus? ▸ How can a function have a limit at certain *x* values even though the function is undefined at that *x* value? ▸ What is the Rule of Four and how can we use this process to solve problems and understand calculus? ▸ How can we use limits to determine the continuity of a function at specific *x* values?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 1.1D: Deduce and interpret behavior of functions using limits.	**Print** Larson and Edwards, section 1.2	**Instructional Activity: Functions Where Limits Fail to Exist** Students use their calculators to explore certain functions where limits fail to exist at certain x values including: 1. Two different values $\lim_{x \to 0} \frac{\lvert x \rvert}{x}$ 2. Unbounded $\lim_{x \to 0} \frac{1}{x^2}$ 3. Oscillating $\lim_{x \to 0} \sin\frac{1}{x}$
LO 1.1B: Estimate limits of functions.	**Print** Larson and Edwards, section 1.3	**Formative Assessment: Various Limit Problems** At this stage of the class students get a quick formative assessment, typically four to five multiple-choice questions from previously released College Board Exams.

After grading this assessment, I go through each problem on the board. If the students have done well on this assessment, we move on with our discussion of limits. If students do not do well, I go back and review each topic that they struggled with.

UNIT 1: LIMITS

BIG IDEA 1
Limits

Enduring Understandings:
- EU 1.1, EU 1.2

Estimated Time:
12 instructional hours

Guiding Questions:

▶ **How can we use limits as the building block for our study of calculus?** ▶ **How can a function have a limit at certain *x* values even though the function is undefined at that *x* value?** ▶ **What is the Rule of Four and how can we use this process to solve problems and understand calculus?** ▶ **How can we use limits to determine the continuity of a function at specific *x* values?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 1.1B: Estimate limits of functions.	**Print** Larson and Edwards, section 1.3	**Instructional Activity: L'Hospital's Rule** As soon as students are comfortable with taking derivatives and their meaning, we revisit limit problems where L'Hospital's Rule applies. To use it we need to have an indeterminate form in our limit, for example, $\frac{0}{0}$ or $\frac{\infty}{\infty}$. For example, we can use L'Hospital's Rule on this limit problem: $$\lim_{x\to 0}\frac{\sin(x)}{x}.$$ If the problem is a candidate for L'Hospital's Rule, we can take the same limit of the derivative of the numerator over the derivative of the denominator. In the problem above, we would get: $$\lim_{x\to 0}(\cos(x))/1.$$ When we substitute in $x = 0$ we get the correct answer of 1. In other limit problems where we can use L'Hospital's Rule, we sometimes have to take the derivative of the numerator and denominator twice (or more!) before we can evaluate the limit.
LO 1.2A: Analyze functions for intervals of continuity or points of discontinuity.	**Print** Larson and Edwards, section 1.4	**Instructional Activity: Is the Function Continuous?** In this activity we explore the continuity of functions and how we "show" that functions are continuous at specific x values. The graphing calculator is very helpful here: I graph various types of functions (including piecewise functions) on my view screen and we discuss the continuity of each function. I ask students where the function is not continuous and why the function is not continuous at these specific x values. Students should have an understanding of asymptotes and points of discontinuity from their precalculus class.

Some students after learning L'Hospital's Rule will "overuse" this rule on many problems. It is important for students to know we can only use L'Hospital's Rule on limit problems where we get an indeterminate form.

UNIT 1: LIMITS

BIG IDEA 1
Limits

Enduring Understandings:
- EU 1.1, EU 1.2

Estimated Time:
12 instructional hours

Guiding Questions:

▶ How can we use limits as the building block for our study of calculus? ▶ How can a function have a limit at certain *x* values even though the function is undefined at that *x* value? ▶ What is the Rule of Four and how can we use this process to solve problems and understand calculus? ▶ How can we use limits to determine the continuity of a function at specific *x* values?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 1.2B: Determine the applicability of important calculus theorems using continuity.	**Print** Larson and Edwards, section 1.4	**Instructional Activity: The Intermediate Value Theorem (IVT) in Action** We discuss real-world problems where the IVT is applicable. One example I use is the velocity of a commercial airline. On the runway before leaving, the velocity of the plane is zero (not moving). Most commercial airlines will reach their maximum velocity of about 500 miles per hour at cruising altitude. We discuss whether this is a continuous function and if so, at some point does the plane have to be flying at 300 miles per hour?
LO 1.1B: Estimate limits of functions. **LO 1.1D:** Deduce and interpret behavior of functions using limits.	**Print** Larson and Edwards, section 3.5	**Instructional Activity: As *x* Goes to Infinity** Students work through a handout containing several limit problems in which x goes to infinity. As we work through limit problems as x approaches infinity, I want students to come up with a strategy for solving this type of limit problem quickly. Hopefully students see that the most important term in the numerator and denominator is the term with the largest exponent. The three conditions that I want my students to explore (and hopefully come up with their own conclusions) are: 1. Largest value for the exponents is the same in the numerator and the denominator: $\lim_{x\to\infty}\frac{x^2}{2x^2}$ 2. Largest value for the exponents is greater in the numerator: $\lim_{x\to\infty}\frac{3+x^3}{4+x^2}$ 3. Largest value for the exponents is greater in the denominator: $\lim_{x\to\infty}\frac{2+x+x^2}{3-x^3}$

Students should have a solid understanding of functions, transformations, and end behavior of functions from their study of precalculus. They should be able to solve for and find vertical and horizontal asymptotes as well as any points of discontinuity of various types of functions.

UNIT 1: LIMITS

BIG IDEA 1
Limits

Enduring Understandings:
- EU 1.1, EU 1.2

Estimated Time:
12 instructional hours

Guiding Questions:

- **How can we use limits as the building block for our study of calculus?** ▸ **How can a function have a limit at certain *x* values even though the function is undefined at that *x* value?** ▸ **What is the Rule of Four and how can we use this process to solve problems and understand calculus?** ▸ **How can we use limits to determine the continuity of a function at specific *x* values?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 1.1D: Deduce and interpret behavior of functions using limits.	**Print** Larson and Edwards, section 3.5	**Summative Assessment: Unit 1 Exam** I have one cumulative summative assessment at the end of this unit covering all the limit concepts we have discussed. This assessment is a traditional test modeled after the AP Exam. Each unit test has a section that allows calculator usage and a section that does not allow a calculator. Each exam has some short-answer questions, multiple-choice questions, and some extended-response questions similar to the free-response questions students will see on the AP Exam. Emphasis is placed on limit problems where students are given piecewise functions, and problems where students are given a graph and asked limit questions from the graph.

This summative assessment addresses all of the guiding questions for the unit.

Mathematical Practices for AP Calculus in Unit 1

The following activities and techniques in Unit 1 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: In "The IVT in Action" instructional activity students practice applying the IVT in real-world applications. The Existence Theorems are important for conceptual understanding of the major concepts in AP Calculus AB.

MPAC 2 — Connecting concepts: In the "As x Goes to Infinity" instructional activity students can visualize how a function can approach a certain y value but never actually "reach" this y value.

MPAC 3 — Implementing algebraic/computational processes: Throughout this unit students realize the importance of algebraic procedures/factoring to help solve certain limit problems.

MPAC 4 — Connecting multiple representations: In the "As x Goes to Infinity" instructional activity students should see the value of the Rule of Four for problem solving with limits and how different approaches help to solve different types of problems.

MPAC 5 — Building notational fluency: In all the activities in this unit students learn how to use limit notation and how to use this notation to help solve problems.

MPAC 6 — Communicating: Students learn how to clearly interpret limit notation and communicate their results into meaningful answers.

