AP® Calculus AB

Course Planning and Pacing Guide

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CollegeBoard

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Welcome to the AP Calculus AB Course Planning and Pacing Guides

This guide is one of several course planning and pacing guides designed for AP® Calculus AB teachers. Each provides an example of how to design instruction for the AP course based on the author's teaching context (e.g., demographics, schedule, school type, setting). These course planning and pacing guides highlight how the components of the *AP Calculus AB and BC Curriculum Framework*, which uses an Understanding by Design approach, are addressed in the course. Each guide also provides valuable suggestions for teaching the course, including the selection of resources, instructional activities, and assessments. The authors have offered insight into the *why* and *how* behind their instructional choices — displayed along the right side of the individual unit plans — to aid in course planning for AP Calculus teachers.

The primary purpose of these comprehensive guides is to model approaches for planning and pacing curriculum throughout the school year. However, they can also help with syllabus development when used in conjunction with the resources created to support the AP Course Audit: the Syllabus Development Guide and the four Annotated Sample Syllabi. These resources include samples of evidence and illustrate a variety of strategies for meeting curricular requirements.

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Instructional Setting

South Florence High School ▶ Florence, South Carolina

School

There are approximately 1,650 students at South Florence High School. It is in a suburban area and is one of three high schools in Florence Public School District One (FDS1). We serve students in grades 9–12. The school's mission is "to prepare students to become responsible, world-class citizens and high-achieving, lifelong learners."

Student population

The student demographics are 47 percent Caucasian, 48 percent African American, 2 percent Hispanic, 2 percent other, and 1 percent Asian. Fifty-six percent of the student population receives free/reduced lunch; 82–85 percent of students go on to a two- to four-year college and we have a 99 percent attendance rate. Approximately 10 percent of students are enrolled in AP classes.

We do not offer a course in computer science, but several business classes are offered at the Florence Career Center, which is on the same campus. This facility offers the following courses: Administrative Support Technology, Computer Programming 1 and 2, Networking 1 and 2, Exploring Computer Science, and Introduction to Engineering. The combined enrollment in these courses is approximately 80 students. We have three computer labs and eight laptop carts.

Eighteen percent of the school population is in the special education program. We are the site for students who are self-contained autistic and severely learning disabled, as well as the site for a separate program for all students who are FSD1 trainable mentally disabled.

Instructional time

The academic year typically begins the third week of August and ends after the first week in June. Students attend for 180 days each year, which gives them approximately 155 days of instructional time and preparation before the AP Calculus Exam. The year is divided into two semesters, and students attend four 90–minute courses each day. For the second semester, my AP Calculus class splits into two classes: AB and BC. Because of the split, I am able to slow down quite a bit for Calculus AB starting in January. I have to work at a quicker pace during the first semester to be sure Calculus BC won't be behind in the second semester.

Students are encouraged to stay after school for study sessions as needed. Each year in April, students from across the region go to the local university to take one full practice AP Calculus Exam.

Instructional Setting (continued)

Student preparation

AP Calculus AB is offered to both juniors and seniors who have met all prerequisites, including Algebra I and II, Geometry, Precalculus, and Differential Calculus Honors. Students are not required to do any summer work, but we typically spend the first two or three days of class reviewing some concepts from precalculus.

Primary planning resources

Finney, Ross L., Franklin D. Demana, Bert K. Waits, and Daniel Kennedy. *Calculus: Graphical, Numerical, Algebraic*. 4th ed. Boston, MA: Pearson, 2012.

Overview of the Course

Mastering calculus, in many ways, can be likened to playing an instrument. To truly excel in the art form, one must have knowledge of the underlying theory, speak the language, and be able to collaborate with others. Students are given essential questions to support enduring understandings as they learn new concepts. I emphasize the importance of using the correct mathematical language and notation on all assignments from the very beginning of the course. The classroom is an open forum with lots of student interaction in class discussion. The 90-minute class period is divided into chunks of time devoted to student exploration, lecture, and collaborative practice. Technology is used liberally throughout the course as a way to verify hypotheses formed by examining data and manipulating graphs. Students are encouraged to analyze problems algebraically, graphically, and numerically.

Students often have difficulty expressing their answers verbally in the mathematics classroom. They can show the work algebraically but aren't sure what it means. Teaching this skill requires communication that is accurate and precise during instruction. If I want my students to say "The derivative of $f(x)$ is positive" instead of "It's positive," then I have to be mindful of that language as I teach. Also, I have found that asking students to simply read problems or answers aloud aides them in their written solutions. Another subject-specific skill students have trouble mastering is the interpretation of graphs. For example, a student might be asked to use the graph of $f'(x)$ to give information about $f(x)$ or $f''(x)$. I have found that students tend to make more connections if they can get involved with the graph kinesthetically. I encourage my students to use their hands to model the behaviors of the functions: their pencils become moving tangent lines! The more learning styles represented in the lesson, the better the experience for students.

I have also debated with myself on the issue of reviewing algebra topics in AP Calculus since I began teaching the course. I decided it's best to review topics as they present themselves. Spending a week or two at the beginning of the course to review prior subject matter wastes valuable time, and I found that I ended up reviewing later in the year anyway. To create a learning environment that meets the needs and learning styles of all students regardless of their prior knowledge, I present material in a variety of ways. Students are often given exploration activities so they can independently explore patterns and make their own conjectures. Using technology makes this process easier because students can manipulate curves and quickly observe changes. In addition to student-led activities, I do some lecturing and give notes for students who are auditory learners.

Formative assessments are the key to interpreting student understanding and meeting their needs. In my classroom, this type of assessment is typically very informal. Homework checks are done periodically to assess individual understanding. Students write the problems on the board that they did not understand from the previous night's homework. Other students are encouraged to solve those problems if they feel they did them correctly. This type of student-led review and discussion gives me more information about their grasp of the topic than if I were doing all the work. I also require students to make individual corrections to all quizzes before unit tests. Students attach these corrections to the quiz and get a separate grade for this work. All corrections require a written explanation of why they missed the problem and how they fixed it. These valuable reflections give insight into their misconceptions so I can clarify before the end of a unit. As students work on corrections, I discuss mistakes with individuals, small groups, and the whole class.

Mathematical Practices for AP Calculus (MPACs)

The Mathematical Practices for AP Calculus (MPACs) capture important aspects of the work that mathematicians engage in, at the level of competence expected of AP Calculus students. They are drawn from the rich work in the National Council of Teachers of Mathematics (NCTM) Process Standards and the Association of American Colleges and Universities (AAC&U) Quantitative Literacy VALUE Rubric. Embedding these practices in the study of calculus enables students to establish mathematical lines of reasoning and use them to apply mathematical concepts and tools to solve problems. The Mathematical Practices for AP Calculus are not intended to be viewed as discrete items that can be checked off a list; rather, they are highly interrelated tools that should be utilized frequently and in diverse contexts.

MPAC 1: Reasoning with definitions and theorems

Students can:

a. use definitions and theorems to build arguments, to justify conclusions or answers, and to prove results;

b. confirm that hypotheses have been satisfied in order to apply the conclusion of a theorem;

c. apply definitions and theorems in the process of solving a problem;

d. interpret quantifiers in definitions and theorems (e.g., "for all," "there exists");

e. develop conjectures based on exploration with technology; and

f. produce examples and counterexamples to clarify understanding of definitions, to investigate whether converses of theorems are true or false, or to test conjectures.

MPAC 2: Connecting concepts

Students can:

a. relate the concept of a limit to all aspects of calculus;

b. use the connection between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, antidifferentiation) to solve problems;

c. connect concepts to their visual representations with and without technology; and

d. identify a common underlying structure in problems involving different contextual situations.

MPAC 3: Implementing algebraic/computational processes

Students can:

a. select appropriate mathematical strategies;

b. sequence algebraic/computational procedures logically;

c. complete algebraic/computational processes correctly;

d. apply technology strategically to solve problems;

e. attend to precision graphically, numerically, analytically, and verbally and specify units of measure; and

f. connect the results of algebraic/computational processes to the question asked.

Mathematical Practices for AP Calculus (MPACs)

MPAC 4: Connecting multiple representations

Students can:

a. associate tables, graphs, and symbolic representations of functions;
b. develop concepts using graphical, symbolical, or numerical representations with and without technology;
c. identify how mathematical characteristics of functions are related in different representations;
d. extract and interpret mathematical content from any presentation of a function (e.g., utilize information from a table of values);
e. construct one representational form from another (e.g., a table from a graph or a graph from given information); and
f. consider multiple representations of a function to select or construct a useful representation for solving a problem.

MPAC 5: Building notational fluency

Students can:

a. know and use a variety of notations (e.g., $f'(x)$, y', $\frac{dy}{dx}$);
b. connect notation to definitions (e.g., relating the notation for the definite integral to that of the limit of a Riemann sum);
c. connect notation to different representations (graphical, numerical, analytical, and verbal); and
d. assign meaning to notation, accurately interpreting the notation in a given problem and across different contexts.

MPAC 6: Communicating

Students can:

a. clearly present methods, reasoning, justifications, and conclusions;
b. use accurate and precise language and notation;
c. explain the meaning of expressions, notation, and results in terms of a context (including units);
d. explain the connections among concepts;
e. critically interpret and accurately report information provided by technology; and
f. analyze, evaluate, and compare the reasoning of others.

Pacing Overview

Unit	Hours of Instruction	Unit Summary
1: Limits and Continuity	15	In this unit, students learn how to evaluate limits and use them to determine the behaviors and characteristics of functions. Emphasis is placed on the connection between asymptotes and limits involving infinity. Students do the "Twizzlers graphs" activity as a formative assessment of their understanding before the unit test.
2: What Is a Derivative?	18	When introducing derivatives, it is essential that students first understand the concept as the limit of the difference quotient. This is when I like to discuss average rate of change versus instantaneous rate of change. Students need to understand the idea of local linearity and study the graphical relationships between a function and its derivative. After they have a conceptual understanding of the derivative, students learn rules for differentiation.
3: Implicit Differentiation and Derivatives of Transcendental Functions	12	Students take their new knowledge of derivatives from the last unit and apply it to find derivatives of composite, trigonometric, exponential, logarithmic, and inverse functions. They also learn how to take the derivative of a function that is not explicitly solved for a single variable. Students do a project on differentiating implicitly in addition to taking a summative unit test.
4: Using Derivatives to Analyze Functions	20	Now that students have a firm understanding of what a derivative is and how to calculate the derivative of various functions, we begin applying the concept. In this unit, students begin to explore the relationship between a function and its first and second derivatives. They use these derivatives to obtain properties about a function that allows for fairly accurate graphing without the use of a graphing device.
5: Applications of Derivatives	15	Students finally get a chance to see derivatives in action! This unit is all about applying the concept of a derivative to real-world situations. Students maximize volume and minimize distance in optimization problems. They also explore rates of change through problems involving related rates.
6: Motion and Rates of Change	10	This is another application of the derivative. Students explore rates of change in the context of rectilinear motion. They explore and examine position, velocity, and acceleration functions, and see how derivatives relate these concepts. This idea is studied further when students learn integration.

Pacing Overview (continued)

Unit	Hours of Instruction	Unit Summary
7: The Definite Integral	16	In this unit, students explore the geometric definition of a definite integral. They study the connections between differential and integral calculus as they learn about the Fundamental Theorem of Calculus. I emphasize the difference between the average value of a function and its average rate of change as students begin to make connections between the differentiation and integration.
8: Differential Equations and Antidifferentiation by Substitution	24	My AP Calculus class splits into two classes — AB and BC — at the beginning of this unit: This schedule allows me to slow down the pacing at this point for my AB students. In this unit, students study equations involving derivatives and apply these types of equations in the context of real-world situations. They sketch and analyze slope fields that model differential equations to determine information about the solution curves. Students also learn how to evaluate indefinite and definite integrals using substitution.
9: Applications of Definite Integrals	27	More emphasis is placed on the accumulation function as students study how the integral can represent net change. They apply the definite integral to determine areas between curves as well as volumes of revolution.

UNIT 1: LIMITS AND CONTINUITY

BIG IDEA 1
Limits

Enduring Understandings:
- EU 1.1, EU 1.2

Estimated Time:
15 instructional hours

Guiding Questions:

▸ In what ways can the limit of a function at a point be determined? ▸ How is finding the limit of a function at a point different from evaluating that function at a point? ▸ How can limits be used to determine the asymptotic and unbounded behavior of a function? ▸ How might limits aide in determining continuity of a function?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 1.1A(a): Express limits symbolically using correct notation. **LO 1.1A(b):** Interpret limits expressed symbolically. **LO 1.1B:** Estimate limits of functions.	**Web** Rahn, "Introduction to Limits"	**Instructional Activity: Introduction to Limits** Students work in small groups to complete this activity using their graphing calculators. The activity requires students to change the table step in their calculators and examine a function at a point for values to the left and right of that point. For example, students begin by examining values to the left and right of $x=1$ with a step of 0.1. Then they'll examine values near $x=1$ with a step of 0.01, which helps them to zoom in on the specified value. The language and notation for a limit has not yet been introduced, so students are exploring the idea for the first time.
LO 1.1A(a): Express limits symbolically using correct notation. **LO 1.1A(b):** Interpret limits expressed symbolically. **LO 1.1B:** Estimate limits of functions.	**Print** Finney et al., chapter 2	**Instructional Activity: Limit Notation and Estimation** I introduce limit notation and give students several limits to estimate. They complete this task by graphing the functions and using the "Graph Trace" option on their calculators to approach values strictly from the left, strictly from the right, and from both sides of a value. We explore and discuss these limits as a class.
LO 1.1A(a): Express limits symbolically using correct notation. **LO 1.1A(b):** Interpret limits expressed symbolically. **LO 1.1B:** Estimate limits of functions.		**Formative Assessment: Using Limit Notation** Students are put into pairs. Each student is given a random set of limits for some function, $f(x)$, at integer x values over a small interval (i.e., $\lim_{x\to 0} f(x)=2$, $\lim_{x\to 3^-} f(x)=-1$, $\lim_{x\to 3^+} f(x)=4$, $\lim_{x\to -1} f(x)=3$, and $\lim_{x\to 1} f(x)=-2$). Individuals must create a graph for $f(x)$ that has all of those limits. Then they switch graphs and must write the symbolic limits that are displayed on the graph they were given. Students discuss their work and check answers with one another.

