Visualizing Differential Equations
Slope Fields

by Lin McMullin

The topic of slope fields is new to the *AP Calculus AB Course Description* for the 2004 exam. Where do slope fields come from? How should we include them? When should we include them?

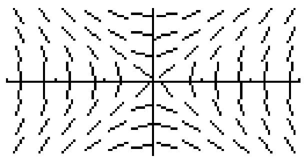
Many real-world phenomena are modeled by differential equations. The *mathematical* use of a slope field is to visualize the graph of the general solution (a family of functions) of a differential equation. Their *pedagogical* use is to help students better understand the solution of differential equations. Multiple representations of all the major ideas in mathematics are the overriding strategy of mathematics teaching today. Writing the equation is one thing, but "seeing" the solutions by plotting slope fields removes the abstractness from the symbolic representation.

What is a slope field? A slope field is the *graphical* representation of a differential equation. It is a graph of short line segments whose slope is determined by evaluating the derivative at the midpoint of the segment. But it's really better to see one…

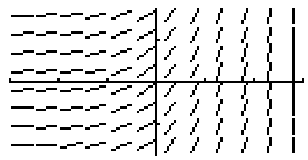
What Are Slope Fields? Introducing Slope Fields to Your Class

- **Graphical Introduction:** Prepare a transparency with a family of functions such as $y = x^2 + B$ for various values of *B*. Use any function you want, of course. Put the transparency on an overhead projector and cover it with another transparency. With a marker pen, moving across the page, draw short segments tangent to the curves. Go down a line and draw another line of segments. Continue down the page. When you are done, remove the first transparency (the one with the curves), and the remaining one is the slope field. This is the strictly graphical introduction to the slope field.
- **Numerical Introduction:** Put a differential equation on the board, perhaps $\frac{dy}{dx} = -\frac{x}{y}$. Give each member of the class one or two points -- the grid points of the square whose diagonal runs from (–3, –3) to (3, 3). Each student calculates the value of $\frac{dy}{dx}$ at his or her point. Then, again on the overhead projector, have each student graph their segments on a coordinate grid, through the point they used to calculate their slope. This is a slope field.
- **Technological Introduction:** At the end of this article is a program for generating slope fields on the TI-83 and TI-83+. These may be used to produce the slope field of a differential equation. Enter the differential equation as the Y1 equation, in the editor, in terms of X and (alpha) Y (not Y1), select a viewing window, and run the program. The slope fields shown below were done with this program. (The TI-86, TI-89, TI-92+, and V 200 have built-in slope field operations.)

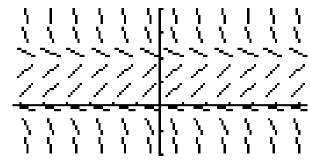
$\frac{dy}{dx} = \frac{x}{y}$



$\frac{dy}{dx} = e^x$

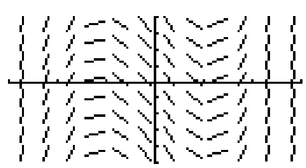


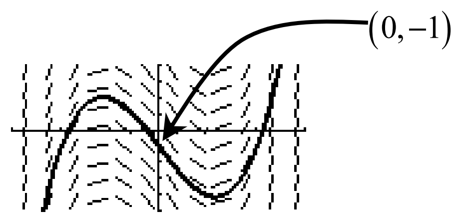
$\frac{dy}{dx} = y(2-y)$



What Do Slope Fields Show You?

The solution curves are hiding in the slope field. Given one point of the particular solution curve, you can sketch the graph from that point, in both directions, to see the graph of the solution. The initial value problem $\frac{dy}{dx} = 3x^2 - 4$ with $y = -1$ when $x = 0$ is shown. Notice how the graph flows through the slope field going both left and right from the starting point (0, –1).





Slope Fields on the AP Exams

The availability of technology to draw slope fields is relatively new. Some textbooks do not mention slope fields, so this is a topic that may need supplementing. Graphing calculators and programs like Winplot will draw slope fields. A little practice looking at slope fields and the associated solutions of the differential equation should help you understand what's going on. Be sure to plot several different initial conditions on the same slope field, especially if there appears to be an asymptote (choose initial conditions from each side of the asymptote).

There are relatively few slope field questions from past AP® Exams since they were not added to the Calculus BC Topic Outline until 1998. Drawing slope fields is *not* one of the things students are permitted to do on the exams with a graphing calculator. Students should know how to sketch slope fields by hand and have been asked to do so on AP Exams in the past. Several styles of questions are possible:

- Draw a slope field by hand at given points -- 1998, BC 4(a), 2000 BC 6(a).
- Given a slope field, identify its differential equation (multiple-choice) -- 1998 BC 24.

- Given a differential equation, identify its slope field (multiple-choice).
- Interpret a slope field -- 2000 BC 6, 2002 BC 5, note also (c) second solution.
- Draw a solution curve on a given slope field -- 2002 BC 5.
- Given a solution curve, identify its slope field.
- Given a slope field, identify its solution curve.