BIG IDEA 2
Derivatives

Essential Understandings:
- EU 2.1, EU 2.2

Estimated Time:
12 instructional hours

Guiding Questions:

▶ **How can we use the tangent line to get the slope of a function at a specific *x* value?** ▶ **How can we use limits and the limit definition of the derivatives to calculate derivatives of functions at specific *x* values?** ▶ **How can we use the different rules for derivatives to get the instantaneous rate of change?** ▶ **How can we use the function graph to get the first derivative graph and vice versa?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.1A: Identify the derivative of a function as the limit of a difference quotient.	**Print** Larson and Edwards, section 2.1	**Instructional Activity: The Tangent Line Using the Graphing Calculator** Using graphing calculators, we explore the tangent line(s) of various functions to estimate the slope of a curve at specific x values. For example, I graph $y = x^2$ on my view screen and we discuss the slope of this function at different x values. I want students to say that the slope changes as we vary our x values moving from left to right on this function.
LO 2.1A: Identify the derivative of a function as the limit of a difference quotient. **LO 2.3B:** Solve problems involving the slope of a tangent line.	**Print** Larson and Edwards, section 2.1	**Instructional Activity: Local Linearity — Is It Really a Line?** Using the zoom-in feature of the graphing calculator I graph $y = sin(x)$ over a very small interval so that it "looks like a line." I ask students to guess the equation of the function they see on the board. The students almost always say $y = x$ and are somewhat surprised when I show them I graphed $y = sin(x)$. We have a discussion about the concept of local linearity and how we can apply this to various functions to get the slope of that function at different x values. We then talk about tangent lines and how the slope of the tangent line is equal to the slope of our function at a specific x value.
LO 2.1A: Identify the derivative of a function as the limit of a difference quotient.	**Print** Larson and Edwards, section 2.1	**Instructional Activity: The Difference Quotient Expanded From Slope Formula In Algebra I** I want students to see that the limit definition of the derivative is basically the same slope formula they saw in Algebra I: $m = \frac{\Delta y}{\Delta x}$. Using appropriate calculus nomenclature, the limit definition of the derivative is basically the same formula students have used before; in calculus our dx (h) is extremely small and approaches zero: $\lim\limits_{x \to \infty} \frac{f(x+h) - f(x)}{(x+h) - (x)}$.

Many textbooks do not mention the concept of local linearity, which is critical for our understanding of a derivative or instantaneous rate of change.

It is important that students see how important limits are to solving for derivatives and the importance of the limit definition of the derivative. I want my students to understand that limits are the basis for why we can use all of the other derivative rules.

UNIT 2: DERIVATIVES (LATE TRANSCENDENTALS)

BIG IDEA 2
Derivatives

Essential Understandings:
- EU 2.1, EU 2.2

Estimated Time:
12 instructional hours

Guiding Questions:

▸ How can we use the tangent line to get the slope of a function at a specific *x* value? ▸ How can we use limits and the limit definition of the derivatives to calculate derivatives of functions at specific *x* values? ▸ How can we use the different rules for derivatives to get the instantaneous rate of change? ▸ How can we use the function graph to get the first derivative graph and vice versa?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.1A: Identify the derivative of a function as the limit of a difference quotient. **LO 2.2B:** Recognize the connection between differentiability and continuity.	**Print** Larson and Edwards, section 2.1	**Instructional Activity: Differentiability And Continuity** Using graphs, we explore the relationship between differentiability and continuity and come up with examples where continuity does not imply differentiability. If a function is differentiable at an x value then it must be continuous at that x value. The converse is not necessarily true and we come up with various functions that are continuous but not differentiable. I graph the following equations and as a class we discuss 1) whether these functions are continuous, and 2) if they are not continuous, at what x values are they not continuous: $y=\sin(x)$ $y=\frac{x}{x=2}$.
LO 2.1C: Calculate derivatives. **LO 2.1D:** Determine higher order derivatives.	**Print** Larson and Edwards, sections 2.2 and 2.3	**Instructional Activity: The Rules for Derivatives, Day One** On day one of this two-day activity, students learn when to use the limit definition of the derivative; rules for the six trigonometric functions; higher-order derivatives, including the second and third derivatives; and the constant, power, product, quotient, chain, and general power rules. I give specific examples of each rule and where/how we use them: for the product rule I give students $y=(x)(x^2)$. At this point students can quickly take the derivative of this function and somebody yells out $\frac{dy}{dx}=3x^2$. We then explore what result we would get if we multiplied the derivative of the first term by the derivative of the second term, which leads into a discussion of how to apply the product rule for derivatives. Students are given a worksheet with many practice problems on this concept, to be completed as a homework assignment.

Many students struggle with some of the algebra and factoring here especially with the quotient rule and the chain rule.

BIG IDEA 2
Derivatives

Essential Understandings:
- EU 2.1, EU 2.2

Estimated Time:
12 instructional hours

Guiding Questions:

- How can we use the tangent line to get the slope of a function at a specific *x* value? ▸ How can we use limits and the limit definition of the derivatives to calculate derivatives of functions at specific *x* values? ▸ How can we use the different rules for derivatives to get the instantaneous rate of change? ▸ How can we use the function graph to get the first derivative graph and vice versa?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.1C: Calculate derivatives. **LO 2.1D:** Determine higher order derivatives.	**Print** Larson and Edwards, sections 2.2 and 2.3	**Instructional Activity: The Rules for Derivatives, Day Two** I use a similar example as on day one of this activity to help students see how to correctly apply the quotient rule for derivatives. I use the example $y = x^3/x$, and students can quickly come up with the correct derivative of $\frac{dy}{dx} = 2x$. Students want to take the derivative of the numerator and divide this by the derivative of the denominator to get $\frac{dy}{dx} = 3x^2/1$. Again they quickly see that this approach is flawed but it leads to a nice discussion of how to apply the quotient rule. At times students ask about rewriting the original function as the product of two terms, one with a negative exponent, and then applying the product rule. Of course this is acceptable and it is great if students suggest this approach. Students are given a worksheet with many practice problems on this concept, to be completed as a homework assignment.	*After I introduce L'Hospital's Rule as a method for solving limit problems we discuss the difference between the quotient rule and L'Hospital's Rule as students often get confused on when to use each method.*
LO 2.1C: Calculate derivatives.		**Formative Assessment: Rules for Derivatives** At this stage of this unit students get a quick formative assessment consisting of four to five multiple-choice questions from previously released College Board Exams. I have found in the past that students struggle with derivatives requiring the chain rule so I make sure that I include at least two problems that address this concept.	*After grading this assessment, I go through each problem on the board. If the students have done well on this assessment, we move to our discussion of derivatives. If students do not do well, I go back and review each topic that they struggled with.*

UNIT 2: DERIVATIVES (LATE TRANSCENDENTALS)

BIG IDEA 2
Derivatives

Essential Understandings:
- EU 2.1, EU 2.2

Estimated Time:
12 instructional hours

Guiding Questions:

▸ How can we use the tangent line to get the slope of a function at a specific *x* value? ▸ How can we use limits and the limit definition of the derivatives to calculate derivatives of functions at specific *x* values? ▸ How can we use the different rules for derivatives to get the instantaneous rate of change? ▸ How can we use the function graph to get the first derivative graph and vice versa?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.2A: Use derivatives to analyze properties of a function. **LO 2.1D:** Determine higher order derivatives.	**Print** Larson and Edwards, sections 2.3 and 2.4	**Instructional Activity: Graphing Functions and Their Derivatives** I use technology (programs on my graphing calculator) to graph the first derivative graph from a function graph and vice versa. The easiest example I use is $y = x^2$. Then we take the first and second derivatives and graph these on one graph, and then on three vertical stacked graphs. Students are asked to explain what each of these graphs tells us about our original function.	*The graphing calculator is a wonderful tool to help students see the relationship between function and derivative graphs.* *Many textbooks graph the function, first, and second derivatives on one graph. Although this is technically wrong, I do think it is helpful for students to see the relationship among these graphs. I want a student to tell me why this is "technically" wrong based on what each graph tells us about our function and what our y axis represents.*
LO 2.1B: Estimate derivatives.	**Print** Larson and Edwards, section 2.1	**Summative Assessment: Unit 2 Exam** I have one cumulative summative assessment at the end of this unit covering all the derivative rules/concepts we have discussed. This assessment is a traditional test modeled after the AP Exam. Each unit test has a section that allows calculator usage and a section that does not allow a calculator. The exams have short-answer questions, multiple-choice questions, and extended-response questions similar to the free-response questions on the AP Calculus AB Exam. Emphasis is put not only on taking various types of derivatives, but in communicating what the derivative tells us (instantaneous rate of change) in a variety of contextual settings.	*There are many excellent College Board multiple-choice and free-response problems that I use to help my students estimate derivatives from tabular data.* *This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 2

The following activities and techniques in Unit 2 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: In the "Local Linearity — Is It Really a Line?" instructional activity students work toward proficiency with the limit definition of the derivative and the various rules/properties for calculating derivatives for a variety of functions.