As students work, I float around the room to assess individual understanding and check work. This could also be done in groups of four so that students have another person to collaborate with as they create graphs.

BIG IDEA 1
Limits

Enduring Understandings:
- EU 1.1, EU 1.2

Estimated Time:
15 instructional hours

Guiding Questions:

▸ In what ways can the limit of a function at a point be determined? ▸ How is finding the limit of a function at a point different from evaluating that function at a point? ▸ How can limits be used to determine the asymptotic and unbounded behavior of a function? ▸ How might limits aide in determining continuity of a function?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 1.1C: Determine the limits of functions.	**Print** Finney et al., chapter 2 **Web** "Calculus: Limits of Functions"	**Instructional Activity: Determining Limits Algebraically** After I give students the basic properties of limits (sum, difference, product, power, etc.), I provide them with a few direct substitution limits to evaluate. I then introduce $\lim_{x\to 1}\frac{x^2+x-2}{x-1}$. They discuss why this limit cannot be solved by direct substitution and determine the limit using the TI-Nspire activity "Calculus: Limits of Functions." Students then brainstorm how they might come to this solution algebraically. After students come to the conclusion that they must use algebra rules to factor and simplify the function, they are split into small groups to practice evaluating limits using algebra rules.
LO 1.1C: Determine the limits of functions.	**Print** Finney et al., chapter 2 **Web** Khan, "Squeeze Theorem or Sandwich Theorem" Kuniyiki, "The Squeeze (Sandwich) Theorem"	**Instructional Activity: Useful Trigonometric Limits and the Squeeze (Sandwich) Theorem** Students estimate $\lim_{x\to 0}\frac{\sin(x)}{x}$ and $\lim_{x\to 0}\frac{1-\cos x}{x}$ graphically and numerically using their graphing calculators. They work with neighbors to evaluate other limits that can be solved using algebra and the limits above (i.e., $\lim_{x\to 0}\frac{\sin x}{x^2-x}$). The last problem students are asked to do, $\lim_{x\to 0}x^2\sin\left(\frac{1}{x}\right)$, cannot be solved with this technique, which leads into a discussion of the squeeze theorem. I really like the food analogy that Sal Khan uses to explain this theorem, so I play the first 3.5 minutes of his *Squeeze Theorem* video as an introduction to this idea. I then guide students through the first two examples from the PDF resource "The Squeeze (Sandwich) Theorem." Students use graphing calculators to support their answers graphically as they work.

Students often forget how to do the algebra required to simplify functions. I do take a few minutes in class to review topics like factoring, multiplying by a conjugate, and getting a common denominator as they work on evaluating limits.

BIG IDEA 1
Limits

Enduring Understandings:
- EU 1.1, EU 1.2

Estimated Time:
15 instructional hours

Guiding Questions:

▸ In what ways can the limit of a function at a point be determined? ▸ How is finding the limit of a function at a point different from evaluating that function at a point? ▸ How can limits be used to determine the asymptotic and unbounded behavior of a function? ▸ How might limits aide in determining continuity of a function?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 1.1D: Deduce and interpret behavior of functions using limits.	**Web** "Limits at Infinity"	**Instructional Activity: Limits Involving Infinity** Students are given a few minutes to work with an elbow partner (person immediately next to them) and brainstorm a list of facts about horizontal and vertical asymptotes. Students are encouraged to add these facts to the board and compile information about asymptotes that they have learned in previous classes. We then use the TI-Nspire activity "Limits at Infinity" to make the connection between a limit as x approaches $\pm\infty$ and horizontal asymptotes. Using tables in the graphing calculator, students investigate the behavior of $f(x)=\frac{1}{x}$ as $x \to 0$ and $x \to 0^{+}$.	*It is helpful for some students to think about what would happen if they "plug in" infinity when evaluating limits toward $\pm\infty$. While this may be helpful, it is important that students understand how to use proper notation. For instance, $\lim_{x\to\infty}\frac{1}{x}=\frac{1}{\infty}$ is not acceptable. It's also still important that they understand what these limits reveal about the function's graph.*
LO 1.1D: Deduce and interpret behavior of functions using limits.	**Web** "Limits at Infinity"	**Formative Assessment: Limits Involving Infinity** As a follow-up to the previous activity, students write, in their own words, an explanation of how limits involving infinity help to determine the behavior of functions. They then share their answers with the class.	*Students receive valuable feedback on their explanations through class discussion. They share answers in small groups as I listen in on the discussions, and we talk about their answers as a whole class. This helps me to clear up any misconceptions about the topic and really drive home the connection between limits and asymptotes.*

BIG IDEA 1
Limits

Enduring Understandings:
- EU 1.1, EU 1.2

Estimated Time:
15 instructional hours

Guiding Questions:

▸ In what ways can the limit of a function at a point be determined? ▸ How is finding the limit of a function at a point different from evaluating that function at a point? ▸ How can limits be used to determine the asymptotic and unbounded behavior of a function? ▸ How might limits aide in determining continuity of a function?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 1.2A: Analyze functions for intervals of continuity or points of discontinuity.	**Web** "One-Sided limits and Continuity with Piece-Wise Defined Functions"	**Instructional Activity: Continuity** Students engage in a discussion of the meaning of a continuous function, with emphasis on the fact that continuous functions are continuous at every point in their domain. Students name functions they know are continuous. We then discuss what is required to be continuous at a single point using sketches of possible scenarios to validate our assumptions. With this new set of requirements for continuity at a point, namely $f(a)$ exists and $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$, students use their graphing calculators to graph piecewise–defined functions and determine if they are continuous or not for given values of x. They must justify their answers using limits.	
LO 1.2A: Analyze functions for intervals of continuity or points of discontinuity.	**Supplies** Pull-and-peel Twizzlers, Lifesavers, M&M's, 5 × 5 coordinate planes	**Formative Assessment: Candy Limits and Continuity** After students are able to identify the four types of discontinuity (removable, jump, infinite, oscillating), they use this information and their prior knowledge of limits to create graphs with candy based on a given description. Twizzlers are used to create the curves, Lifesavers are open points on the graph, and M&M's are defined points. I show students statements like "$f(x)$ is continuous for all x except at $x = -1$ where it has a nonremovable discontinuity" or "$f(3)$ exists but $\lim_{x \to 3} f(x)$ does not." Students use candy to create a graph that meets the criteria and I walk around and observe as they work.	*As students do this activity, I encourage them to compare their graphs with those around them so they realize there isn't just one right answer. I circle the room and verify answers as they work. I usually find at this point that students have trouble interpreting and representing something like "$f(3)$ exists but $\lim_{x \to 3} f(x)$ does not." So this activity typically leads to a review of limits of a function versus defined values of a function.*
LO 1.2B: Determine the applicability of important calculus theorems using continuity.	**Web** "Intermediate Value Theorem"	**Instructional Activity: Intermediate Value Theorem** Students are given the Intermediate Value Theorem. They use the interactive Geogebra tool to answer two discussion questions: (1) Does the function have to be continuous on the interval $[a, b]$? Justify your answer; and (2) Are there some values of N for which there is more than one c such that $f(c) = N$? Does this contradict the theorem? Explain.	*This activity can be pulled up on students' smartphones as well as on computers.*

UNIT 1: LIMITS AND CONTINUITY

BIG IDEA 1
Limits

Enduring Understandings:
- EU 1.1, EU 1.2

Estimated Time:
15 instructional hours

Guiding Questions:

▸ In what ways can the limit of a function at a point be determined? ▸ How is finding the limit of a function at a point different from evaluating that function at a point? ▸ How can limits be used to determine the asymptotic and unbounded behavior of a function? ▸ How might limits aide in determining continuity of a function?

Learning Objectives	Materials	Instructional Activities and Assessments	
All of the learning objectives in this unit are addressed.		**Summative Assessment: Limits and Continuity Test** Students are given a test with some multiple-choice questions and some free-response questions. Like the AP Calculus Exam, a portion of the test is noncalculator and the other portion allows a calculator. The multiple-choice questions assess students' ability to estimate and evaluate limits, find points of discontinuity, and identify types of discontinuity. The first free-response question deals with students' ability to apply the Intermediate Value Theorem to determine the behavior of a continuous function on an interval. The second free-response question requires students to find limits and justify answers with graphs and tables.	*This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 1

The following activities and techniques in Unit 1 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: The “Candy Limits and Continuity” assessment requires students to apply the definition of a limit and continuity to model a given scenario graphically. In the “Intermediate Value Theorem” instructional activity, students must use technology to explore and make conjectures.

MPAC 2 — Connecting concepts: The “Limits Involving Infinity” formative assessment asks students to connect the idea of a limit to the visual representation of an asymptote on a graph.

MPAC 3 — Implementing algebraic/computational processes: In the “Determining Limits Algebraically” instructional activity, students must select appropriate mathematical strategies to evaluate limits. The “Continuity” instructional activity requires students to connect the results of computational processes to the question being asked (“Is the function continuous at the given point?”).

MPAC 4 — Connecting multiple representations: The “Introduction to Limits” instructional activity requires students to develop concepts using graphical and numerical representations with technology. In the “Useful Trigonometric Limits and the Squeeze (Sandwich) Theorem” instructional activity, students develop the concept graphically, symbolically, and using numerical representations.

MPAC 5 — Building notational fluency: In the “Using Limit Notation” formative assessment, students must connect limit notation to different representations as they graph and discuss verbally with their partners.

MPAC 6 — Communicating: Students analyze, evaluate, and compare one another’s reasoning in the “Using Limit Notation” instructional activity as they check one another’s work. In the “Limits Involving Infinity” formative assessment, students must be able to explain the connection between limits involving infinity and a function’s asymptotic behavior. The “Candy Limits and Continuity” activity assessment requires students to be able to compare the reasoning of others as they compare their graphs to their classmates’ graphs.

UNIT 2: WHAT IS A DERIVATIVE?

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 2.1, EU 2.2, EU 2.3

Estimated Time:
18 instructional hours

Guiding Questions:

- **In what ways might you explain what a derivative is?** ▸ **How does a function's graph relate to the graph of its derivative?** ▸ **How might you determine which rule to apply when differentiating a function?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.1A: Identify the derivative of a function as the limit of a different quotient. **LO 2.3A:** Interpret the meaning of a derivative within a problem. **LO 2.3C:** Solve problems involving the slope of a tangent line.	**Web** Rahn, "Introducing Instantaneous Rate of Change"	**Instructional Activity: Rates of Change** On the electronic whiteboard, I show a video I have made of someone pushing open a door in our hallway and allowing it to swing shut. We discuss the rate at which the door moves, and I ask the students if they think the door moved at the same rate over the entire period. After we spend a few minutes talking about this, I let them break into small groups to complete the activity. They use tables of values and average rates of change to narrow in on the rate of change of the door at exactly $t=1$ second. Through discussion, students deduce that instantaneous rate of change is the limit of average rate of change as the time interval approaches zero.	*The video I create is seven seconds long, which aligns nicely with the activity because $0 \le t \le 7$ is the domain of the function modeling the motion of the door with respect to time.*
LO 2.1A: Identify the derivative of a function as the limit of a different quotient. **LO 2.3A:** Interpret the meaning of a derivative within a problem. **LO 2.3C:** Solve problems involving the slope of a tangent line.	**Web** "Derivatives with Secant Lines" "Local Linearity"	**Instructional Activity: Graphical Representations of the Limit Definition of a Derivative** I use the Geogebra manipulative to supplement discussion on how the derivative of a function is the slope of a tangent line at a point. Students are asked to move point C close to point D from either side of D and observe the changes to the slope of the line. The goal is for students to visualize $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ and understand that this is the derivative at the point $x=a$. The "Local Linearity" activity adds to this discussion by showing students that functions begin to "merge" with their tangent lines as you zoom in on a point, which helps students to understand that the slope of a curve at a point is equivalent to the slope of the tangent line at that point.	*I also give students $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ when discussing the limit definition of a derivative and we practice using both definitions. I ask questions like "When is the slope of a secant line between two points a good estimate for the slope of the tangent line at a single point?"* *This could also be a good manipulative to use again later when introducing the Mean Value Theorem.*

UNIT 2: WHAT IS A DERIVATIVE?

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 2.1, EU 2.2, EU 2.3

Estimated Time:
18 instructional hours

Guiding Questions:

▸ **In what ways might you explain what a derivative is?** ▸ **How does a function's graph relate to the graph of its derivative?** ▸ **How might you determine which rule to apply when differentiating a function?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.1A: Identify the derivative of a function as the limit of a different quotient. **LO 2.3A**: Interpret the meaning of a derivative within a problem. **LO 2.3C**: Solve problems involving the slope of a tangent line.	**Web** "The Calculus Controversy" *Calculus: The Musical!*	**Instructional Activity: Notation for Derivatives** I show students the different notations for derivatives. I always hope that one curious student will ask why there are so many different versions, but if not, I trigger this conversation. Then, I show "The Calculus Controversy" video, which gives students a little bit of history about the subject and briefly explains the reason for varying notations. For fun, I also let students listen to the song "Differentiabul" from *Calculus: The Musical!* This song is a great tool for committing the limit of the difference quotient to memory and it makes mention of notation.
LO 2.1A: Identify the derivative of a function as the limit of a different quotient. **LO 2.3A**: Interpret the meaning of a derivative within a problem. **LO 2.3C**: Solve problems involving the slope of a tangent line.		**Formative Assessment: Derivatives and Tangent Lines** By this time, students have a formula for the derivative of a function at $x = a$, they know how to express a derivative using proper notation, and they understand that derivatives give the slope of a tangent line at $x = a$. I put students in groups and give each group different problems that they'll present to the class. Most problems involve finding the derivative at some point and writing the equation of the tangent line at that point. Students are asked to put their tangent lines in point-slope form. When students have completed the assignment, we discuss how graphing the function and tangent line on the same coordinate plane can help to verify our solutions.

As students prepare their presentations, I check in with each group to be sure they understand the concept and have done the work correctly. After the presentations, I discuss with the class common mistakes I noticed while they worked, like not properly distributing the negative through $f(a)$ when evaluating $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ for a function.