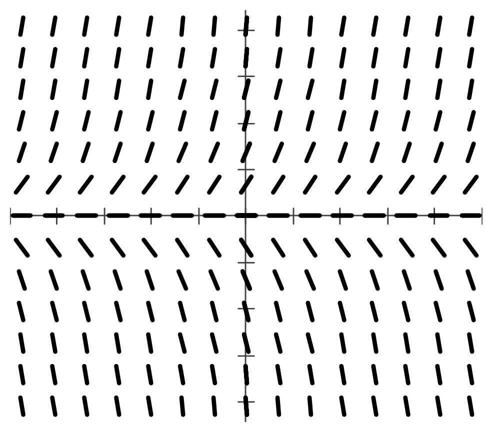
Problems and Examples

1. 1998 AP Exam BC 4 (draw slope field + Euler + full solution) (See http://apcentral.collegeboard.com/members/article/1,3046,152-171-0-8031,00.html) *You must be a registered member of AP Central® to access this page.*

2. 2000 AP Exam BC 6 (draw slope field + interpret slope field + analytic solution + use solution and slope field together) (See http://apcentral.collegeboard.com/members/article/1,3046,152-171-0-8031,00.html) *You must be a registered member of AP Central to access this page.*

3. 2002 AP Exam BC 5 (See http://apcentral.collegeboard.com/members/article/1,3046,152-171-0-8031,00.html) *You must be a registered member of AP Central to access this page.*

4. The slope field for a differential equation is shown at the right. Which statement is true for solutions of the differential equation?



   I. For $x < 0$ all solutions are decreasing.

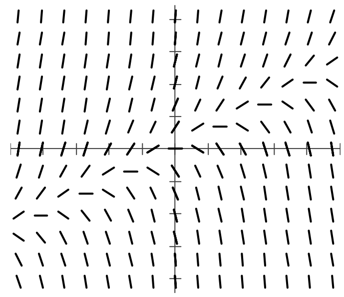
   II. All solutions level off near the x-axis.

   III. For $y > 0$ all solutions are increasing.

   (A) I only (B) II only (C) III only

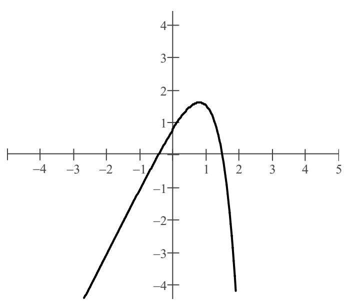
   (D) II and III only (E) I, II, and III

5.

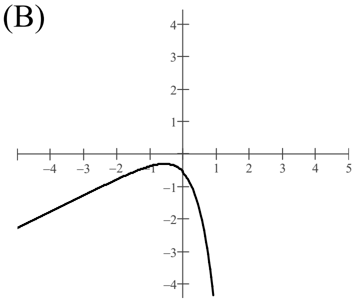


Which one of the following could be the graph of the solution of the differential equation whose slope field is shown above?

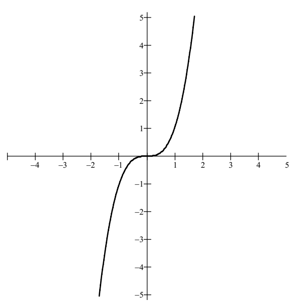
(A)



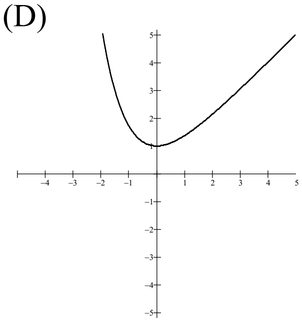
(B)



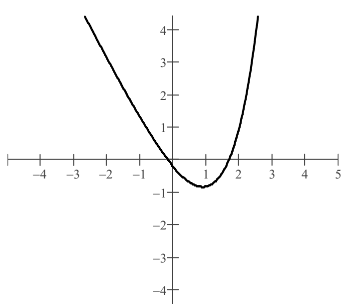
(C)



(D)



(E)



6. The slope field for the differential equation $\frac{dy}{dx} = \frac{x^2 y + y^2}{4x + 2y}$ will have vertical segments when

(A) $y = 2x$, only

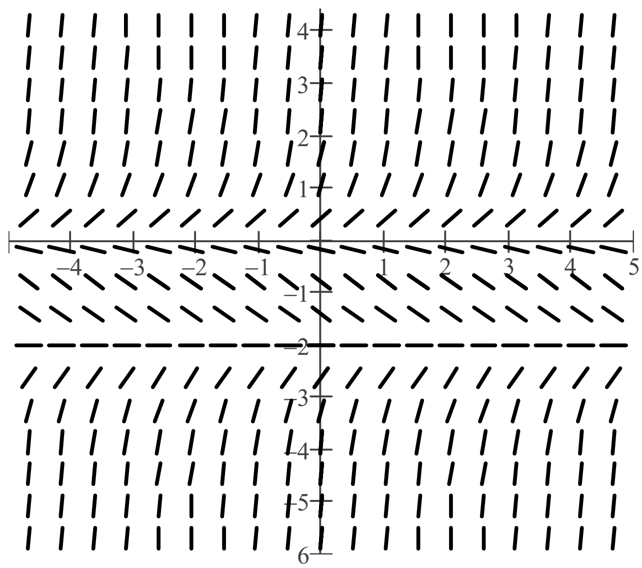
(B) $y = -2x$, only

(C) $y = -x^2$, only

(D) $y = 0$, only

(E) $y = 0$ or $y = -x^2$

7.