MPAC 2 — Connecting concepts: In the "The Tangent Line Using the Graphing Calculator," instructional activity students gain the understanding that we are really using limits when calculating derivatives and recognize the importance of the tangent line and local linearity.

MPAC 3 — Implementing algebraic/computational processes: In "The Rules for Derivatives" instructional activities students see how the product rule for x^3 versus $(x)(x^2)$ is a great example of sequencing computational procedures logically and correctly.

MPAC 4 — Connecting multiple representations: In the "Graphing Functions and Their Derivatives" instructional activity students should be able to estimate/calculate derivatives from equations/graphs/tables and be able to correctly interpret both first and second derivative graphs.

MPAC 6 — Communicating: Students practice their writing to clearly explain and justify their answers in the context of a problem and how this information allows us to make decisions contextually. In the "Graphing Functions and Their Derivatives" instructional activity students need to communicate what each of these graphs tells us about the behavior of the function.

BIG IDEA 2
DERIVATIVES

Essential Understandings:
- EU 2.1, EU 2.3

Estimated Time:
10 instructional hours

Guiding Questions:

- **How can we expand on our understanding of derivatives using equations that are not mathematical functions?**
- **How can we use implicit differentiation to find the instantaneous rate of change of an equation at a specific point, not just a specific *x* value?**
- **How can we take the derivative of an equation with respect to time?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.1C: Calculate derivatives.	**Print** Larson and Edwards, section 2.5	**Instructional Activity: Implicit Differentiation Formally Defined** As we get closer to implicit differentiation I start to use the notation of $\frac{dy}{dx}$ for the derivative. To see if students can come up with their own conclusions about taking derivatives of equations that are not functions, or when y cannot be explicitly written in terms of x, I have them each take the derivative of $y=x$, $y=2x$, $y=x^3$, $y=(3x+4)^2$, and $x=y^2$. By now this should be rather easy for the first four equations and I ask that students use the $\frac{dy}{dx}$ format. When they get to the fifth equation the interesting discussions begin. Some students correctly take the derivative of x with respect to y and get $\frac{dy}{dx}=2y$. Others struggle as they have always taken the derivative when the equation had "just y" in it. I tell students to just invert this answer to get the derivative, $\frac{dy}{dx}=\frac{1}{2y}$.	*Students should be comfortable with $f'(x), y', \frac{dy}{dx}$ as notation for a derivative. Students should also feel comfortable with taking the derivative of any variable with respect to any other variable, not just x and y.* *Many students will struggle with implicit differentiation especially with trigonometric functions. I make sure students have plenty of practice using this technique for taking derivatives.*
LO 2.3A: Interpret the meaning of a derivative within a problem.	**Print** Larson and Edwards, section 2.5	**Formative Assessment: Implicit Differentiation** At this stage of this unit students get a quick formative assessment, typically four to five multiple-choice questions from previously released College Board Exams. This assessment tests students' understanding that when taking a derivative using implicit differentiation our answer should be interpreted as any type of derivative, slope of a curve at a point, or instantaneous rate of change at a specific point.	*Depending on the results of this assessment, I may move on to new material or double back and do some review work.*

UNIT 3: IMPLICIT DIFFERENTIATION AND RELATED RATES

BIG IDEA 2
DERIVATIVES

Essential Understandings:
- EU 2.1, EU 2.3

Estimated Time:
10 instructional hours

Guiding Questions:

- **How can we expand on our understanding of derivatives using equations that are not mathematical functions?** ▸ **How can we use implicit differentiation to find the instantaneous rate of change of an equation at a specific point, not just a specific *x* value?** ▸ **How can we take the derivative of an equation with respect to time?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.3C: Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion. **LO 2.3D:** Solve problems involving rates of change in applied contexts.	**Print** Larson and Edwards, section 2.6	**Instructional Activity: Related Rates** In this activity I introduce the concept of related rates problems and I encourage my students to draw a picture that represents the problem we are solving. My initial problem is dropping a stone in a body of water, and we draw on the board how the concentric circles get larger as time progresses. We make a list of what variables are changing with respect to time and what we want to solve for.
All of the learning objectives in this unit are addressed.		**Summative Assessment: Unit 3 Exam** I have one cumulative summative assessment at the end of this unit covering implicit differentiation and related rates. Both topics tend to be difficult for students and I want this assessment to focus on just these two topics. Related rate problems are covered well in most textbooks so it is easy to find many good problems that address this skill. All summative assessments will contain problems/questions from previously discussed topics, but the focus will be on these two topics.

Students need to understand that units are required for all related rates problems. Many times when solving the problem I will not include units during each step. When we have finally calculated our final answer then we discuss as a class what units are appropriate for the specific problem.

Knowing that students will likely struggle with these two concepts, I make sure to include these types of problems in future assessments. There will also be at least one day of review covering them in late April as we are reviewing/preparing for the upcoming AP Exam. This assessment addresses all of the guiding questions for this unit.

UNIT 3: IMPLICIT DIFFERENTIATION AND RELATED RATES

BIG IDEA 2
DERIVATIVES

Essential Understandings:
- EU 2.1, EU 2.3

Estimated Time:
10 instructional hours

Guiding Questions:

▸ **How can we expand on our understanding of derivatives using equations that are not mathematical functions?** ▸ **How can we use implicit differentiation to find the instantaneous rate of change of an equation at a specific point, not just a specific *x* value?** ▸ **How can we take the derivative of an equation with respect to time?**

Learning Objectives	Materials	Instructional Activities and Assessments
All of the learning objectives in this unit are addressed.	**Web** Free-response questions available at AP Central	**Summative Assessment: Calculus Lab Unit 3** Beginning with Unit 3, for some units I give students a calculus lab to work on as a second summative assessment. This lab consists of three to four previously released College Board free-response questions related to the topics included in the unit. Students work in groups of two and have a week to complete the lab. For this unit, I use free-response questions for implicit differentiation and related rates as many students struggle with these two important concepts.

This assessment addresses all of the guiding questions for this unit.

Mathematical Practices for AP Calculus in Unit 3

The following activities and techniques in Unit 3 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: In the "Implicit Differentiation Formally Defined" instructional activity students use the "form" of the derivative to help them understand the definition of implicit differentiation. Students use this reasoning to help understand this new type of derivative.

MPAC 2 — Connecting concepts: In the "Related Rates" instructional activity students learn to use visual representations to help set up and solve problems.

MPAC 3 — Implementing algebraic/computational processes: Students need to be able to use algebraic/computational processes correctly to solve a variety of implicit differentiation problems.

MPAC 5 — Building notational fluency: Students should be able to connect appropriate notation to different representations and interpret this notation in a contextual setting. In the "Related Rates" instructional activity students learn how important it is to take the derivative with respect to time.