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 2.1, EU 2.2, EU 2.3

Estimated Time:
18 instructional hours

Guiding Questions:

- **In what ways might you explain what a derivative is?** ▸ **How does a function's graph relate to the graph of its derivative?** ▸ **How might you determine which rule to apply when differentiating a function?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.2A: Use derivatives to analyze properties of a function. **LO 2.1B:** Estimate derivatives.	**Web** "Sign of a Derivative"	**Instructional Activity: Introduction to Derivative Graphs** I display the "Sign of a Derivative" TI-Nspire document on the board and ask for student observations as I move point D on the curve. Then I turn on the "Geometry Trace" option, and students see the derivative graph being sketched out as I drag point D on the curve. They make the connection that that y value of the derivative graph is the slope at each point on the function graph.	
LO 2.2A: Use derivatives to analyze properties of a function. **LO 2.1B:** Estimate derivatives.	**Print** Finney et al., chapter 3	**Instructional Activity: Graphing the Derivative** I give each student the graph of a polynomial function, such as $f(x)=x^3-x^2-6x+2$ or $f(x)=x^2-1$. On the interval $[-3,3]$ I have the students draw tangent lines on the curve for each integer value of x to estimate the slope at each point. Then I have the students sketch the derivative graph on a separate grid. This isn't meant to be a very accurate sketch of a derivative: the intent is to drive home the relationship between these graphs.	
	Web "Graphical Derivatives"	**Formative Assessment: Graphical Derivatives** In this Texas Instruments document, students analyze two graphs on a split screen and determine which is the function and which is the derivative. They are also given a function graph and must sketch the graph of a derivative or vice versa. I let the students work in pairs and float around the room answering questions as they work.	*I give verbal feedback by interacting with student pairs as they work. I check their sketches and leave written feedback about their accuracy. The day after this activity, we compile a general list of patterns and relationships students observed between function graphs and derivative graphs.*

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 2.1, EU 2.2, EU 2.3

Estimated Time:
18 instructional hours

Guiding Questions:

▸ In what ways might you explain what a derivative is? ▸ How does a function's graph relate to the graph of its derivative? ▸ How might you determine which rule to apply when differentiating a function?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.2B: Recognize the connection between differentiability and continuity.	**Web** "Derivatives with Piece-Wise Defined Functions" *Calculus: The Musical!*	**Instructional Activity: Differentiability and Continuity** I ask students to work together to come up with a list of places on a graph where a unique tangent line could not be drawn. Students typically give answers such as "at a jump" or "at an asymptote." We eventually get a complete list of points on a curve where the derivative would fail to exist. I play the song "Ain't Differentiable" from *Calculus: The Musical!* a few times and ask students to jot down parts of the lyrics that include facts about differentiability and continuity. We share and discuss our favorite parts. Then students work together on the "Derivatives with Piece-Wise Defined Functions" activity. They graph the functions, give the derivatives, and state the points for which the derivative fails to exist and why.
LO 2.1C: Calculate derivatives.		**Instructional Activity: Power Rule** Students use the TI-Nspire CAS graphing calculator to take derivatives of various polynomial functions. I ask them to write down their observations and work with neighbors to come up with a rule for differentiating polynomials. This leads to a very informal version of the power rule. I give them the formal rule and we discuss how to differentiate sums and differences.
LO 2.1C: Calculate derivatives.	**Web** *Calculus: The Musical!*	**Instructional Activity: Product and Quotient Rules** I find that the best way to teach and learn rules for differentiation is simply by practice and repetition. I give students the product and quotient rules and play the "Product Rule" and "Quotient Rule" songs from *Calculus: The Musical!* I sometimes have students sing the songs as a class. If you have a particularly musical group, the "Quotient Rule" song can be sung as a round. After becoming familiar with the rules, I give the students a few functions and allow them to work together differentiating those functions using the power, product, and quotient rules.

In the "Derivatives with Piece-Wise Defined Functions" activity I usually allow students to take the derivative of the functions by using a CAS graphing calculator. However, taking the derivatives by hand could be a better way to get students to truly think about left- and right-hand derivatives.

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 2.1, EU 2.2, EU 2.3

Estimated Time:
18 instructional hours

Guiding Questions:

- **In what ways might you explain what a derivative is?** ▸ **How does a function's graph relate to the graph of its derivative?** ▸ **How might you determine which rule to apply when differentiating a function?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.1C: Calculate derivatives.		**Instructional Activity: Chain Rule** I give the students the function $f(x)=(2x+3)^2$ and ask them what rule they would use to differentiate this function. They usually want to use the power rule, so I let them do that and simplify the answer. Then we expand the function to a trinomial and differentiate again. The students notice that the answers are different, so I give them a few more similar functions so that they can try to find the missing piece. Once they notice that the derivative of the inner function is needed to use the power rule, I give them the formal version of the chain rule. I stress that this rule is used for compositions of functions.	*My students always need a quick review of what it means to be a composite function. I usually stop at some point during this lesson to give them some composite functions and let them practice naming the two functions that make up the composite. I need to know that my students can identify the inner function and the outer function before we begin using the chain rule.*
LO 2.1C: Calculate derivatives.	**Web** Schwartz, "Differentiation by the Chain Rule"	**Formative Assessment: Rules for Differentiation** Students use the "Differentiation by the Chain Rule" worksheet to practice using the power, product, quotient, and chain rules. For each problem they complete, I ask them to identify what rules they'll use before taking the derivative. Students work in groups and each group completes a different set of problems. After a few minutes of working, I have each group present and explain the problem they found most challenging from their set. Students have rich discussions both in their groups and as a class about when to use which rule and why. Finally, students complete an exit ticket by writing a few sentences to explain the process they go through as they decide when it's appropriate to use each rule.	*Students get a lot of feedback from one another in small groups as they work together on the practice problems. I also move from group to group and confirm or support student work as needed. Students receive written feedback on their exit tickets the next day. The group work helps me to target the types of problems for which they need more practice, and I give them more of those the next day.*

UNIT 2: WHAT IS A DERIVATIVE?

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 2.1, EU 2.2, EU 2.3

Estimated Time:
18 instructional hours

Guiding Questions:

▸ In what ways might you explain what a derivative is? ▸ How does a function's graph relate to the graph of its derivative? ▸ How might you determine which rule to apply when differentiating a function?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.1D: Determine higher order derivatives.		**Instructional Activity: Higher-Order Derivatives** To introduce the idea of higher-order derivatives, I put students into small groups and give each group three graphs in no particular order. I tell them that one graph is the function $f(x)$, one graph is the derivative of $f(x)$, and one graph is the derivative of that derivative, and ask them to label the graphs appropriately. Once they have completed this activity, I explain that the last graph is called a second derivative. We discuss how this process can be repeated to find higher-order derivatives, and I present students with the notation for these derivatives.
All of the learning objectives in this unit are addressed.		**Summative Assessment: Test on Derivatives** Students take a test on derivatives that includes some multiple-choice, short-answer, and free-response questions. Most of the test is noncalculator, but I do allow them to use a calculator for a few problems involving graphs. The test includes questions about derivatives represented algebraically, graphically, and numerically. For one of the free-response questions, I give students a table of values for f, f', f'' and g, g', g'' at two different x values. Using this information, students find derivatives for functions like $H(x)=\frac{f(x)}{g(x)}$ or $P(x)=f(x)\cdot g(x)$. This tests their knowledge of derivative rules while giving them practice using tables. Another important part of the test is matching function graphs to derivative graphs. The multiple-choice questions test their skills with applying the rules for differentiation and their knowledge of differentiability and continuity.

This summative assessment addresses all of the guiding questions for the unit.

Mathematical Practices for AP Calculus in Unit 2

The following activities and techniques in Unit 2 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: In the "Rates of Change" instructional activity, students use technology to explore the connection between limits and instantaneous rate of change to make conjectures. The "Introduction to Derivative Graphs" instructional activity gets students to develop conjectures based on the manipulation of a function and its tangent line using the graphing calculator.

MPAC 2 — Connecting concepts: Students must connect the concepts of limits and derivatives in "Derivatives with Secant Lines" as they begin to understand that a derivative is the limit of the difference quotient.

MPAC 3 — Implementing algebraic/computational processes: In "Graphing the Derivative," students must strive for accuracy and precision when creating tangent lines and computing slope in order to produce an appropriate graph of the derivative. The "Rules for Differentiation" formative assessment requires students to select the appropriate derivative rule to apply and complete the process correctly.

MPAC 4 — Connecting multiple representations: The "Differentiability and Continuity" instructional activity allows students to explore differentiability both algebraically using the function and graphically using a graphing calculator.

MPAC 5 — Building notational fluency: Students learn a variety of notations for derivatives and a bit about their history from the video in the "Notation for Derivatives" instructional activity.

MPAC 6 — Communicating: The "Graphical Derivatives" formative assessment always leads to lots of class discussion. Students have to explain connections they are making about this new concept of a derivative. They also need to present their conclusions clearly and analyze and evaluate the reasoning of their partner to see if they agree in their choices.

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 2.1, EU 2.2

Estimated Time:
12 instructional hours

Guiding Questions:

▸ How might you determine which rule to apply when differentiating functions? ▸ What are some similarities and differences between differentiating functions implicitly and explicitly? ▸ What are some relationships between inverse functions and derivatives of inverse functions?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.1C: Calculate derivatives.		**Instructional Activity: Derivatives of Trigonometric Functions** I ask students to graph $f(x)=\sin(x)$ and $f(x)=\cos(x)$ on their graphing calculators. We discuss the characteristics of each graph and students deduce that the cosine function is the derivative of the sine function. With some more graphical investigation, students will also conclude that the derivative of the cosine function is the negative sine function. After this activity, I ask the students to work together to come up with the derivatives for $f(x)=\tan(x)$, $f(x)=\csc(x)$, $f(x)=\sec(x)$, and $f(x)=\cot(x)$ using known trigonometric identities. They compare their answers as a class and we compile a list of the derivatives of the six basic trigonometric functions.
LO 2.1C: Calculate derivatives.	**Web** Schwartz, "Differentiation of Trigonometric Functions"	**Formative Assessment: Derivatives of Trigonometric Functions** Students work in groups of three or four to practice taking derivatives of trigonometric functions. If the problem requires the use of the chain rule, I ask that students identify the inner and outer functions before taking the derivative. These problems require the use of the product, quotient, power, and chain rules. For example, students differentiate $f(x)=\sin(x)\cos(x)$, $h(x)=\sin^2 x$, and $g(x)=\dfrac{x}{\sin(x)}$.

I sometimes encourage students to rewrite trigonometric functions like $f(x)=\sin^3 x$ as $f(x)=(\sin x)^3$. This format seems to help them more easily identify the composition of functions and properly take the derivative. This activity is a great review of all the rules for derivatives they've learned.

UNIT 3: IMPLICIT DIFFERENTIATION AND DERIVATIVES OF TRANSCENDENTAL FUNCTIONS

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 2.1, EU 2.2

Estimated Time:
12 instructional hours

Guiding Questions:

▶ How might you determine which rule to apply when differentiating functions? ▶ What are some similarities and differences between differentiating functions implicitly and explicitly? ▶ What are some relationships between inverse functions and derivatives of inverse functions?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.1C: Calculate derivatives.		**Instructional Activity: Implicit Differentiation** I give students an independent warm-up assignment: define the words "implicit" and "explicit" and state how they might be used to describe functions. I allow students to use their phones and/or tablets to help them define these words and connect them to mathematics. After thinking independently, they pair up to discuss their results. Then they share their definitions as a class. Together, we take the equation $x^2 + y^2 = 25$ and differentiate it both explicitly and implicitly. When differentiating implicitly, I emphasize that y must be treated as an implicit function of x so that students understand the use of the chain rule. Several more examples are shown and students are given an opportunity to practice differentiating implicitly as a class.
LO 2.1C: Calculate derivatives.	**Web** "Inverse Derivative"	**Instructional Activity: Inverse Derivative** I show the students the TI-activity "Inverse Derivative" on the interactive whiteboard. Students observe the relationship between a function's derivative and its inverse's derivative. Students then work together to solve problems given that if $f^{-1}(x)$ is the inverse of $f(x)$, then $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$. I focus on table problems, where students are given f, g, f', and g' for specific x values. I ask students to find the derivative of $f^{-1}(x)$ or $g^{-1}(x)$ at a specified x value.
LO 2.1C: Calculate derivatives.	**Web** Spector, "Derivatives of Inverse Trigonometric Functions"	**Instructional Activity: Derivatives of Inverse Trigonometric Functions** I give the students the equation $y = \sin x$ and ask how we might get the inverse. The students typically remember to switch the x and y from our previous review of inverse functions. I then ask them to take the derivative of that inverse function implicitly. I guide them toward using the Pythagorean identity and they eventually conclude that $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

We begin class by working in small groups to compile a list of facts about inverse functions. This serves as a quick review before getting into derivatives of inverses.

UNIT 3: IMPLICIT DIFFERENTIATION AND DERIVATIVES OF TRANSCENDENTAL FUNCTIONS

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 2.1, EU 2.2

Estimated Time:
12 instructional hours

Guiding Questions:

- **How might you determine which rule to apply when differentiating functions?** ▸ **What are some similarities and differences between differentiating functions implicitly and explicitly?** ▸ **What are some relationships between inverse functions and derivatives of inverse functions?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.1C: Calculate derivatives.	**Web** Spector, "Derivatives of Inverse Trigonometric Functions"	**Formative Assessment: Derivatives of Inverse Trigonometric Functions** After seeing a proof for the derivative of the arcsine function, students are put into pairs and given a laptop. They use a web resource from The Math Page to go through the rest of the proofs of inverse trigonometric functions at their own pace and practice problems. After all groups have completed the activity, we review answers to the practice problems as a class and answer any questions.	*There are nine practice problems on this Web page that I have students do in pairs. I walk around the room and answer questions as needed. It is important to stress to students to read the page in order and not skip the proofs. The answers to the problems are there, but students are encouraged to try them before peeking at the solutions.*
LO 2.1C: Calculate derivatives.	**Print** Finney et al., chapter 4	**Instructional Activity: The Derivative of Exponential Functions** I ask students to sketch the exponential function $f(x)=e^x$ and draw a few tangent lines on the curve at various integer values. Using these tangent lines, I ask the students to list some characteristics of the exponential function's derivative and make a conjecture about what the derivative might be. Students should note that the derivative is always positive and increasing. We confirm our guesses using the TI-Nspire CAS graphing calculator to take the derivative of $f(x)$. Then, the students find the derivative of $g(x)=a^x$ by rewriting the function as $g(x)=e^{x\ln a}$. I emphasize the importance of using the chain rule with exponential derivatives and show examples where this is necessary, for example, $f(x)=e^{4x}$ and $g(x)=5^{x^2-4}$.	
LO 2.1C: Calculate derivatives.	**Web** "The Derivatives of Logs"	**Instructional Activity: The Derivative of Logs** Students work in pairs to complete this activity. They use the TI-Nspire to create a scatterplot of the derivative graph of $y=\ln(x)$. This graphical investigation leads to $\frac{d}{dx}\ln(x)=\frac{1}{x}$. They also use the change-of-base formula and the derivative of the natural logarithm to come up with the derivative for $y=\log_b x$ on their own and use the calculator to confirm their results.	*I typically don't have my students do problem 3 in this activity. It's an extension that can be done if time allows.*

UNIT 3: IMPLICIT DIFFERENTIATION AND DERIVATIVES OF TRANSCENDENTAL FUNCTIONS

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 2.1, EU 2.2

Estimated Time:
12 instructional hours

Guiding Questions:

▸ How might you determine which rule to apply when differentiating functions? ▸ What are some similarities and differences between differentiating functions implicitly and explicitly? ▸ What are some relationships between inverse functions and derivatives of inverse functions?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.1C: Calculate derivatives. **LO 2.3B:** Solve problems involving the slope of a tangent line.	**Web** "Tangent Curves" "Equation Explorer"	**Summative Assessment: Historical Curves Project** Students find equations for tangent lines to special curves at indicated points. They use the online implicit grapher "Equation Explorer" to graph the function with its tangent line to verify their results. For each of the five equations, students must neatly show their work for finding the derivatives of the implicitly defined curves and the equations of the tangent lines. They print out each graph with its tangent line to include as a part of their work. This project is assigned after we cover implicit differentiation and is not due until the end of the unit.	*This summative assessment addresses the following guiding question:* ▸ What are some similarities and differences between differentiating functions implicitly and explicitly? *For bonus points, I give students a short quiz on the historical information about the curves on the day the project is due. I ask questions like "How did the 'Witch of Agnesi' curve get its scary name?" and "The Folium of Descartes is a plane curve proposed by Descartes to challenge which mathematician's extreme-finding techniques?"*
All of the learning objectives in this unit are addressed.		**Summative Assessment: Test on Derivatives of Transcendental Functions** Students take a test on derivatives of transcendental functions that includes some multiple-choice and short-answer questions. This assessment is somewhat shorter than most unit tests as this is a shorter unit and students have already been assessed on the topic of implicit differentiation with a project. Questions require calculating derivatives of trigonometric, exponential, logarithmic, and inverse functions. Some problems are given algebraically and some are given in the form of a table.	*This summative assessment addresses the following guiding questions:* ▸ How might you determine which rule to apply when differentiating functions? ▸ What are some relationships between inverse functions and derivatives of inverse functions?

Mathematical Practices for AP Calculus in Unit 3

The following activities and techniques in Unit 3 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: In the “Derivatives of Trigonometric Functions” instructional activity, students develop conjectures based on their graphical investigations and apply definitions to solve problems. A similar approach is taken in the “Derivatives of Exponential Functions” and “Derivatives of Logs” instructional activities where students must explore graphs and make conjectures about derivatives.

MPAC 2 — Connecting Concepts: Students use the graphs of trigonometric functions in the “Derivatives of Trigonometric Functions” instructional activity to connect the visual representation of a derivative to the algebraic one. Students use the underlying structures of trigonometric functions to determine unknown derivatives.

MPAC 3 — Implementing algebraic/computational processes: The “Derivatives of Trigonometric Functions” formative assessment requires students to attend to precision and apply the appropriate mathematical strategy to take the derivative of a function.

MPAC 4 — Connecting multiple representations: Students develop concepts using graphical and numerical representations as they create a scatterplot for the derivative of the natural logarithm function in the “Derivative of Logs” instructional activity.

MPAC 5 — Building notational fluency: When applying the chain rule to a trigonometric function, students must know and understand the notation in order to calculate the derivative accurately in the “Derivatives of Trigonometric Functions” formative assessment.

MPAC 6 — Communicating: The “Implicit Differentiation” instructional activity makes use of the think-pair-share technique. Students explain connections among the concepts and compare their reasoning to that of others.

BIG IDEA 2
Limits

Enduring Understandings:
- EU 1.1, EU 2.2, EU 2.3, EU 2.4

Estimated Time:
20 instructional hours

Guiding Questions:

▸ What must you do differently when finding global extrema instead of local extrema? ▸ How are the first and second derivative tests the same? How are they different? ▸ What can you conclude about a function by studying its derivative? ▸ What can you conclude about a function by studying its second derivative?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 1.1C: Determine limits of functions.	**Web** "A Tale of Two Lines" **Print** Finney et al., chapter 5	**Instructional Activity: L'Hospital's Rule** We do this activity as a class. Students are asked to approximate the ratios of the y values of two functions for several different x values. Then, they determine the ratio of the slopes of the two functions. These functions appear to be linear because the graph is zoomed in to a narrow window around the point $(a,0)$ where the curves intersect. Students determine that the ratio of the y values is equivalent to the ratio of the slopes near $x=a$. From this visual interpretation, students can understand how L'Hospital's Rule works. We discuss the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$, which are necessary for applying L'Hospital's Rule, practice applying it, and then evaluate limits of those forms using problems from the textbook.
LO 2.2A: Use derivatives to analyze properties of a function.	**Web** "Critical Points and Local Extrema"	**Instructional Activity: Critical Points and Local Extrema** Students work in pairs to complete the TI-Nspire activity. This exploration of critical points guides students to the conclusion that local maxima and minima must occur at a critical point but all critical points are not necessarily the location for extrema. Students zoom in on various curves and, using the concepts of local linearity, determine the slope at a specified point.

This topic cannot be covered until students have an understanding of how to evaluate limits and calculate derivatives, which is why it is not included in Unit 1. Instead of teaching this topic with applications of derivatives, you may want to wait until all units are covered. Teaching L'Hospital's Rule at the end of the course will allow students to brush up on evaluating limits and calculating derivatives. It's a new topic that comes back full circle to the beginning units.

UNIT 4: USING DERIVATIVES TO ANALYZE FUNCTIONS

BIG IDEA 2
Limits

Enduring Understandings:
- EU 1.1, EU 2.2, EU 2.3, EU 2.4

Estimated Time:
20 instructional hours

Guiding Questions:

- **What must you do differently when finding global extrema instead of local extrema?**
- **How are the first and second derivative tests the same? How are they different?**
- **What can you conclude about a function by studying its derivative?**
- **What can you conclude about a function by studying its second derivative?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.2A: Use derivatives to analyze properties of a function.		**Instructional Activity: Absolute and Relative Extrema** I give the students the function $f(x)=e^{-x^2}$. I then ask them to determine all extrema on the interval $[-1,1]$ and identify them as absolute or relative. They know from the previous activity and class discussion to set the derivative equal to zero to get critical points and evaluate the function for the endpoints as well. Next, I ask students to determine whether the critical point at $x=0$ is relative or absolute over the function's entire domain. This exploration leads to a nice review of limits toward infinity and how end behavior can help bring some clarity when determining the shape of a curve. After discussion and algebraic investigation, students are allowed to graph the function on their graphing calculators.	*This is a good time to introduce the Extreme Value Theorem.* *Because $f(x)=e^{-x^2}$ is an exponential curve, some students believe there is only a relative maximum at $x=0$ for $(-\infty,\infty)$ before they actually see a graph. This is why I chose this particular function. It emphasizes the importance of determining domain and end behavior when identifying extrema.*
LO 2.2A: Use derivatives to analyze properties of a function.	**Print** Finney et al., chapter 5	**Formative Assessment: Finding Extreme Values** I give students several functions and ask them to find all extreme values. They must label them as absolute or relative. Most of the problems I give students have a specific interval. For those that do not, I require students to state the end behavior of the function and make a rough sketch of what it might look like when graphed. After students have completed their work, I allow them to check their own answers by graphing the functions using TI-Nspire.	*As students work, I walk around the room and assist as needed. Instead of having students work on the same problems at the same time, you could give groups particular problems and have them present their work to the class to practice communicating mathematical results.*

BIG IDEA 2
Limits

Enduring Understandings:
- EU 1.1, EU 2.2, EU 2.3, EU 2.4

Estimated Time:
20 instructional hours

Guiding Questions:

- **What must you do differently when finding global extrema instead of local extrema?**
- **How are the first and second derivative tests the same? How are they different?**
- **What can you conclude about a function by studying its derivative?**
- **What can you conclude about a function by studying its second derivative?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.4A: Apply the Mean Value Theorem to describe the behavior of a function over an interval.	**Web** "Mean Value Theorem"	**Instructional Activity: Mean Value Theorem** In this activity, students manipulate a tangent line on a function to match its slope to that of a secant line. They are given a variety of functions, some being differentiable and some that are not. Through this exploration, students determine that the conditions for applying the Mean Value Theorem are differentiability and continuity. There is also an extension into Rolle's Theorem that helps students determine whether or not an extreme value occurs on an interval (a,b) when $f(a)=f(b)$.
LO 2.2A: Use derivatives to analyze properties of a function.		**Instructional Activity: The First Derivative Test** At this point, students already know that extrema occur at critical points. They also know from Unit 2 that functions increase where their derivatives are positive and decrease where they are negative. So introducing this test simply puts a name to these ideas and gives students an organized way to determine these properties for a function. I simply show students the test and how to organize their findings on a first derivative number line. If students seem to be having trouble with this concept, I show a split screen of some functions and their derivatives to review the graphical relationships between these functions. I emphasize the important of determining the domain of a function before applying this test.

After analyzing graphs, we have a fun discussion about this theorem's application in the real world. I usually ask, "If my average speed from school to home was 55 miles per hour and the speed limit on this trip is 50 miles per hour, was I necessarily speeding?" Students apply this new concept to answer the problem.

UNIT 4: USING DERIVATIVES TO ANALYZE FUNCTIONS

BIG IDEA 2
Limits

Enduring Understandings:
- EU 1.1, EU 2.2, EU 2.3, EU 2.4

Estimated Time:
20 instructional hours

Guiding Questions:

▸ What must you do differently when finding global extrema instead of local extrema? ▸ How are the first and second derivative tests the same? How are they different? ▸ What can you conclude about a function by studying its derivative? ▸ What can you conclude about a function by studying its second derivative?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.2A: Use derivatives to analyze properties of a function.	**Print** Finney et al., chapter 5 **Supplies** Construction paper	**Instructional Activity: Graphic Organizer for Applying Derivatives to Analyze Functions** Students take a sheet of construction paper and fold it "hotdog" style. They cut one side of the fold so that they create five flaps and on each flap write "Critical Points," "Inflection Points," "First Derivative Test," "Second Derivative Test," and "Test for Concavity." Each flap has two sections inside. On one, they must write what each concept is used to determine. On the other, they must write a definition for the word or a list of steps for applying the test. For example, students research "Test for Concavity." They find that it determines the curvature of a function. They also find that applying this test requires taking the second derivative and determining where it is positive and negative.
LO 2.2A: Use derivatives to analyze properties of a function.	**Web** Bogley and Robson, "Using the First and Second Derivative Tests"	**Formative Assessment: Using the First and Second Derivative Tests** Students are given online problems to practice finding critical points, extrema, intervals of increasing and decreasing, inflection points, and intervals of concave up and concave down for a function. They are allowed to work in groups and compare answers as they work. Students are not allowed to use a calculator until their work is complete and they'd like to compare their answers with the graphs of the functions. Finding all of this information for one function can be a bit overwhelming to students at first, so it's important that they get some practice and label what they are finding as they go.

As students are working, I walk around the room to check their work and listen to their group discussions. I correct their mistakes as they work and I may pull example problems from the textbook if they seem to need more practice.

BIG IDEA 2
Limits

Enduring Understandings:
- EU 1.1, EU 2.2, EU 2.3, EU 2.4

Estimated Time:
20 instructional hours

Guiding Questions:

▸ What must you do differently when finding global extrema instead of local extrema? ▸ How are the first and second derivative tests the same? How are they different? ▸ What can you conclude about a function by studying its derivative? ▸ What can you conclude about a function by studying its second derivative?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.2A: Use derivatives to analyze properties of a function.	**Print** Finney et al., chapter 5	**Formative Assessment: Curve-Sketching Activity** I put students into small groups and give each group a different f' function. They use this function to algebraically determine various characteristics of the f function and then create a first derivative and a second derivative number line. I have them stack the number lines on top of each other and divide the number lines at each critical point and each possible point of inflection to create small intervals. For each interval, students must sketch what the graph would look like. They put these pieces of graph together to create a single sketch of what f will look like over the entire interval.	*I like to have students in each group work independently at first and share their results after graphing. I check their work and give one-on-one feedback as they proceed. Within each group, they will see that their graphs have similar shape, but may have different y values. I make sure to emphasize the importance of this observation because it is helpful later on when we discuss antiderivatives.*
LO 2.2A: Use derivatives to analyze properties of a function.		**Instructional Activity: Determining Properties of a Function from Its Derivative Graphically** I give students a sketch of a continuous polynomial function f' on the interval $[-4,4]$. As a class, we examine the graph to determine the following for f: where its extrema occur, where its inflection points occur, where it increases and decreases, and where it is concave up and concave down. Based on this information, we sketch what we think f might look like. Afterward, students are asked to sketch their own f' graph, switch with a neighbor, and repeat the activity with the new curve.	