MPAC 6 — Communicating: In the "Related Rates" instructional activity students learn how to explain the connection between the concepts of rate of change and how variables are changing with respect to time.

BIG IDEA 2
Derivatives

Essential Understandings:
- EU 2.2, EU 2.4

Estimated Time:
10 instructional hours

Guiding Questions:

- **What do the first and second derivatives of a function tell us about its behavior?** ▸ **How can using the derivatives of a function help us accurately sketch the function?** ▸ **What role do the Existence Theorems have in our justifications of function behavior?** ▸ **How can derivatives be used to solve optimization problems?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.2A: Use derivatives to analyze properties of a function.	**Print** Larson and Edwards, sections 3.1–3.4	**Instructional Activity: We Really Can Graph a Function Without Our Graphing Calculators** In this activity students apply their knowledge of taking derivatives to specific functions to see how the function behaves and how to graph these functions. We use $f(x)=x^4-4x^3$ to take the first and second derivatives and set each of these functions to zero. We find all the critical x values/numbers and apply the first and second derivative tests to determine what these x values represent: they can be either maximums, minimums, or points of inflection. Substituting these critical x values into our original function we can get the associated y values to get the max/min points and/or the points of inflection. Using this information about our function (where the function is increasing/decreasing and its concavity) we can very accurately sketch this function without our graphing calculators.	*Students need to understand that functions increase/decrease over intervals, not at specific x values.*
LO 2.2A: Use derivatives to analyze properties of a function.		**Formative Assessment: Curve Sketching** At this stage of this unit students get a quick formative assessment that consists of one to two functions. Students are required to sketch these functions without their calculators: using the first and second derivative tests and their knowledge of function behavior, students should be able to do this fairly accurately. Depending on the results of this assessment, we typically will move on to new material but still emphasize the applications of the derivative.	*Depending on how students do on this assessment, I usually try to do at least one to two problems on the board. If students do well on this assignment, we move on. If not, I include a few more sample problems for homework and will include more of these types of problems on the unit summative assessment.*

BIG IDEA 2
Derivatives

Essential Understandings:
- EU 2.2, EU 2.4

Estimated Time:
10 instructional hours

Guiding Questions:

▸ What do the first and second derivatives of a function tell us about its behavior? ▸ How can using the derivatives of a function help us accurately sketch the function? ▸ What role do the Existence Theorems have in our justifications of function behavior? ▸ How can derivatives be used to solve optimization problems?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.4A: Apply the Mean Value Theorem to describe the behavior of a function over an interval.	**Print** Larson and Edwards, sections 1.4, 3.1, and 3.2	**Instructional Activity: Why Do We Need the Four Existence Theorems in AP Calculus?** I introduce and sketch the four Existence Theorems that students need to know: the Intermediate Value Theorem (IVT), Extreme Value Theorem (EVT), Rolle's Theorem, and Mean Value Theorem (MVT) for Derivatives. These theorems simply tell us that certain x values exist, assuming all the appropriate conditions are met. These theorems DO NOT tell us how to find these x values. For example, the EVT tells us that over a closed interval, the function must have a minimum and a maximum value somewhere, but it does not tell us how to find these values and/or what these values are.	*Students struggle on the AP Exam with the multiple-choice questions that address these four Existence Theorems and also on the free-response questions that ask students to justify their answers using the Existence Theorems.*
LO 2.4A: Apply the Mean Value Theorem to describe the behavior of a function over an interval.	**Print** Larson and Edwards, sections 1.4, 3.1, and 3.2	**Formative Assessment: Existence Theorems** After discussing these Existence Theorems I give the students a quick quiz where they need to sketch each theorem, correctly describing what each tells us and what conditions need to be met to apply these theorems. I include two multiple-choice questions that address these four theorems to show the students how these concepts could be tested on the AP Exam. Typically the multiple-choice questions on the Existence Theorems score very low on the AP Exams and students do not recognize these problems as testing the Existence Theorems.	*Students may struggle with this assessment because even though we teach these Existence Theorems, we honestly do not use them very often in AP Calculus AB. I stress these with my students and try to use them as often as I can in future discussions.*

UNIT 4: APPLICATIONS OF DIFFERENTIATION/CURVE SKETCHING/OPTIMIZATION

BIG IDEA 2
Derivatives

Essential Understandings:
- EU 2.2, EU 2.4

Estimated Time:
10 instructional hours

Guiding Questions:

- **What do the first and second derivatives of a function tell us about its behavior?** ▸ **How can using the derivatives of a function help us accurately sketch the function?** ▸ **What role do the Existence Theorems have in our justifications of function behavior?** ▸ **How can derivatives be used to solve optimization problems?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.3C: Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion. **LO 2.3D**: Solve problems involving rates of change in applied contexts.	**Print** Larson and Edwards, section 3.7	**Instructional Activity: What Size Enclosure Can We Build to Maximize Area?** In this activity students are given a problem where they have 500 feet of fencing and want to enclose some farm space around a barn. Since this space is contiguous to the barn it will only need to be a three-sided fence. Students each need to come up with two equations using two variables (typically they select X and Y). Eliminating one variable and then taking the derivative of this new equation, students use their knowledge of the first and second derivative tests to arrive at the correct dimensions for their enclosure.	*I typically teach related rates problems in Unit 3, but like to include some review work on them in this unit as students can struggle with these types of problems.*
All of the learning objectives in this unit are addressed.		**Summative Assessment: Unit 4 Exam** I have one cumulative summative assessment at the end of this unit covering applications of the derivative, the Existence Theorems, and optimization problems. All summative assessments contain problems/questions from previously discussed topics, but the focus is on these three topics for this assessment. Students are given a function (maybe cubic) and have to very accurately draw this function using the topics from this unit. Calculators are not allowed for these types of problems.	*When teaching the Existence Theorems students tend to find these intuitive and easy to understand. Since we do not use these theorems that frequently in this course, I make sure we include these theorems in our class discussions and students practice many free-response questions where these theorems are included/tested.* *This summative assessment addresses the following guiding question*: ▸ What role do the Existence Theorems have in our justifications of function behavior?

UNIT 4: APPLICATIONS OF DIFFERENTIATION/CURVE SKETCHING/OPTIMIZATION

BIG IDEA 2
Derivatives

Essential Understandings:
- EU 2.2, EU 2.4

Estimated Time:
10 instructional hours

Guiding Questions:

▸ What do the first and second derivatives of a function tell us about its behavior? ▸ How can using the derivatives of a function help us accurately sketch the function? ▸ What role do the Existence Theorems have in our justifications of function behavior? ▸ How can derivatives be used to solve optimization problems?

Learning Objectives	Materials	Instructional Activities and Assessments
All of the learning objectives in this unit are addressed.		**Summative Assessment: Curve-Sketching Lab** I also include a calculus lab assessing the students' understanding of the topics in this unit. I print two to four previously released College Board free-response questions and give them to the students. They work in groups of two and have a week to complete these questions. I grade each question on a 0- to 9- point scale just like the AP Exam.

Mathematical Practices for AP Calculus in Unit 4

The following activities and techniques in Unit 4 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: In the instructional activity “Why Do We Need the Four Existence Theorems in AP Calculus?” students develop an understanding of how important these theorems are for solving problems and justifying their answers.

MPAC 2 — Connecting concepts: In the instructional activity “We Really Can Graph a Function Without Our Graphing Calculators” students learn how we can use the first and second derivative tests to understand how to analyze a function’s behavior and how to accurately sketch/draw these functions.

MPAC 3 — Implementing algebraic/computational processes: Students continue to see how important derivatives are for analyzing and understanding how functions behave. In the “Curve-Sketching Lab” assessment students learn to analyze functions using derivatives.