UNIT 4: USING DERIVATIVES TO ANALYZE FUNCTIONS

BIG IDEA 2
Limits

Enduring Understandings:
- EU 1.1, EU 2.2, EU 2.3, EU 2.4

Estimated Time:
20 instructional hours

Guiding Questions:

- **What must you do differently when finding global extrema instead of local extrema?** ▸ **How are the first and second derivative tests the same? How are they different?** ▸ **What can you conclude about a function by studying its derivative?** ▸ **What can you conclude about a function by studying its second derivative?**

Learning Objectives	Materials	Instructional Activities and Assessments	
All of the learning objectives in this unit are addressed.	**Web** McMullin, "AP Calculus Free Response Question Type Analysis and Notes"	**Summative Assessment: Test on Using Derivatives to Analyze Functions** Students take a test that includes some multiple-choice, short-answer, and free-response questions. The multiple-choice questions deal with using L'Hospital's Rule to determine limits, finding extreme values, inflection points, intervals of increasing and decreasing, and intervals where a function is concave up or concave down. Short-answer questions include analyzing a function's derivative graph to determine properties of the function. I also give students a table of information and have them sketch a function using that table. I typically pick one or two free-response questions from previous AP exams to include. Topics 4 and 6 in Lin McMullin's AP guide make finding appropriate questions much easier (you may find you'll need to exclude some parts of the questions as they deal with topics not yet covered).	*This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 4

The following activities and techniques in Unit 4 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: In the "Critical Points and Local Extrema" instructional activity, students use technology to explore the concept of critical points and make conjectures about the connection to local extrema. The "Mean Value Theorem" instructional activity ensures that students consider the hypotheses of the Mean Value Theorem before applying their conclusion.

MPAC 2 — Connecting concepts: The "Absolute and Relative Extrema" instructional activity helps students to connect the idea of a limit to determining properties of functions.

MPAC 3 — Implementing algebraic/computational processes: Students must attend to precision numerically, analytically, and graphically as they solve problems in the "Using the First and Second Derivative Tests" formative assessment.

MPAC 4 — Connecting multiple representations: In the "Curve-Sketching Activity" formative assessment, students extract and interpret mathematical content from a given graph of a derivative in order to sketch an antiderivative.

MPAC 5 — Building notational fluency: As students work with a function f, its derivative f', and its second derivative f'', in the "Determining Properties of a Function from Its Derivative Graphically" instructional activity and the "Curve-Sketching Activity" formative assessment, they must use proper notation to label their work and connect that notation to algebraic and graphical representations.

MPAC 6 — Communicating: The "Graphic Organizer for Applying Derivatives to Analyze Functions" instructional activity gives students the opportunity to research and use accurate and precise language to define mathematical terms. It also requires students to explain connection among concepts.

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 2.3

Estimated Time:
15 instructional hours

Guiding Questions:

- **When solving word problems that involve derivatives, what are some key words or phrases that might help you take the right approach?** ▸ **Why is it important to consider the domain when optimizing?** ▸ **When solving a related rates problem, how do you distinguish quantities that are constant from those that change over time?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.3C: Solve problems involving related rates, optimization, and rectilinear motion.	**Web** Bevelacqua and Kennedy, "Dog Pen Problem"	**Instructional Activity: The Dog Pen Problem** Because this is the first optimization problem the students have seen, we go through this TI-activity as a class. I have students take turns standing at the board and controlling the activity. They drag a vertex of the rectangular pen and collect data on the various dimensions of the pen. I have students make conjectures about what dimensions will give the most area, and they confirm these using a scatterplot of an area-width graph. After viewing the graph, I ask how calculus could be used to find the width that created a maximum area. Students connect the idea of optimizing with taking a derivative and finding critical points.	*There is an extension to this activity where the pen is divided into three congruent parts. I typically let students work this part out algebraically rather than going through it in the TI document.*
LO 2.3C: Solve problems involving related rates, optimization, and rectilinear motion.	**Web** "The Classic Box Problem"	**Instructional Activity: The Classic Box Problem** Each student has this TI-Nspire document on their calculators, but we do the lesson as a class. I guide the students as they manipulate the document and explore various dimensions the box could have. Students make conjectures as to what height of the box gives the maximum volume for the box. The animations in this activity make the mathematics come to life for students so that they can clearly visualize the problem. I don't let students get to page 5.1, which gives the formula for the box for any height: x. I try to have students come up with the formula on their own and optimize to see if their solution is consistent with their conjecture from before.	*The document shows a box made by cutting four congruent squares out of the corners of a rectangle and folding up the sides. Students must determine the dimensions of the squares that create a box with the largest volume possible.*
LO 2.3C: Solve problems involving related rates, optimization, and rectilinear motion.	**Web** Carter, "Maximizing Area"	**Instructional Activity: Maximizing Area** Students work in pairs as they go through this activity with little teacher assistance. They are given a rectangle bounded by the x-axis and a parabola and must maximize the area of the rectangle. They use their calculators to capture data and create a scatterplot that allows them to find the x value that maximizes the rectangle's area.	*This and the next activity do not come with a handout. Therefore, I have the students do the work to solve these problems algebraically on sheets of paper once they've completed the exploration on the calculator.*

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 2.3

Estimated Time:
15 instructional hours

Guiding Questions:

▸ When solving word problems that involve derivatives, what are some key words or phrases that might help you take the right approach? ▸ Why is it important to consider the domain when optimizing? ▸ When solving a related rates problem, how do you distinguish quantities that are constant from those that change over time?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.3C: Solve problems involving related rates, optimization, and rectilinear motion.	**Web** Carter, "Minimizing Distance"	**Instructional Activity: Minimizing Distance** Students work in pairs as they go through this activity with little teacher assistance. They are given a curve and a point and must find the minimum distance between the two. After using their calculators to create a scatterplot of possible distances between the curve and the point, students use the trace tool to estimate the minimum distance. They verify their answer by using the calculator to take the derivative of the distance equation, set the derivative equal to zero, and solve.
LO 2.3A: Interpret the meaning of a derivative within a problem. **LO 2.3C:** Solve problems involving related rates, optimization, and rectilinear motion.	**Web** Brown, "Optimization Practice," "Optimization Class Examples"	**Formative Assessment: Optimization Problems** After having done a few examples of optimization problems using the graphing calculator, I give students several examples to try in groups. I pull problems from Brown's "Optimization Practice" document and the "Optimization Class Examples" document as well. I assign problems to small groups and have students present their work to the class. Some of the problems are assigned as homework. For most of the problems, students are allowed to use a calculator.
LO 2.3A: Interpret the meaning of a derivative within a problem. **LO 2.3C:** Solve problems involving related rates, optimization, and rectilinear motion.	**Web** Schwartz, "Related Rates Classwork"	**Instructional Activity: Introduction to Related Rates** As a way to introduce the concept and notation of related rates problems, I use examples 1 and 2 from the "Related Rates Classwork" document. Students should already understand the derivative as a rate of change. So, with some guidance they should be able to take a phrase like "the radius of a circle is increasing at a rate of 3 in / hr" and express it mathematically as $\frac{dr}{dt}=3$. In example 2, students must recall how to take derivatives implicitly to find the rates of change for perimeter and area.

Be sure to give students a guideline as to what work you expect them to show when solving optimization problems. I require my students to show the calculus. Even if they used a calculator to take the derivative, they must still write down the steps and justify that they've found a maximum or minimum value. This is typically done using the first derivative test. I also require answers in complete sentences.

UNIT 5: APPLICATIONS OF DERIVATIVES

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 2.3

Estimated Time:
15 instructional hours

Guiding Questions:

- When solving word problems that involve derivatives, what are some key words or phrases that might help you take the right approach? ▸ Why is it important to consider the domain when optimizing? ▸ When solving a related rates problem, how do you distinguish quantities that are constant from those that change over time?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.3A: Interpret the meaning of a derivative within a problem. **LO 2.3C:** Solve problems involving related rates, optimization, and rectilinear motion.	**Web** Burke, "Related Rates Graphic Organizer" Lewis, "Related Rates 1	**Instructional Activity: Graphic Organizer for Related Rates** Because students often have trouble knowing when to substitute values into related rates problems, I give them a graphic organizer that helps distinguish "general/constant" values from "point in time" information. They use this graphic organizer as a guide through the problem in the Related Rates 1 video. Students work in pairs to solve the problem before I play the solution.
LO 2.3A: Interpret the meaning of a derivative within a problem. **LO 2.3C:** Solve problems involving related rates, optimization, and rectilinear motion.	**Web** "Related Rates Matching Lab"	**Formative Assessment: Related Rates Matching Lab** Students work in groups of four to complete the matching activity. They are given seven word problems, formulas, and derivatives of those formulas. They must match the formula with its derivative and the correct word problem. After matching all problems correctly, students work out the solutions using their graphic organizer as a guide.
LO 2.3A: Interpret the meaning of a derivative within a problem. **LO 2.3C:** Solve problems involving related rates, optimization, and rectilinear motion.	**Print** Kamischke, "How Many Licks?" in *A Watched Cup Never Cools* **Supplies** Tootsie Pops and dental floss	**Instructional Activity: Tootsie Pop Lab** Students suck on a Tootsie Pop and wrap dental floss around the lollipop to help determine the radius every 30 seconds. The data are recorded in a table. Using the data, students can create a regression in their graphing calculators to model the radius of the lollipop as a function of time. With this information, they determine the rate of change of the lollipop's radius and calculate how fast the volume is decreasing when the radius is three-fourths its original value.

This topic lends itself to some really excellent geometry review. Students may need to be refreshed on formulas for area and volume as well as setting up and solving proportions.

I require students to label values as "general/constant" or "point in time" before they work out the problems. While they work on these examples, I help out as needed.

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 2.3

Estimated Time:
15 instructional hours

Guiding Questions:

- **When solving word problems that involve derivatives, what are some key words or phrases that might help you take the right approach?** - **Why is it important to consider the domain when optimizing?** - **When solving a related rates problem, how do you distinguish quantities that are constant from those that change over time?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.3A: Interpret the meaning of a derivative within a problem. **LO 2.3C:** Solve problems involving related rates, optimization, and rectilinear motion.	**Web** "Can Project" **Supplies** Cans of food	**Summative Assessment: Optimization Can Project** Students bring a can of food to class and take its measurements. They use these measurements to determine whether or not the can was created with optimal dimensions in mind. This means students will need to determine the dimensions of a can that would hold the same amount of food but have minimal surface area. Students display their work and findings in a portfolio. They label all work and design a can using their new, optimal dimensions.	*This summative assessment addresses the following guiding questions:* - When solving word problems that involve derivatives, what are some key words or phrases that might help you take the right approach? - Why is it important to consider the domain when optimizing?
LO 2.3A: Interpret the meaning of a derivative within a problem. **LO 2.3C:** Solve problems involving related rates, optimization, and rectilinear motion.		**Summative Assessment: Related Rates Project** Students are asked to come up with their own related rates problem. The problem must make use of a geometric concept such as proportions, Pythagorean theorem, volume, area, distance, and so on. To be sure the problems in the class vary, I put these concepts in a hat and have students draw one to see which concept they must incorporate. I ask students to create a presentation using Google presentations, Educreations, or PowerPoint presentation to illustrate their problem and show all work to obtain the solution. In addition to solving their own problem, students must solve four of their classmates' problems as well. This does not have to be a part of their presentation and can be turned in separately.	*This summative assessment addresses the following guiding question:* - When solving a related rates problem, how do you distinguish quantities that are constant from those that change over time?

Mathematical Practices for AP Calculus in Unit 5

The following activities and techniques in Unit 5 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: Several activities in this unit require students to use technology to explore problems and make conjectures. Some of these include "The Classic Box Problem," "Maximizing Area," and "Minimizing Distance" instructional activities.

MPAC 2 — Connecting concepts: The "Introduction to Related Rates" instructional activity helps students to further make the connection between derivatives and a rate of change.

MPAC 3 — Implementing algebraic/computational processes: The "Tootsie Pop Lab" instructional activity requires students to apply technology strategically to model their radius and determine its rate of change.

MPAC 4 — Connecting multiple representations: Students use a scatterplot to construct a graph from their table of collected data points in the "Tootsie Pop Lab" instructional activity in order to make a continuous graph and obtain a function.

MPAC 5 — Building notational fluency: The "Introduction to Related Rates" instructional activity helps students to assign meaning to notation when they take phrases from the word problem and represent them using a derivative with respect to time.

MPAC 6 — Communicating: The "Optimization Can Project" summative assessment gives students the opportunity to present their methods and conclusions both in written form and mathematically.

UNIT 6: MOTION AND RATES OF CHANGE

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 2.2, EU 2.3

Estimated Time:
10 instructional hours

Guiding Questions:

▸ How do position, velocity, and acceleration relate to one another mathematically? ▸ How are velocity and speed the same? How are they different? ▸ How can concepts like roots, extrema, and concavity be interpreted in terms of particle motion?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.3A: Interpret the meaning of a derivative within a problem. **LO 2.3C:** Solve problems involving related rates, optimization, and rectilinear motion.	**Web** Ross, "AP Calculus: Motion"	**Instructional Activity: AP Calculus Motion Module** Though this module includes five worksheets, I only use the first three in this unit because the others deal with integration. These activities take students through new vocabulary terms, such as "velocity" and "acceleration," and show how motion might be studied numerically, graphically, and analytically. They also help students to understand the concept of speed and how it relates to velocity and acceleration. For example, one worksheet in particular guides students with graphs and tables to determine that a particle is speeding up when velocity and acceleration have the same sign and slowing down when velocity and acceleration have different signs. This resource makes it easier to organize a discussion about motion since most textbooks don't have a unit solely devoted to this topic.
LO 2.3A: Interpret the meaning of a derivative within a problem. **LO 2.3C:** Solve problems involving related rates, optimization, and rectilinear motion.	**Web** Ross, "AP Calculus: Motion"	**Formative Assessment: Rectilinear Motion** Based on the new vocabulary introduced in the "AP Calculus: Motion Module," I give students a short fill-in-the-blank vocabulary quiz on position, velocity, and acceleration. I also give a fill-in-the-blank quiz about speed.

Because the module takes three to four hours to complete as a class, I give these two assessments at separate times. Both of these quick assessments guide me in my instructional decisions following completion of the module.

UNIT 6: MOTION AND RATES OF CHANGE

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 2.2, EU 2.3

Estimated Time:
10 instructional hours

Guiding Questions:

- **How do position, velocity, and acceleration relate to one another mathematically?**
- **How are velocity and speed the same? How are they different?**
- **How can concepts like roots, extrema, and concavity be interpreted in terms of particle motion?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.3A: Interpret the meaning of a derivative within a problem. **LO 2.3C:** Solve problems involving related rates, optimization, and rectilinear motion.	**Web** "Position, Distance, Velocity"	**Instructional Activity: Motion on a Straight Line** In this activity, I display the TI-Nspire on the interactive whiteboard. The activity models a situation where a particle has moved for 10 seconds. There is a split screen so students can see changes on the position or velocity graph as well as changes in the motion of the particle. Students observe changes to the graphs as they watch the particle move over the 10-second interval. I like this activity because there is a small white dot that moves the way the particle would move based on a given graph. This simple animation helps students make the connection that the type of motion we are studying is linear, which can be hard to understand when studying nonlinear position and velocity graphs.
LO 2.3A: Interpret the meaning of a derivative within a problem. **LO 2.3C:** Solve problems involving related rates, optimization, and rectilinear motion.	**Web** McMullin, "AP Calculus Free Response Question Type Analysis and Notes"	**Formative Assessment: AP-Style Motion Practice** In McMullin's free-response guide, topic 2 is motion on a line. I take some of these free-response problems and give them to groups of students. I have to look at the problems beforehand and possibly exclude some parts if they involve integration or Riemann sums. As students work these problems, we discuss neatness, organization of work, appropriate use of notation, and proper justification of answers. As I walk around the room to answer questions and assess student understanding, I also give tips about formatting for the AP Exam. For example, I remind students to label their functions just as they are labeled in the problem, round to the thousandths place, and use equals and approximation signs appropriately.

For some of these problems, you may want to give extra information. For example, if the problem asks students to find the total distance traveled and only gives the velocity function, you could give them the position function. This is one way around leaving that question out altogether since they can't integrate yet.

BIG IDEA 2
Derivatives

Enduring Understandings:
- EU 2.2, EU 2.3

Estimated Time:
10 instructional hours

Guiding Questions:

▸ How do position, velocity, and acceleration relate to one another mathematically? ▸ How are velocity and speed the same? How are they different? ▸ How can concepts like roots, extrema, and concavity be interpreted in terms of particle motion?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.3A: Interpret the meaning of a derivative within a problem. **LO 2.3C:** Solve problems involving related rates, optimization, and rectilinear motion.	**Web** "Xtreme Calculus: Part 2"	**Instructional Activity: Understanding Distance-Time and Velocity-Time Graphs** This final activity is given to students to clarify any misconceptions about motion before their final assessment. I have them complete the activity without much assistance from me, although I do allow collaboration among the class. The activity walks students through two scenarios: one modeled by a distance-time graph and the other modeled by a velocity-time graph. Students are able to review the concepts of line average and instantaneous velocity, and make graphical connections with tangent lines. They also review the concept of speed and how its increasing or decreasing nature depends on both velocity and acceleration.	*There is an extension activity that is a part of this activity that involves the CBR 2 device. I do not have one of these devices, but sample data are given so that teachers without this tool can still have their students complete the extension activity.*
All of the learning objectives in this unit are addressed.		**Summative Assessment: Test on Rectilinear Motion** Students are assessed on their conceptual understanding of the relationships among position, velocity, acceleration, and speed. The assessment includes problems that are presented numerically with a table, graphically, and algebraically. Some test items are formatted to model AP free-response questions.	*This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 6

The following activities and techniques in Unit 6 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 2 — Connecting concepts: In the "AP Calculus Motion Module" instructional activity, students connect concepts from Unit 4 concerning properties of functions to the idea of rectilinear motion. For instance, zeros of the derivative function are now times when a particle is at rest. Students begin to understand that there are the same underlying concepts in both units.

MPAC 3 — Implementing algebraic/computational processes: The "Understanding Distance-Time and Velocity-Time Graphs" instructional activity requires students to select appropriate mathematical strategies to solve problems and connect the results of their algebraic processes to the concept of motion to answer questions accurately.

MPAC 4 — Connecting multiple representations: The "AP Calculus Motion Module" instructional activity presents motion numerically, algebraically, and graphically. The "Motion on a Straight Line" instructional activity allows students to develop concepts graphically and numerically using technology.

MPAC 5 — Building notational fluency: While students are working on the "AP-Style Motion Practice" formative assessment problems, they must assign meaning to different notations and use them appropriately.

MPAC 6 — Communicating: As students work together to solve the "AP-Style Motion Practice" formative assessment problems, they must explain their reasoning to others. As they discuss the problems, they come to a consensus on how to solve each one. These questions also ask that students include and explain units in the context of the problem.

UNIT 7: THE DEFINITE INTEGRAL

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.3, EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
16 instructional hours

Guiding Questions:
- **What connections can be made between differentiation and integration?**
- **How might one approximate area under a curve?**
- **What are some real-world applications for the definite integral?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.2B: Approximate a definite integral. **LO 3.4C**: Apply definite integrals to problems involving motion.		**Instructional Activity: Connecting Area Under a Curve and Motion** As an introduction to antidifferentiation, I begin with a motion problem. I ask students to determine, both algebraically and graphically, how far a train has traveled if it was going 75 miles per hour from 7 to 9 a.m. Through graphical analysis of a velocity/time graph, students realize that the area under a velocity curve gives distance traveled. Using this connection and prior knowledge of rectilinear motion, students begin to understand the concept of antidifferentiation geometrically. Then I give students various velocity/time graphs and ask them to find total distance traveled over specific time intervals. Some of the graphs are linear and some are not. For those that are not, I have students come up with their own method for approximating the solution.
LO 3.2B: Approximate a definite integral. **LO 3.4C**: Apply definite integrals to problems involving motion.	**Web** "TI-Activity: Riemann Sums"	**Formative Assessment: Riemann Sums** This activity allows students to visualize left endpoint, right endpoint, and midpoint approximations of definite integrals or finite areas under curves. As students increase the number of rectangles under a curve, they can compare the area to the actual solution of the definite integral and see that the approximation becomes more accurate. This activity also requires students to consider how the curvature of a graph and the type of Riemann sum affects whether the result will be an over- or underapproximation.

I use the nonlinear curves to discuss the most efficient methods for approximating areas under curves. I have students compare their methods with other groups to see which are more precise.

Although students have not yet explored the definite integral as the limit of a Riemann sum, this activity does introduce them to definite integral notation and how to interpret the notation as the area under a curve. Students find the limit of a Riemann sum and sigma notation difficult to understand, so I don't start off with that as their first glimpse into definite integrals. Instead, I prefer students make the geometry connection first and then discuss limits.

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.3, EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
16 instructional hours

Guiding Questions:

- What connections can be made between differentiation and integration? ▶ How might one approximate area under a curve? ▶ What are some real-world applications for the definite integral?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.2A(a): Interpret the definite integral as the limit of a Riemann sum. **LO 3.2A(b):** Express the limit of a Riemann sum in integral notation.	**Web** "Riemann Sums: HMC Calculus Tutorial" Downey, "Riemann Sums"	**Instructional Activity: The Limit of a Riemann Sum** I put students into groups with a laptop and assign two websites to explore. They read the information about Riemann sums from the Harvey Mudd College Mathematics website. This website takes them through a very formal definition of a definite integral so that they interpret it as the limit of a Riemann sum. The second website revisits this limit idea but is more interactive. Students take notes as they review each website. I ask them to write down connections they've made, important definitions, and questions they still have. They share their notes with another group to see if they might have different information or answers to their questions. After all groups are done, we come together to answer any questions that students might still have.
LO 3.2B: Approximate a definite integral. **LO 3.2C:** Calculate a definite integral using areas and properties of definite integrals.	**Web** "Approximating Area Under a Curve"	**Formative Assessment: Approximating Area Under a Curve** I have students work in small groups to practice applying left endpoint, right endpoint, midpoint, and trapezoidal approximations of areas under curves. In this activity, students use those different approximation methods to evaluate all or part of the area under the function $f(x)=\sqrt{64-x^2}$. At the end of the activity, students are asked to find the exact area under the semicircle using geometry. I ask students to discuss their solutions as they work and decide which method they think might be most accurate. If time allows, I add a few integration problems to this assessment that students can solve with geometry. For example, I will give students the integral of a linear function $f(x)=2x-5$, absolute value function $f(x)=\|x-3\|$, or a shifted semicircle $f(x)=\sqrt{16-x^2}+2$.

Different intervals are used for each method of approximation on this worksheet. For example, the students do a left endpoint approximation over the interval $[0,4]$ and a right endpoint approximation from $[0,8]$. You may decide to use the same interval throughout the entire activity so that students can make comparisons in the end to the actual solution.

Also, this is a good time to review some geometry basics such as the properties of a trapezoid and how to find its area.

UNIT 7: THE DEFINITE INTEGRAL

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.3, EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
16 instructional hours

Guiding Questions:

▸ What connections can be made between differentiation and integration? ▸ How might one approximate area under a curve? ▸ What are some real-world applications for the definite integral?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.3D: Solve problems involving rates of change in applied contexts. **LO 3.4A:** Interpret the meaning of a definite integral within a problem.	**Web** "Swimming Pool Problem"	**Instructional Activity: Interpreting a Definite Integral** Students are given a scenario in which water is leaking out of a swimming pool. They have a function and its graph for the rate at which water leaks from the pool. The activity guides students through the process of estimating the amount of water that has leaked out of the pool after t minutes using Riemann sums. Students interpret their mathematical results in relation to the situation throughout the activity. They use the TI-Nspire CAS calculators to find the exact value of definite integrals to compare with their approximations. The overall goal of this activity is for students to understand that the definite integral of the rate of change of the water gives the amount of water lost over a specific time period.
LO 3.2C: Calculate a definite integral using areas and properties of definite integrals.	**Web** Downey, "Properties of Definite Integrals" **Print** Finney et al., chapter 6	**Instructional Activity: Properties of Definite Integrals** I give pairs of students a laptop and have them go through the first six examples on the "Properties of Definite Integrals" interactive applet. As they work through each example I have them take notes on the properties and write an explanation, in their own words, for each one. When all groups are done, we share what we've learned as a class. Then I give the students some examples from the book to practice applying the properties they've learned.

UNIT 7: THE DEFINITE INTEGRAL

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.3, EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
16 instructional hours

Guiding Questions:

▸ **What connections can be made between differentiation and integration?** ▸ **How might one approximate area under a curve?** ▸ **What are some real-world applications for the definite integral?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.1A: Recognize antiderivatives of basic functions. **LO 3.3A:** Analyze functions defined by an integral. **LO 3.3B(a):** Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals.	**Web** "Fundamental Theorem of Calculus, Part I" "Fundamental Theorem of Calculus, Part II"	**Instructional Activity: The Fundamental Theorem of Calculus** I have students get into groups of four. Within each group, students choose a partner and get a laptop. Each pair watches a different video about the Fundamental Theorem of Calculus. One video explains Part I of the theorem while the other video explains Part II. The student pairs take notes on their video and take turns teaching the other pair what they learned. When all groups have finished, the whole class shares what they've learned, takes collective notes, and corrects any misunderstandings.	*Because this is the first time students have been introduced to this topic, they might have difficulty explaining the theorem. Some students will not fully understand this theorem until examples have been provided. If your students struggle with this idea, work with the pairs individually and show them concrete examples that might enhance their understanding.*
LO 3.1A: Recognize antiderivatives of basic functions. **LO 3.3A:** Analyze functions defined by an integral. **LO 3.3B(a):** Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals.	**Print** Finney et al., chapter 6	**Formative Assessment: Practice Applying the Fundamental Theorem of Calculus** Students should understand how to determine the antiderivatives of basic functions from the previous activity. We do examples as a class before I give them problems to try from the book. For example, I ask them to state the antiderivative for functions such as $f(x)=3x^2$, $g(x)=\sec^2 x$, or $h(x)=\frac{1}{x}$. I only give examples that they can mentally work backward using their knowledge of derivatives. Then, I have students work together to apply both parts of the Fundamental Theorem of Calculus to evaluate definite integrals and find the derivative of functions defined as integrals. To apply Part I, I ask the students to find the derivative of a function like $f(x)=\int_0^x (t^2+t)dt$ and to apply Part II I would ask students to evaluate a definite integral like $\int_0^3 x^2\,dx$.	*As students are working on practice problems from the textbook, I walk around the room checking answers and correcting mistakes. I find that when students are taking the derivative of an integral they forget to correctly apply the chain rule.* *During this practice time I sometimes stop the students to review some basic derivatives they should have memorized so they can properly evaluate the given definite integrals.*

UNIT 7: THE DEFINITE INTEGRAL

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.3, EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
16 instructional hours

Guiding Questions:

- What connections can be made between differentiation and integration? ▸ How might one approximate area under a curve? ▸ What are some real-world applications for the definite integral?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.1A: Recognize antiderivatives of basic functions. **LO 3.3A:** Analyze functions defined by an integral. **LO 3.3B(a):** Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals.	**Web** Schwartz, "The Accumulation Function"	**Instructional Activity: Area Accumulation** Students use tables and graphs to analyze the integral $\int_0^x f(t)dt$. They plug in different values for x and, given the $f(t)$ graph, observe the accumulating area. We do this activity as a class and discuss how the accumulation function can be applied to real-world situations using the first example of the worksheet involving snowfall and the accumulation of snow on the ground. In this example, students should see that the definite integral of the rate at which the snow falls gives the amount of snow fallen over a specified time interval. They are also asked to determine when the amount of snow on the ground was increasing, decreasing, and greatest using concepts they learned when studying derivatives.	*I don't use this entire packet of worksheets. I typically only use the first two pages for instruction. I may give example 2 on page 3 as a homework assignment to assess for understanding.*
LO 3.4C: Apply definite integrals to problems involving motion.	**Web** Ross, AP Central Calculus: "Motion"	**Instructional Activity: Definite Integrals and Motion** Worksheet 4 from the AP Central Calculus "Motion" module guides students to understanding motion from a new perspective. It's mostly fill in the blank and uses precise vocabulary so students understand how to interpret a definite integral of velocity and how to find total distance traveled by a particle as opposed to displacement.	
LO 3.4B: Apply definite integrals to problems involving the average value of a function.	**Web** "Average Rate of Change vs. Average Value"	**Instructional Activity: Average Value of a Function** Students take a geometric approach to understanding the average value of a function. Several different polynomial graphs and one noncontinuous graph are given. Students manipulate the graphs to match the area of a rectangle to the area under a curve. They work together to take notes on the conditions under which they can make the two areas equal. For instance, students determine from the noncontinuous function that a graph must be continuous in order to apply the Mean Value Theorem for integrals. They analyze the numbers they get when the areas are equal, and this exploration leads to the Mean Value Theorem for integrals. Students also make a connection between this theorem and the Mean Value Theorem for derivatives by manipulating formulas.	*At the end of this activity, students are asked to explain how $f(c)=\frac{1}{b-a}\int_a^b f(x)dx$ and $f'(c)=\frac{f(b)-f(a)}{b-a}$ are equivalent. Simply taking the derivative of $f(c)$ will show this, which will help students to make the connection between the Mean Value Theorem for derivatives and the Mean Value Theorem for integrals.*