MPAC 4 — Connecting multiple representations: In the curve-sketching part of this unit, students can connect the algebraic component of derivatives with the graphical interpretation of the function and first and second derivative graphs.

MPAC 6 — Communicating: This MPAC is very important in each unit, and as the students learn more about calculus concepts, I want them to make connections and applications among the various topics/units to problem solve and discuss/justify their final answers. Using the Existence Theorems students learn how to use these theorems as justifications for how functions behave.

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Essential Understandings:
- EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
20 instructional hours

Guiding Questions:

- **What is the relationship between differentiation and integration?** ▸ **How does the area under a curve relate to the idea of an accumulation process?** ▸ **How can the Fundamental Theorem of Calculus be used to solve problems?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.1A: Recognize antiderivatives of basic functions. **LO 3.3B(a):** Calculate antiderivatives. **LO 2.3F:** Estimate solutions to differential equations.	**Print** Larson and Edwards, section 4.1	**Instructional Activity: Why Do We Need to Add a Constant of Integration?** When first introducing the antiderivative I use the following example and tell students we "are going in the other direction, from a derivative to the function": $\frac{dy}{dx}=2x$. Students can easily see that the function is $F(x)=x^2$. We then discuss what the derivative is of the following functions. Students quickly see we get the same derivative for many different functions and will see the need for our $+C$ term when taking an antiderivative: $F(x)=x^2+2$ $F(x)=x^2+5$ $F(x)=x^2-4$ $F(x)=x^2-12$

Once students are comfortable with antiderivatives and adding $+C$, I briefly introduce the concept of a slope field and the graphical representation of the constant of integration $(+C)$. We will come back and visit slope fields again in the future but I want my students to "see" what a slope field represents.

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Essential Understandings:
- EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
20 instructional hours

Guiding Questions:

▸ **What is the relationship between differentiation and integration?** ▸ **How does the area under a curve relate to the idea of an accumulation process?** ▸ **How can the Fundamental Theorem of Calculus be used to solve problems?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.3B(a): Calculate antiderivatives.		**Instructional Activity: Why Do We Need *U*-Substitution?** In this activity students learn how to integrate more complicated functions using u-substitution. I start with an easy function and ask the students to integrate $\int(\cos(2x))dx$. Usually I get the answer $F(x)=\sin(2x)+C$ but I hope some students will ask about the 2 in the original integrand. I then introduce **u**-substitution to help integrate functions that are more complicated than what we've learned to date: $u=(2x)$ $du/dx=2$ $du=2dx$ $du/2=dx$. The original problem can be rewritten as: $\int\cos(u)(du/2)$ $\frac{1}{2}\int\cos(u)du$ and students can quickly see how to "handle the $2x$" in the original integrand, correctly getting: $F(x)=1/2(\sin(2x))+C$.

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Essential Understandings:
- EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
20 instructional hours

Guiding Questions:

- **What is the relationship between differentiation and integration?**
- **How does the area under a curve relate to the idea of an accumulation process?**
- **How can the Fundamental Theorem of Calculus be used to solve problems?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.2A(a): Interpret the definite integral as the limit of a Riemann sum. **LO 3.2A(b):** Express the limit of a Riemann sum in integral notation. **LO 3.2B:** Approximate a definite integral.	**Print** Larson and Edwards, sections 4.2 and 4.6	**Instructional Activity: Using Rectangles to Approximate Area Under a Curve** Building on skills learned in geometry, we discuss using known geometrical shapes (rectangles and trapezoids) to approximate the area under a curve. I start with the function $F(x)=x^2+1$ and I ask students to sketch this graph and divide the x axis under the curve into four equal sections from zero to four. Using four rectangles we can approximate the area under this curve: we discuss the different ways we can draw the rectangles and which method is the most accurate. I then introduce Riemann sums and the different approximation techniques and how we can calculate the area under this curve using these different methods. We have a class discussion about which technique is the most accurate (all of them are more accurate with more rectangles).

The common abbreviations that I use are:

- *LRAM = left rectangular approximation method*
- *RRAM = right rectangular approximation method*
- *MRAM = midpoint rectangular approximation method*
- *TRAPAP = trapezoidal approximation (although technically not a Riemann sum since we are not using rectangles, it is logical to include this approximation technique at this time).*

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Essential Understandings:
- EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
20 instructional hours

Guiding Questions:

▸ What is the relationship between differentiation and integration? ▸ How does the area under a curve relate to the idea of an accumulation process? ▸ How can the Fundamental Theorem of Calculus be used to solve problems?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.3B(b): Evaluate definite integrals.	**Print** Larson and Edwards, section 4.3	**Instructional Activity: Deriving the Evaluative Component of the Fundamental Theorem of Calculus (FTC)** After my students are comfortable with the approximation techniques for estimating the area under a curve I use this problem to help them derive the evaluative component of the FTC. I ask them to sketch $\frac{dy}{dx}=2x$ from $x=0$ to $x=4$, which I label as a first derivative. Students can now see the function as $F(x)=x^2$ and quickly calculate the area under the first derivative graph as equal to 16. I then ask them to think about the upper/lower bounds and whether they see any pattern. Usually one student will yell out that $x^2=16$ and that they can just plug in the upper bound into our function to get the area under the curve. I then ask them to find the area under the first derivative graph from $x=1$ to $x=4$ and this is equal to 15: $$\int_a^b f(x)=F(b)-F(a).$$	*Some textbooks call this the FTC Part 1, other textbooks call this the FTC Part 2. I never use this language as I find it confusing for students.* *With some discussion and direction from me, usually some students will come up with the evaluative component of the FTC by themselves, which is really cool.*
		Formative Assessment: Fundamental Theorem of Calculus At this stage of this unit students get a quick formative assessment, typically four to five multiple-choice questions from previously released College Board Exams. All of these questions address the FTC in different ways.	*The FTC is an important unit in my class and after this assessment, I typically go back and review all the questions and do some reteaching as needed. I will not move on to another concept until all students are proficient with this concept.*

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Essential Understandings:
- EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
20 instructional hours

Guiding Questions:

- **What is the relationship between differentiation and integration?**
- **How does the area under a curve relate to the idea of an accumulation process?**
- **How can the Fundamental Theorem of Calculus be used to solve problems?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.3A: Analyze functions defined by an integral. **LO 3.4A:** Interpret the meaning of a definite integral within a problem.	**Print** Larson and Edwards, section 4.4 **Web** Free-response questions available at AP Central	**Instructional Activity: Functions Defined by an Integral** As part of the FTC, students need to be able to recognize and work with functions defined by an integral such as $g(x)=\int_a^x f(t)dt$. At first many students struggle with this equation and are not sure what it tells us. If we integrate $f(t)$ we get $F(t)$, which has to equal the left side of the equation or $g(x)$. It is important that students can make this connection; this concept is often tested on the AP Calculus Exam, many times with a graphical approach. I also want my students to be able to take the derivative of $g(x)=\int_a^x f(t)dt$ to get $g'(x)=f(x)$ since $d/dx(g(x))=\frac{d}{dx}\int_a^x f(t)dt$, which is the FTC in action! On the right-hand side of the equation above the derivative of the integral with respect to x is $f(x)$.

When writing a function defined by an integral we need to make a subtle but important change. Since our upper limit of integration is a variable now (typically x), our integrand needs to be a different variable, typically t (sometimes called a dummy variable).

At this point I want students to see that differentiation and integration are inverse operations. Using some previously released College Board multiple-choice and free-response questions the students work through a number of problems in small groups.

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Essential Understandings:
- EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
20 instructional hours

Guiding Questions:
▸ **What is the relationship between differentiation and integration?** ▸ **How does the area under a curve relate to the idea of an accumulation process?** ▸ **How can the Fundamental Theorem of Calculus be used to solve problems?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.4B: Apply definite integrals to problems involving the average value of a function. **LO 3.4C:** Apply definite integrals to problems involving motion. **LO 3.4E:** Use the definite integral to solve problems in various contexts.	**Print** Larson and Edwards, section 4.4	**Instructional Activity: Average Value of a Function** Often in the real world we want to calculate the average value of a function over a certain interval. For example, a plane starts at a velocity of zero, speeds up to a velocity of 500 miles/hour, lands, and then stops at the gate. What was your average velocity over your time frame/interval? Students quickly remember this formula as the area of a rectangle, $A=b*h$, and we can quickly rewrite this formula as $A/b=h$. In this case h would be your average value or height and then I can introduce this calculus formula for students, $F(x)=\int_a^b f(x)dx/(b-a)$, where: $h=F(x)$ (average value) and $\int_a^b f(x)dx/(b-a)$ (area/base), which is a nice easy connection for students to make from their study of geometry.	*If I can get my students to see the average value as similar to the area of a rectangle from geometry, this calculus formula is much easier to understand and remember.*
All of the learning objectives in this unit are addressed.	**Web** Free-response questions available at AP Central	**Summative Assessment: Unit 5 Exam** For an assessment at the end of this unit, I give students three to four previously released College Board free-response questions related to the FTC. Students have a week to work on these problems and each student works with a partner. I grade these free-response questions using the College Board scoring guidelines (rubrics) and this assessment has the same weight as an exam.	*The FTC is such an important concept in the study of calculus that it's one unit I want to take my time with. There are many different types of problems where we can use the FTC in different ways and I want to expose my students to a wide variety of problems.* *This summative assessment addresses all of the guiding questions for the unit.*

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Essential Understandings:
- EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
20 instructional hours

Guiding Questions:

▸ What is the relationship between differentiation and integration? ▸ How does the area under a curve relate to the idea of an accumulation process? ▸ How can the Fundamental Theorem of Calculus be used to solve problems?

Learning Objectives	Materials	Instructional Activities and Assessments
All of the learning objectives in this unit are addressed.		**Summative Assessment: Fundamental Theorem of Calculus Lab** Students receive an end-of-unit exam including some multiple-choice, short-answer, and free-response questions. I always have a section that allows calculator usage and a section that does not allow a calculator, similar to the AP Calculus Exam. For this assessment I emphasize problems on the FTC and average value.

This summative assessment addresses all of the guiding questions for the unit.

Mathematical Practices for AP Calculus in Unit 5

The following activities and techniques in Unit 5 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: In all the activities throughout the unit students develop the ability to use the various FTC formulas to problem solve in a variety of different problems.

MPAC 2 — Connecting concepts: In the "Functions Defined by an Integral" instructional activity students discover that the derivative and integration are inverse operations of each other and see just how powerful the FTC is.

MPAC 3 — Implementing algebraic/computational processes: The first activity with integrating $cos(2x)$ involves selecting appropriate strategies and completing the algebraic computations (in the u-substitution) correctly.

MPAC 4 — Connecting multiple representations: In the "Average Value of a Function" instructional activity students can relate this new calculus formula to a very simple formula they studied in geometry, making the new formula much easier to remember.

MPAC 5 — Building notational fluency: In the "Deriving the Evaluative Component of the Fundamental Theorem of Calculus" instructional activity students learn how important correct notation can help them understand problems and the relationship between the first derivative and its corresponding function.

MPAC 6 — Communicating: Students needs to be able to effectively communicate their final answer in a real-world contextual setting and this skill is always tested on the free-response portion of the AP Exam. Students learn how to interpret the results from their graphing calculators for solving problems where their answers need precision.

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Essential Understandings:
- EU 2.1, EU 2.2, EU 3.1

Estimated Time:
10 instructional hours

Guiding Questions:
- **How can we use the skills learned so far to work with transcendental functions?**
- **How can the derivative of a function equal its first derivative (and other higher-order derivatives)?**
- **How is using the chain rule for derivatives of transcendental functions similar to and different from using the chain rule for polynomial functions?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.1C: Calculate derivatives.	**Print** Larson and Edwards, sections 5.1 and 5.3	**Instructional Activity: The Definition of Euler's Number *e*** In this activity students learn another definition for Euler's number e, which they should have studied in precalculus. I write on the board that we can define e as $$\ln(e)=\int_1^e\left(\frac{1}{t}\right)dt=1$$ and therefore $$\ln(x)=\int_1^x\left(\frac{1}{t}\right)dt$$ and ultimately to the general formula $$\frac{d}{dx}(\ln u)=u'/u\,.$$

Some textbooks present these concepts as "early transcendentals" and other textbooks present them later in the textbook ("later transcendentals"). It really doesn't matter when students learn these skills and it might make sense to follow the order as presented in your textbook.

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Essential Understandings:
- EU 2.1, EU 2.2, EU 3.1

Estimated Time:
10 instructional hours

Guiding Questions:

▶ **How can we use the skills learned so far to work with transcendental functions?** ▶ **How can the derivative of a function equal its first derivative (and other higher-order derivatives)?** ▶ **How is using the chain rule for derivatives of transcendental functions similar to and different from using the chain rule for polynomial functions?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.2A: Use derivatives to analyze properties of a function.	**Print** Larson and Edwards, section 5.1	**Instructional Activity: The Derivative of $y = e^x$** This is such an interesting function, as the first derivative (and subsequent higher-order derivatives) is equal to the function. I usually graph $y = e^x$ on my calculator view screen and use the features of this calculator to graph the first derivative on the same viewing window. Students quickly see only one graph and I want a student to call out that the first derivative of e^x is equal to the function $y = e^x$. I also make a point that this derivative can be written as $\frac{dy}{dx}$ *(ln base)(original function)(Derivative of exponent)*, which would give us $\frac{dy}{dx} = (\ln e)(e^x)(1)$. This is not really needed for this function, but later when we take the derivative of more complicated functions, this process makes the derivative much easier for students: $y = 2^{(3x+2)} \quad \int (1)dy = \int (2x)dx$ $\frac{dy}{dx} = (\ln 2)(2^{3x+2})(3)$.	*I call this my three-step "recipe": I find my students can remember this algorithm and they make fewer mistakes with these types of derivatives.*
LO 2.1C: Calculate derivatives.		**Formative Assessment: Natural Log** Students are given a number of problems where they have to both differentiate and integrate some transcendental functions. These formative assessments usually take 10–15 minutes of class time.	*I always go through a couple of these problems on the board so all students can see the solution and if needed, will go back and review/refresh these concepts.*

UNIT 6: TRANSCENDENTAL FUNCTIONS — DERIVATIVES AND INTEGRALS

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Essential Understandings:
- EU 2.1, EU 2.2, EU 3.1

Estimated Time:
10 instructional hours

Guiding Questions:

▸ How can we use the skills learned so far to work with transcendental functions? ▸ How can the derivative of a function equal its first derivative (and other higher-order derivatives)? ▸ How is using the chain rule for derivatives of transcendental functions similar to and different from using the chain rule for polynomial functions?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.1A: Recognize antiderivatives of basic functions. **LO 3.2C:** Calculate a definite integral using areas and properties of definite integrals.	**Print** Larson and Edwards, sections 5.2 and 5.4	**Instructional Activity: Integrating e^x** Students should quickly see that when we integrate the function $\int e^x dx$ we get $F(x) = e^x + c$. For more complicated functions we can use u-substitution or pattern recognition to arrive at the correct antiderivative. The rules for definite integrals still apply and we can use these same rules with these transcendental functions.	*Many students actually feel working with these transcendental functions is easier than some of the derivatives/integration we have done earlier in the course. I like the later transcendentals as students need to take both derivatives and integrals in this unit: as we get closer to the AP Exam it is important for students to be able to solve a variety of problems in different scenarios.*
All of the learning objectives in this unit are addressed.		**Summative Assessment: Unit 6 Exam** Students receive an end-of-unit exam including some multiple-choice, short-answer, and free-response questions. I always have a section that allows calculator usage and a section that does not allow a calculator, similar to the AP Exam. Special emphasis on this summative assessment is on transcendental functions where students have to use the chain rule when taking derivatives and u-substitution when integrating.	*I like to introduce these transcendental concepts later in the course. Many books/teachers introduce them earlier in the year. In this unit students have to take derivatives and integrate, and as we get closer to the AP Exam it is good practice to have a mix of problems with regard to differentiation and integration.* *This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 6

The following activities and techniques in Unit 6 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: Students need to be able to make conjectures on tasking derivatives of transcendental functions where the "rules" are different from polynomial functions.