UNIT 7: THE DEFINITE INTEGRAL

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.3, EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
16 instructional hours

Guiding Questions:

- What connections can be made between differentiation and integration?
- How might one approximate area under a curve?
- What are some real-world applications for the definite integral?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.4C: Apply definite integrals to problems involving motion.	**Web** Ross, AP Central Calculus: "Motion"	**Formative Assessment: Definite Integrals and Motion** Worksheet 5 from the AP Central Calculus "Motion" module contains three examples that I have students complete in groups. Each example is presented differently: graphically, analytically, and numerically. These examples require students to not only evaluate definite integrals but also interpret their meaning in terms of particle motion. It is a great review of the entire unit while also reviewing the concept of motion from the previous unit.	*Because motion is not a new topic for students, they need very little guidance as they work. I let them work through the problems without much help unless I see big conceptual misunderstandings. If that is the case, the next instructional activity can be used to help students better understand the material.*
LO 3.4C: Apply definite integrals to problems involving motion.	**Web** "Applied Fundamental Theorem of Calculus"	**Instructional Activity: Applied Fundamental Theorem of Calculus Using Motion** The students work on the TI-Nspire activity in groups to refine what they've learned about the Fundamental Theorem of Calculus, the accumulation function, and motion. This activity requires students to understand and explain that the area under a velocity curve represents a change in distance over a specified time interval. They compare two graphs to determine which graph is the antiderivative and how it compares with its derivative graph, all while interpreting the concepts in terms of motion.	
All of the learning objectives in this unit are addressed.		**Summative Assessment: Test on Definite Integrals** This assessment requires students to be able to apply the Fundamental Theorem of Calculus and properties of definite integrals to evaluate definite integrals and find the derivatives of functions defined as integrals. They must also be able to approximate definite integrals using the Riemann sums. Students are assessed on their conceptual understanding of the relationships among position, velocity, and acceleration with a focus on the accumulation function and areas under curves. Some test items are formatted to model AP multiple-choice and free-response questions but many are short-answer questions.	*This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 7

The following activities and techniques in Unit 7 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: The "Riemann Sums" formative assessment allows students to visually explore what happens to the area approximation when the number of rectangles is increased under a curve. This calculator exploration allows students to make connections between the concepts of limits and integration. The "Properties of Definite Integrals" instructional activity has students develop conjectures by manipulating an online applet. In the "Average Value of a Function" instructional activity students manipulate a variety of graphs and must determine the hypotheses of the Average Value Theorem for integrals based on this investigation.

MPAC 2 — Connecting concepts: In the "Connecting Area Under a Curve and Motion," instructional activity students must connect what they learned about area from geometry to the concepts they learned in the unit on rectilinear motion to determine that the area under a velocity curve gives distance traveled.

MPAC 3 — Implementing algebraic/computational processes: The "Area Accumulation" instructional activity requires some computation in order to fill in tables, so students must attend to precision as the values build from one to the next.

MPAC 4 — Connecting multiple representations: The "Approximating Area Under a Curve" formative assessment requires students to determine approximate areas under curves numerically but also has students visualize the process by drawing rectangles under a curve.

MPAC 5 — Building notational fluency: "The Limit of a Riemann Sum" instructional activity introduces students to the notation for integration and aids them in connecting this notation to the limit of a Riemann sum.

MPAC 6 — Communicating: The "Interpreting a Definite Integral" instructional activity has students not only evaluate and approximate definite integrals but also explain what they represent in the context of the situation in which water leaks from a pool.

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.3, EU 3.5

Estimated Time:
24 instructional hours

Guiding Questions:

- **What are some things you can determine about a curve based on the slope field of its differential equation?**
- **What role does the chain rule play in evaluating integrals using substitution?**
- **How might you determine what technique to use when integrating a function?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO3.3B(a): Calculate antiderivatives.		**Instructional Activity: Indefinite Integrals and +C** Though students know what an antiderivative is, they have not evaluated indefinite integrals. So I emphasize the importance of adding a constant when evaluating an indefinite integral by having students determine the derivative of a succession of similar polynomials, for example, $F(x)=3x^2+6x+5$, $F(x)=3x^2+6x-12$, and $F(x)=3x^2+6x+1$. They notice that although the constants are different, the derivatives are the same. This leads to the conclusion that there isn't just one antiderivative of $f(x)=6x+6$, so a constant must be added when determining an indefinite integral.
LO3.3B(a): Calculate antiderivatives.		**Instructional Activity: Indefinite Integrals** Instead of giving students a long list of basic indefinite integrals to memorize, I try to have students come up with the rules on their own by working backward using their knowledge of derivatives. From their work in the last chapter students should already know how to determine the antiderivative of a polynomial, but there are some other functions they may not have considered. I have them type four specific functions into the TI-Nspire CAS to evaluate and then determine the general rule. For instance, students will evaluate $\int e^x\,dx$, $\int e^{3x}\,dx$, $\int e^{\frac{x}{2}}dx$, and $\int e^{\pi x}dx$ using their calculators. Then I have them fill in the blank for $\int e^{kx}dx=$ ______. This can be done for integrals involving logarithmic, trigonometric, and inverse trigonometric functions.

One reason this activity is helpful is so students can see the role the chain rule plays in determining an antiderivative. Not properly dealing with constants is a common mistake and determining the rules for integration rather than simply being given a list to memorize should help students to make those common errors less frequently.

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.3, EU 3.5

Estimated Time:
24 instructional hours

Guiding Questions:

- What are some things you can determine about a curve based on the slope field of its differential equation?
- What role does the chain rule play in evaluating integrals using substitution?
- How might you determine what technique to use when integrating a function?

Learning Objectives	Materials	Instructional Activities and Assessments
LO3.3B(a): Calculate antiderivatives.	**Web** Kouba, "The Method of U-Substitution" Kouba, "Integration of Trigonometric Integrals" **Print** Finney et al., chapter 7	**Instructional Activity: Integration by Substitution** I use examples (from the University of California, Davis, Web pages) to show students how to use substitution to make complicated integration problems look like the basic ones they worked in the previous activity. I emphasize the fact that u-substitution typically works best when you look at the problem with compositions of functions in mind. For instance, when integrating $\int 3x^2(x^3-4)^5\,dx$, we determine what functions make up the problem and then decide how we might form a substitution. After doing two or three examples as a class, I have the students work in small groups to complete the rest of the examples together.
LO3.3B(a): Calculate antiderivatives.	**Web** Kouba, "The Method of U-Substitution" Kouba, "Integration of Trigonometric Integrals" **Print** Finney et al., chapter 7	**Formative Assessment: Evaluating Integrals Using Substitution** Students can use a laptop or their own devices to pull up the Web pages. When the group completes an example, I encourage them to look at the detailed solution to check their work. If they made a mistake, I have them cross through the incorrect work and write what they did wrong beside the mistake. Then they redo the problem correctly.

I have found that the best way for students to master this topic is through lots of practice. As students work together on these problems, I determine common mistakes by reading the explanations of their incorrect work as they progress. If several groups are making the same mistake, I usually stop the class and discuss the problem with the whole class.

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.3, EU 3.5

Estimated Time:
24 instructional hours

Guiding Questions:

- **What are some things you can determine about a curve based on the slope field of its differential equation?** ▸ **What role does the chain rule play in evaluating integrals using substitution?** ▸ **How might you determine what technique to use when integrating a function?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.5A: Analyze differential equations to obtain general and specific solutions.	**Web** McMullin, "Differential Equations"	**Instructional Activity: Solving Differential Equations** I give each student a handout to accompany the video on differential equations. Students watch the first seven minutes of the video and take notes on what a differential equation is. After doing this independently, I have the students discuss what they learned with neighbors. Then we put together important facts as a class for our notes about differential equations. I work through the example problems from the video with the class instead of having them watch McMullin do it so we can move at our own pace and stop for questions as needed.
LO 2.3E: Verify solutions to differential equations. **LO 2.3F:** Estimate solutions to differential equations.	**Web** Stephenson, AP Central Calculus: "Slope Fields" Lindenmuth, "Slope Fields"	**Instructional Activity: Slope Fields** We start by discussing ways of graphing functions. Students typically respond with making tables and plotting points. We take this idea and use it to graph differential equations, forming a slope field. I have students graph slope fields on the first page of the AP Central Calculus "Slope Fields" handout. They check their work by entering the differential equations into the slope field generator in Geogebra. We discuss the different characteristics of each slope field and make notes. For instance, students should notice that differential equations that only depend on y values have the same slopes horizontally, whereas differential equations that only depend on x values have the same slopes vertically.
LO 2.3E: Verify solutions to differential equations. **LO 2.3F:** Estimate solutions to differential equations.	**Web** Stephenson, AP Central Calculus: "Slope Fields" Lindenmuth, "Slope Fields" Free-response questions available at AP Central	**Formative Assessment: Slope Fields** Students work in groups to complete the AP Central Calculus "Slope Fields" handout. They match differential equations with the correct slope field and do two free-response questions in which they must sketch a slope field, draw solution curves through particular points, and find particular solutions to differential equations with given initial conditions. After all groups have completed the work, we review the problems as a class.

After looking at student work on this handout, I can determine if students need more practice with slope fields and solving differential equations. If they do, I give them more AP-style examples, such as free-response questions from old AP Exams. Two good problems are number 6 from the 2010 exam and number 5 from the 2008 exam.

UNIT 8: DIFFERENTIAL EQUATIONS AND ANTIDIFFERENTIATION BY SUBSTITUTION

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.3, EU 3.5

Estimated Time:
24 instructional hours

Guiding Questions:

- **What are some things you can determine about a curve based on the slope field of its differential equation?**
- **What role does the chain rule play in evaluating integrals using substitution?**
- **How might you determine what technique to use when integrating a function?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.5B: Interpret, create, and solve differential equations from problems in context.	**Web** "Radioactive Decay of Carbon-14"	**Instructional Activity: Exponential Growth and Decay** We begin studying exponential growth and decay with a video about the science behind radioactive decay. Then I present the fact that the rate at which an element decays is directly proportional to the amount of the element that is present. Students are asked to write this relationship as a differential equation. We solve the differential equation together given the initial condition $y = y_0$ when $t = 0$. Using the fact that the solution to the differential equation $\frac{dy}{dt} = -ky$ is $y = y_0 e^{-kt}$, students can solve problems involving exponential growth and decay.
LO 3.5B: Interpret, create, and solve differential equations from problems in context.	**Supplies** Vernier EasyTemp temperature probe and TI-Nspire teacher software	**Instructional Activity: Newton's Law of Cooling** Using the Vernier EasyTemp temperature probe and the TI-Nspire teacher software, I show students what a temperature versus time graph looks like. I usually warm the temperature probe up in my hand and begin collecting data when I release the probe. As the probe cools, it collects data and displays a graph. Students observe that the graph is best modeled by an exponential function. After this example, I ask students, "To what is the rate of change of temperature directly proportional?" After some discussion, students are presented with this relationship and the formula for Newton's Law of Cooling.
LO 3.5B: Interpret, create, and solve differential equations from problems in context.	**Print** Finney et al., chapter 7	**Formative Assessment: Exponential Growth and Decay practice** Students work in groups to complete real-world application problems from the textbook involving exponential growth and decay. Problems include bacteria growth, radioactive decay, and heating and cooling temperatures. While all groups will complete all assigned problems, each group is given one problem that they must present and explain to the entire class. This gives students the opportunity to justify and confirm their answers.

I use the class presentations to assess student understanding. I ask that all students in the group take part in presenting the problem: I try to keep the groups small so that everyone can participate. If students need more practice after presentations, I give them more problems to try.

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.3, EU 3.5

Estimated Time:
24 instructional hours

Guiding Questions:

▶ **What are some things you can determine about a curve based on the slope field of its differential equation?** ▶ **What role does the chain rule play in evaluating integrals using substitution?** ▶ **How might you determine what technique to use when integrating a function?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.5B: Interpret, create, and solve differential equations from problems in context.	**Supplies** Vernier EasyTemp temperature probe, TI-Nspire teacher software, and various types of coffee cups	**Instructional Activity: Keeping Your Coffee Warm** Students break into small groups and experiment to see which type of coffee cup keeps their coffee warm for the longest amount of time. Each group has at least one TI-Nspire calculator and one Vernier EasyTemp temperature probe. They determine the temperature of the room and then pour coffee in the cups one at a time and use the temperature probe to collect data on how the coffee cools. The students use the calculator to obtain an exponential function to fit the data. They use the function to determine how long it takes for the coffee to cool to a reasonable drinking temperature, which they will determine by researching online. Students share with the class which cup kept coffee above drinking temperature longest.	*I give each group a regular coffee mug, a stainless steel insulated travel mug, and a Tervis tumbler cup to test. I did this activity with thermometers borrowed from the science department before I had technology, and it still worked. Students just have to collect data, make a table of values, and create a function for themselves. The technology yields more accurate results and saves time.*
All of the learning objectives in this unit are addressed.		**Summative Assessment: Test on Differential Equations and Integration by Substitution** The test is made up of multiple-choice, short-answer, and free-response questions. Students must be able to evaluate definite and indefinite integrals using substitution. They should also be able to find the particular solution to a differential equation given an initial condition. This is part of a free-response problem in which they must also sketch a slope field and approximate values for $y = f(x)$. The short answer includes some application problems of differential equations, including exponential decay and Newton's Law of Cooling.	*This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 8

The following activities and techniques in Unit 8 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: In the “Indefinite Integrals” instructional activity, students use technology along with definitions they learned from studying derivatives to make conjectures about the basic integration problems.