MPAC 2 — Connecting concepts: In the "Integrating e^x" instructional activity students quickly see the relationship between differentiation and integration and how the transcendental functions behave differently from the polynomial functions they studied earlier in the year.

MPAC 3 — Implementing algebraic/computational processes: In the "Integrating e^x" instructional activity students learn how important it is to correctly use algebraic rules when computing derivatives of transcendental functions (and hopefully all types of functions).

MPAC 5 — Building notational fluency: In "The Definition of Euler's Number e" instructional activity students learn how to interpret various nomenclatures and to correctly differentiate and integrate the different transcendental functions.

MPAC 6 — Communicating: In all the activities in this unit, students need to be able to communicate their final answer in a real-world contextual setting to make a decision.

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Essential Understandings:
- EU 3.1, EU 3.5

Estimated Time:
10 instructional hours

Guiding Questions:

- **How can we use a slope field to learn about the function and how it behaves?**
- **How can we build upon our knowledge of integration/antiderivatives to solve more complicated differential equations (DEs) using the procedure of separation of variables?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.5A: Analyze differential equations to obtain general and specific solutions. **LO 2.3E:** Verify solutions to differential equations. **LO 2.3F:** Estimate solutions to differential equations.	**Print** Larson and Edwards, sections 6.1–6.3	**Instructional Activity: What Is a Slope Field?** In this activity students learn what a slope field is and how to interpret/read one. In the simplest of definitions, a slope field is a small window of slope segments representing the function graphically. I put the following differential equation (DE) on the board and ask the students "What is the function defined by this DE?" $\frac{dy}{dx}=2x$ Students say that the function is $F(x)=x^2$ and I sketch the following functions: $F(x)=x^2+2$ $F(x)=x^2-3$ $F(x)=x^2+5.$ Students quickly see that these could also be solutions to the DE and I sketch these graphs on my whiteboard. Adding the constant of integration, $F(x)=x^2+c$, is very logical to students. Using one of my fingers, I mark the continuous solutions on the board, a slope field consisting of small slope segments.

I like to introduce slope fields on the first day of antiderivatives, way back in section 4.1 in my textbook.

In my classroom we affectionately call the constant of integration a "cupcake." Don't forget your cupcake!!

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Essential Understandings:
- EU 3.1, EU 3.5

Estimated Time:
10 instructional hours

Guiding Questions:

▸ How can we use a slope field to learn about the function and how it behaves? ▸ How can we build upon our knowledge of integration/antiderivatives to solve more complicated differential equations (DEs) using the procedure of separation of variables?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.5A: Analyze differential equations to obtain general and specific solutions. **LO 2.3E:** Verify solutions to differential equations. **LO 2.3F:** Estimate solutions to differential equations.	**Print** Larson and Edwards, sections 6.1–6.3	**Formative Assessment: Slope Fields** I give students a number of slope field graphs where they need to match the slope field with the associated DE. I make sure I include one slope field graph where students need to match this graph with the function, not the DE.	*I actually introduce slope fields quickly when we first take antiderivatives. Depending on how students do on this assessment, I may have to give them more sample problems to work on and many times we actually create a slope field at 20–30 different points on a piece of graph paper.*
LO 3.5B: Interpret, create, and solve differential equations from problems in context. **LO 2.3E:** Verify solutions to differential equations.	**Print** Larson and Edwards, sections 6.2 and 6.3	**Instructional Activity: We've Been Separating Variables All Year!** In this activity I want my students to see that they have really been separating the variables all year, at least since they started finding antiderivatives. Using $\frac{dy}{dx}=2x$ as a very simple DE, I ask my students if we can multiply both sides of the equation by dx, getting $dy=(2x)dx$. I then ask them if we can integrate both sides of the equation, the left side with respect to y and the right side with respect to x: $$\int(1)dy=\int(2x)dx.$$ Again, students seem comfortable with this even though they think it might be "too much work" for this easy DE, arriving at the correct solution: $y=x^2+c$. Using this logic of separating the variables, we quickly move on to more complicated DEs such as $\frac{dy}{dx}=(y/2x^2)$.	*On many AP Exams one of the free-response questions is related to a DE and separation of variables. Unfortunately, many students do not perform well on these questions and they have a hard time "separating correctly the variables." Make sure your students have lots of practice with these types of questions and can algebraically separate the variables correctly.*

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Essential Understandings:
- EU 3.1, EU 3.5

Estimated Time:
10 instructional hours

Guiding Questions:

▸ How can we use a slope field to learn about the function and how it behaves? ▸ How can we build upon our knowledge of integration/antiderivatives to solve more complicated differential equations (DEs) using the procedure of separation of variables?

Learning Objectives	Materials	Instructional Activities and Assessments	
All of the learning objectives in this unit are addressed.		**Summative Assessment: Unit 7 Exam** Students take an end-of-unit exam including some multiple-choice, short-answer, and free-response questions. I always have a section that allows calculator usage and a section that does not allow a calculator, similar to the AP Exam. On this assessment I emphasize problems where students need to separate the variables, which tends to be a tough skill for students. When given a DE, this is a good time for students to see the Rule of Four in action. They can solve the equation algebraically, graph a slope field to see what the function looks like, use a table of values, and communicate their results in a real-world setting to draw a conclusion.	*This summative assessment addresses all of the guiding questions for the unit.*
All of the learning objectives in this unit are addressed.	**Web** Free-response questions available at AP Central	**Summative Assessment: Differential Equation Lab** For an assessment at the end of this unit, I give students three to four previously released College Board free-response questions related to the FTC. Students have a week to work on these problems and each student works with a partner. I grade these free-response questions using the College Board scoring guidelines (rubrics) and this assessment has the same weight as an exam.	*This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 7

MPAC 1 — Reasoning with definitions and theorems: Students learn to build arguments about antiderivatives with $+C$ creating a family of solutions rather than a single solution.

MPAC 2 — Connecting concepts: From the slope field activity students should be able to match both the associated differential equation (DE) and function connecting these two concepts.

MPAC 3 — Implementing algebraic/computational processes: In both the slope field and DE activities, students learn the importance of correctly using the rules of algebra to separate the variables and then integrating using good mathematics.

MPAC 4 — Connecting multiple representations: Using slope fields, students learn to visualize these graphs as the "family" of vertically stacked functions and how this is represented by the constant of integration $+C$.

MPAC 5 — Building notational fluency: Students must be comfortable using proper notation with differential equations and then correctly solving for their final answer in the format $y=$.

MPAC 6 — Communicating: In this unit students learn how to explain the relationship between a DE and the function, as well as communicating the results they get from a slope field.

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Essential Understandings:
- EU 3.4

Estimated Time:
14 instructional hours

Guiding Questions:

- How can we find the area between two curves? ▸ How can we find the volume of a solid of revolution? ▸ How can we find the volume of a solid using other geometric shapes besides a circle?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.2C: Calculate a definite integral using areas and properties of definite integrals. **LO 3.4D:** Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.	**Print** Larson and Edwards, section 7.1	**Instructional Activity: Area Between Two Curves** In this activity students learn to find the area between two curves and hopefully realize they have been doing this since they learned about the definite integral. I start with an easy function and ask the students to find the area under the function $f(x)=x^2$, from $x=0$ to $x=2$. Students quickly compute this value and I ask them about the lower curve. Hopefully a student recognizes that the “lower curve” is also a function and has always been: $g(x)=0$. Extending this realization to develop the following formula is very intuitive for most students: $A=\int_a^b (f(x)-g(x))dx$. Because we are finding the area between two curves, we no longer need to worry about positive or negative accumulations (areas).
LO 3.4D: Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.	**Print** Larson and Edwards, section 7.2 **Supplies** A large Hershey’s Kiss, access to a deli slicer	**Instructional Activity: How Do We Find the Volume of a Large Hershey’s Kiss?** In this activity I want students to visualize slicing a 3-D solid into circular slices and how we can accumulate these circular slices to arrive at a volume. I buy a large Hershey’s Kiss and ask the students what the volume is of this piece of chocolate. Students struggle with this question and then I ask them if we can slice the kiss into smaller circular slices. I ask the cafeteria staff to put the kiss on a deli slicer and to cut it into as many circular slices as possible. When we have our circular slices, we add them all up (put the Hershey’s Kiss back together). After this activity, students are usually very comfortable with this formula: $$V=\int_a^b \pi(r)^2 dx$$ where $f(x)=r$.

Students will never be asked a question on the AP Exam where they have to use cylindrical shells to solve for a volume. All the problems can be solved using either disks or washers. Of course teachers can teach shells to their students (and students can use this method on the AP exam) but shells are not in the AP Calculus AB curriculum.

UNIT 8: AREA/VOLUME OF REVOLUTION — APPLICATIONS OF INTEGRATION

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Essential Understandings:
- EU 3.4

Estimated Time:
14 instructional hours

Guiding Questions:

- **How can we find the area between two curves?**
- **How can we find the volume of a solid of revolution?**
- **How can we find the volume of a solid using other geometric shapes besides a circle?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.4D: Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.		**Instructional Activity: Revolving Around the *Y* Axis** In this activity students learn how to rotate a function around the y axis. In these types of problems students need to solve for x in terms of y, determine the bounds in terms of y, and use the formula: $V=\int_c^d \pi(r)^2 dy$ where $f(y)=r$. I start with an easier function like $y=x^{\wedge}2$ from $x=0$ to $x=2$ and ask them to revolve this equation around the y axis. Students need to solve for x and also solve for the upper/lower bounds in terms of y. I usually solve one problem on the board and then have the students work on some problems in small groups.
LO 3.4D: Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.		**Instructional Activity: Revolving Around Any Horizontal or Vertical Line** For many problems students need to revolve a function, $f(x)=x^2$, around a vertical (or horizontal) line such as $y=4$ from $x=0$ to $x=2$. In this example, there will be both an inner and outer radius and our volume formula will be: $$V=\int_0^2 \pi((R)^2-(r)^2)dx$$ $$V=\int_0^2 \pi((2)^2-(2-(x^2))^2)dx.$$

Students should feel comfortable solving problems around any vertical or horizontal line, not just the x and y axes. Many problems have both an "inner and outer" radius.

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Essential Understandings:
- EU 3.4

Estimated Time:
14 instructional hours

Guiding Questions:

▸ How can we find the area between two curves? ▸ How can we find the volume of a solid of revolution? ▸ How can we find the volume of a solid using other geometric shapes besides a circle?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.4D: Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.		**Formative Assessment: Area Volume** At this stage of this unit students get a quick formative assessment, typically four to five multiple-choice questions from previously released College Board Exams, including at least one to two problems on area between two curves and one to two volume-of-revolution problems.
LO 3.4D: Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.	**Supplies** Multiple protractors of various sizes	**Instructional Activity: Volume with Known Cross-Sectional Area Using Protractors** In this activity students get to "build" a 3-D solid on the whiteboard using protractors. I draw a large right triangle on the board and tell students this is the base of a 3-D solid. The solid is "coming out of the board" and we'll use semicircles to get a volume. We find all the different-sized protractors in the room (and there are always plenty as I also teach geometry) and I ask students to come up to the board and fit their protractor on the sides on the triangle. When I get six to eight students and the different-sized protractors students can easily "see" what this solid will look like. Deriving this formula for volume is an easy next step for students: $V=\int_a^b(\pi r^2/2)dx$.

These concepts are always tested on the AP Exam and I want all of my students to be proficient with these concepts. Typically this is the last unit I teach and I will slow down and reteach/reintroduce these concepts if needed.

UNIT 8: AREA/VOLUME OF REVOLUTION — APPLICATIONS OF INTEGRATION

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Essential Understandings:
- EU 3.4

Estimated Time:
14 instructional hours

Guiding Questions:

▸ How can we find the area between two curves? ▸ How can we find the volume of a solid of revolution? ▸ How can we find the volume of a solid using other geometric shapes besides a circle?

Learning Objectives	Materials	Instructional Activities and Assessments	
All of the learning objectives in this unit are addressed.		**Summative Assessment: Unit 8 Exam** Students receive an end-of-unit exam including some multiple-choice, short-answer, and free-response questions. I always have a section that allows calculator usage and a section that does not allow a calculator, similar to the AP Exam. Emphasis in this assessment will be on volume-of-revolution problems where students revolve a function(s) around both the x, y axes as well as other x, y values.	*Volume of revolution is an important topic and students should feel comfortable revolving functions around the x, y axes as well as other horizontal and vertical lines where there could be both an inner and outer radius. These types of problems are almost always tested on the free-response section of the AP Exam and students should feel comfortable solving these problems both with and without their graphing calculators.* *This summative assessment addresses all of the guiding questions for the unit.*
All of the learning objectives in this unit are addressed.	**Web** Free-response questions available at AP Central	**Summative Assessment: Area/Volume Lab** For an assessment at the end of this unit, I give students three to four previously released College Board free-response questions related to the FTC. Students have a week to work on these problems and each student works with a partner. I grade these free-response questions using the College Board scoring guidelines (rubrics) and this assessment has the same weight as an exam.	*It is important for students to be able to find the volume of a solid with a variety of function types in revolutions/cross sections.* *This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 8

MPAC 1 — Reasoning with definitions and theorems: In all of the activities for this unit, students get to see how/why the formulas we use for area/volume work and how they are derived.

MPAC 2 — Connecting concepts: In the Hershey's Kiss activity students get to see firsthand how we can "slice" a solid into circles (disks) and add these all up to get a 3-D solid. At the end of this activity of course they get to eat this solid as an added bonus!

MPAC 3 — Implementing algebraic/computational processes: Students need to feel comfortable working with the area and volume-of-revolution formulas to solve many different types of problems. Students also need to feel comfortable with the different algebraic manipulations needed to solve these types of problems.

MPAC 4 — Connecting multiple representations: Students learn how to use the formulas for volume of a solid to help them visualize what the 3-D solid looks like.

MPAC 5 — Building notational fluency: In both the "How Do We Find the Volume of a Large Hershey's Kiss" and the "Volume with Known Cross-Sectional Area Using Protractors" instruction activities students can easily connect the definite integral to a Riemann sum type of scenario where we add up an infinite number of shapes, circles/disks, washers, squares, semicircles, and so on.

MPAC 6 — Communicating: Students need to be able to correctly use units in their final answers and explain their final answer in a contextual setting: i.e. for area problems units should be squared feet, and for volume problems units should be feet cubed.

Resources

General Resources

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Supplementary Resources

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