MPAC 2 — Connecting concepts: By analyzing and matching slope fields to their differential equations in the “Slope Fields” instructional activity and formative assessment, students are making a further connection between differentiation and antidifferentiation.

MPAC 3 — Implementing algebraic/computational processes: When students do the “Evaluating Integrals Using Substitution” formative assessment, they have to determine how to set up the substitution as well as how to complete the algebraic process correctly.

MPAC 4 — Connecting multiple representations: The “Slope Fields” instructional activity connects the concept of differential equations to their visual representation in the form of a slope field.

MPAC 5 — Building notational fluency: During the “Exponential Growth and Decay” instructional activity we have a lot of discussion about what each part of the formulas represents and how they can be applied in various situations.

MPAC 6 — Communicating: In the “Evaluating Integrals Using Substitution” formative assessment, students are required to use accurate and precise language when explaining the mistakes in their work. Checking their work against online solutions requires them to compare their reasoning with the reasoning of others.

UNIT 9: APPLICATIONS OF DEFINITE INTEGRALS

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.3, EU 3.4

Estimated Time:
27 instructional hours

Guiding Questions:

▸ How do you interpret the definite integral of a rate of change? ▸ How can you find the area between two curves? ▸ For a function that is both positive and negative over the interval $[a,b]$, how is finding the area between the curve and the x-axis from $x=a$ to $x=b$ different from simply integrating the function over the same interval? ▸ How do you find the volume of a solid of revolution when the enclosed region is not bounded by the axis of rotation over the entire interval?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.3D: Solve problems involving rates of change in applied contexts. **LO 3.4A:** Interpret the meaning of a definite integral within a problem. **LO 3.4E:** Use the definite integral to solve problems in various contexts.	**Web** McMullin, "Rate and Accumulation Type Problems"	**Instructional Activity: Rate and Accumulation Problems** In the previous unit students learned that the definite integral of a rate of change yields the net change. This was mostly studied from a motion perspective, so this activity allows them to solve definite integral problems in other contexts. Lin McMullin presents three different accumulation-type problems in a video. I put the students in groups of four and present them with one problem at a time. Groups are given 15–20 minutes to work on a problem. Then I show the class the video so they can check their work and hear the reasoning behind the correct answers. I also show students the scoring guidelines for the problem and have them score themselves.
LO 3.4D: Apply definite integrals to problems involving area and volume.		**Instructional Activity: Representing Areas Between Curves** I give students a sheet with several different graphs. Some have one function and others have two. The first example is a continuous, positive function labeled $f(x)$, and students must write a definite integral that gives the area bounded by the curve and the x-axis from $x=2$ to $x=6$. They use a highlighter to color the specified area. Another example includes a function that has some negative values over the given interval, and students follow similar instructions. Eventually, students see an example in which they must write a definite integral to represent the area bounded between two positive functions over a given interval. They use two highlighters to represent the areas bounded by each curve and the x-axis and then determine an appropriate integral representation.

You don't have to actually give students the functions — just $f(x)$ or $g(x)$ is enough. This is not about getting a numerical answer but rather how to properly set up the integrals.

UNIT 9: APPLICATIONS OF DEFINITE INTEGRALS

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.3, EU 3.4

Estimated Time:
27 instructional hours

Guiding Questions:

▸ How do you interpret the definite integral of a rate of change? ▸ How can you find the area between two curves? ▸ For a function that is both positive and negative over the interval $[a,b]$, how is finding the area between the curve and the x-axis from $x = a$ to $x = b$ different from simply integrating the function over the same interval? ▸ How do you find the volume of a solid of revolution when the enclosed region is not bounded by the axis of rotation over the entire interval?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.4D: Apply definite integrals to problems involving area and volume.	**Web** "Area Between Two Curves"	**Formative Assessment: Finding Areas Between Curves** Students work together to practice finding areas between curves using examples from the Visual Calculus Web page. They must sketch a graph of the functions and set up an integral to find the desired area. Depending on the problem, I have students sketch the graphs either with or without the aid of a graphing calculator. They can also click on the graph icon on the page to see a graph of the area they are finding. When all groups have finished the work, they each present one assigned problem to the class and explain their reasoning as they work. I encourage students to check their solutions against the solutions on the Web page only after completing the problem.	*Before we begin this practice, I review with students how to determine the intersection between two functions both algebraically and using the graphing calculator. If I feel more practice is needed, I assign additional problems for homework from the textbook.*
LO 3.4D: Apply definite integrals to problems involving area and volume. **LO 3.4D:** Apply definite integrals to problems involving area and volume.		**Instructional Activity: Putting the Function in Terms of y to Determine Area** To show students that generating an integral to represent a bounded area is sometimes a creative and flexible process, I ask them to find the area in the first quadrant bounded by the functions $f(x) = x - 2$ and $g(x) = \sqrt{x}$. Many students notice that this cannot be done with one definite integral. After allowing them to work the problem using the sum of two definite integrals, I rotate the graph on an interactive whiteboard to show that this problem can be done with one integral if it is rewritten in terms of y. This way, the linear function is the upper bound while the other function is the lower bound over the entire interval. Students work the problem in terms of y and compare with their previous answer.	*Putting functions in terms of y to determine area can be challenging for some students. They aren't sure when this is appropriate and can make careless errors, such as using limits from the x-axis when they should use limits from the y-axis. If I find my students struggling with this concept, I will sketch several examples and ask them to simply state if they'd find the area in terms of x or in terms of y.*

UNIT 9: APPLICATIONS OF DEFINITE INTEGRALS

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.3, EU 3.4

Estimated Time:
27 instructional hours

Guiding Questions:

▶ How do you interpret the definite integral of a rate of change? ▶ How can you find the area between two curves? ▶ For a function that is both positive and negative over the interval $[a,b]$, how is finding the area between the curve and the x-axis from $x = a$ to $x = b$ different from simply integrating the function over the same interval? ▶ How do you find the volume of a solid of revolution when the enclosed region is not bounded by the axis of rotation over the entire interval?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.4D: Apply definite integrals to problems involving area and volume.	**Web** "The Area Between"	**Formative Assessment: The Area Between** This TI-activity puts students into a real-world situation in which they must calculate the area of a walkway in order to determine how much concrete is needed to make the walkway. Students are given a standard height of one-third of a foot to help calculate the volume. This activity has some very nice visuals and gives students a lot of practice using the calculator to evaluate definite integrals. The students work through three situations, two of which have given limits and one in which students must determine the intersection points of two graphs before finding the area of the object.	*The real-world examples and visuals on this activity are appealing, but I don't necessarily have students follow the directions given. They use the functions and visuals to practice finding areas between two curves, but I let them figure out their own methods for inputting the problems into the calculator for evaluation.*
LO 3.4D: Apply definite integrals to problems involving area and volume.	**Web** Rahn, "Volumes of Solids with Known Cross-Sections" **Supplies** Glue or tape	**Instructional Activity: Volumes of Solids with Known Cross-Sections** I put students into five groups. While the groups have the same base region to create a solid, they each have cross-sections of a different shape (square, rectangle, semicircle, equilateral triangle, and isosceles right triangle). Students cut out their cross-sections and glue or tape them upright on the base region to create a three-dimensional figure. The questions in the activity guide students toward finding the volume of their solid by summing up the area of their cross-section. When all groups have completed their work, I have each group share their results and we compare areas to be sure the values make sense. For instance, the solid with semicircle cross-sections looks much smaller than the one with rectangular cross-sections. Therefore, the volumes should compare respectively.	*This activity is only available by purchase through the Teachers Pay Teachers website.*

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.3, EU 3.4

Estimated Time:
27 instructional hours

Guiding Questions:

▸ How do you interpret the definite integral of a rate of change? ▸ How can you find the area between two curves? ▸ For a function that is both positive and negative over the interval $[a,b]$, how is finding the area between the curve and the x-axis from $x = a$ to $x = b$ different from simply integrating the function over the same interval? ▸ How do you find the volume of a solid of revolution when the enclosed region is not bounded by the axis of rotation over the entire interval?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.4D: Apply definite integrals to problems involving area and volume.	**Web** "The Disk Method for Volumes of Solids of Revolution" **Supplies** Honeycomb tissue and paper balls	**Instructional Activity: The Disk Method** Students are given class time to read and review the Math Demos Web page on the disk method for volumes of solids of revolution. Students write down key facts and formulas from the Web page as they read. When they complete their reading, each group is given a honeycomb tissue paper party decoration that opens up to create a sphere. Both the information on the website and the party decoration simply serve as visuals for how the disk method creates a solid figure. We discuss the information they gathered as a class and clarify any misunderstandings before trying practice problems.	*The honeycomb paper tissue balls can be found online at a fairly reasonable price or in a party decorations store.*
LO 3.4D: Apply definite integrals to problems involving area and volume.	**Print** Finney et al., chapter 7	**Formative Assessment: Practice Finding Volumes of Solids** I create different groups or stations in the classroom. Each group has a different set of practice problems. Two groups have problems that require the disk method, two more have problems finding the volume of a solid generated by known cross-sections, and the last two are problems that could be more easily solved in terms of y. I don't tell the students what the stations are, but they should figure this out after one or two problems. Students work at their own pace to move from station to station and complete all the problems. They may collaborate with others at the same station as they work.	*The problems for the stations come mostly from the textbook. As students move around the room completing the assignment, I monitor their progress and give help as needed. I ask students to initial the problems at each station that were most difficult, and we review those as a class when everyone has completed their work.*

UNIT 9: APPLICATIONS OF DEFINITE INTEGRALS

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.3, EU 3.4

Estimated Time:
27 instructional hours

Guiding Questions:

▸ How do you interpret the definite integral of a rate of change? ▸ How can you find the area between two curves? ▸ For a function that is both positive and negative over the interval $[a,b]$, how is finding the area between the curve and the x-axis from $x = a$ to $x = b$ different from simply integrating the function over the same interval? ▸ How do you find the volume of a solid of revolution when the enclosed region is not bounded by the axis of rotation over the entire interval?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.4D: Apply definite integrals to problems involving area and volume.	**Web** "Visualizing Solids of Revolution — Washers"	**Instructional Activity: Visualizing Solids of Revolution — Washers** Students work in pairs as they explore the washer method for finding the volume of a solid of revolution. In this activity, students see a visual representation of what a solid of revolution might look like and what one washer cross-section looks like inside the shape. Students can move the cross-section to see what it looks like anywhere in the interval. The activity leads students to determine that adding the areas of an infinite amount of washers inside a solid will result in finding the volume of the solid. Students make conjectures and form a definite integral representation for the shape.
LO 3.4D: Apply definite integrals to problems involving area and volume.	**Web** "AP Calculus AB: Exam Practice" Free-response questions available at AP Central	**Formative Assessment: Areas Between Curves and Solids of Revolution Practice** I picked out some old free-response questions from former AP Calculus Exams and made a collection of problems for students to try. Some of the problems include 2005 question 1, 2011 question 3, and 2010 question 4. Students work together to complete these problems and justify their work as needed. When everyone has completed the work, I show students the scoring guidelines and work any parts of the problem I feel need further explanation. Students switch papers and try to score each other's work according to the scoring guidelines.

As students work on these questions, I take note of which parts gave the majority of students trouble. I make sure to do these parts of the problem when we review as a class. I also go through the scores students gave each other, verify their scoring, and leave notes about what they could improve in terms of notation, showing work, formatting, and so on.

UNIT 9: APPLICATIONS OF DEFINITE INTEGRALS

BIG IDEA 3
Integrals and the Fundamental Theorem of Calculus

Enduring Understandings:
- EU 2.3, EU 3.4

Estimated Time:
27 instructional hours

Guiding Questions:

▸ How do you interpret the definite integral of a rate of change? ▸ How can you find the area between two curves? ▸ For a function that is both positive and negative over the interval $[a,b]$, how is finding the area between the curve and the x-axis from $x = a$ to $x = b$ different from simply integrating the function over the same interval? ▸ How do you find the volume of a solid of revolution when the enclosed region is not bounded by the axis of rotation over the entire interval?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.4D: Apply definite integrals to problems involving area and volume.	**Materials** Honeycomb tissue paper balls	**Instructional Activity: Making Your Own Solid** Students use honeycomb paper tissue party decorations to create their own solid of revolution. They create a function, print it out using Geogebra to produce a graph, and trace it onto their tissue ball. Then, they cut the tissue ball to the shape of their function and open it to see what kind of solid they've created. After creating the solid, students must write a definite integral that gives its volume. Most students end up creating a solid that requires the disk method to determine the volume, but they could create regions that require the washer method to find the volume of their solid.	
All of the learning objectives in this unit are addressed.		**Summative Assessment: Test on Applications of Definite Integrals** This assessment includes some multiple-choice and free-response questions, but most are short-answer questions. Students must be able to find areas between curves and volumes of solids of revolution and solids with known cross-sections. There is one accumulation-type problem that involves oil leaking into a lake. A portion of this test requires a graphing calculator.	*This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 9

The following activities and techniques in Unit 9 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: The "Visualizing Solids of Revolutions — Washers" instructional activity guides students toward making conjectures about how to determine the volume of a solid of revolution when its cross-section is not solid.

MPAC 2 — Connecting concepts: As students create definite integrals to correspond to the graphs they are given in the "Representing Areas Between Curves" instructional activity, they are connecting the concept of a definite integral to the visual representation of area between two curves. The "Volumes of Solids with Known Cross-Sections" instructional activity helps students to relate the idea of a limit to finding the volume of a solid by adding an infinite amount of areas of cross-sections.

MPAC 3 — Implementing algebraic/computational processes: The "Putting the Function in Terms of y to Determine Area" instructional activity requires students to attend to precision as they set up two different definite integrals (one in terms of y, the other in terms of x) that correctly give the area of the bounded region.

MPAC 4 — Connecting multiple representations: "The Disk Method" instructional activity uses computer-generated animation to guide students toward developing the concept using graphs.

MPAC 5 — Building notational fluency: In the "Representing Areas Between Curves" instructional activity, students connect the definite integral notation to its graphical representation. They assign meaning to this notation by using it to represent area.

MPAC 6 — Communicating: As students work together on the "Rate and Accumulation Problems" instructional activity, they must clearly present their methods and justification of their work. They analyze the work of their peers and compare it with their own.

Resources

General Resources

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