Which statement is true about the solutions $y(x)$, of a differential equation whose slope field is shown above?

I. If $y(0) > 0$ then $\lim_{x\to\infty} y(x) \approx 0$.

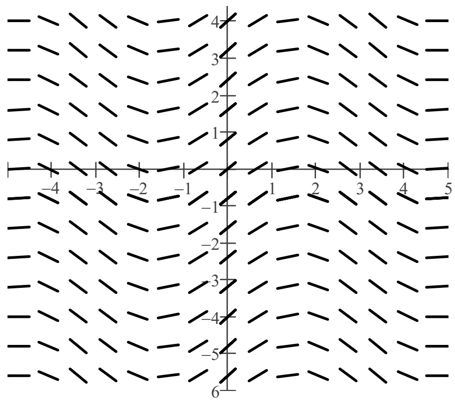
II. If $-2 < y(0) < 0$ then $\lim_{x\to\infty} y(x) \approx -2$.

III. If $y(0) < -2$ then $\lim_{x\to\infty} y(x) \approx -2$.

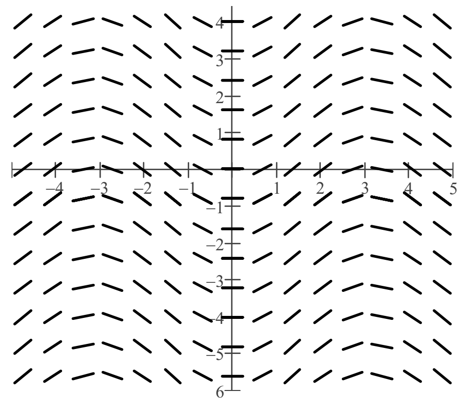
(A) I only (B) II only (C) III only (D) II and III only (E) I, II, and III

9. Which choice represents the slope field for $\dfrac{dy}{dx} = \cos x$?

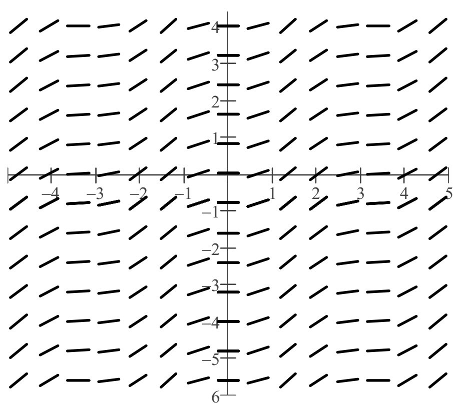
(A)



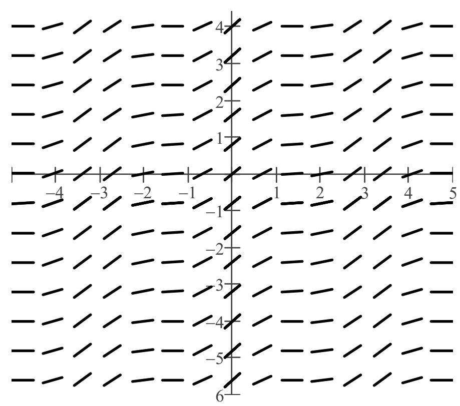
(B)



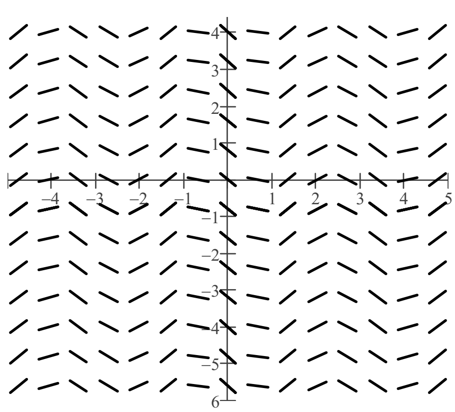
(C)



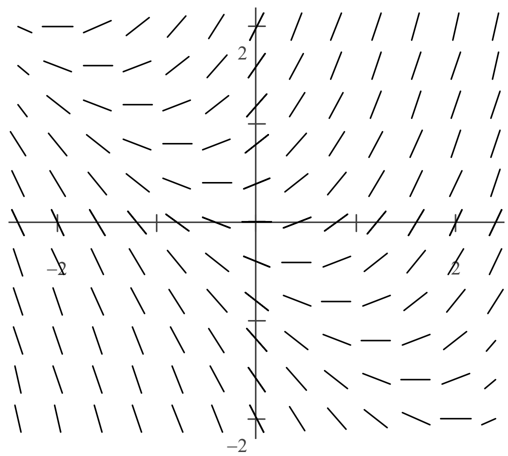
(D)



(E)



10.
1998 BC 24



Shown above is the slope field for which of the following differential equations?

(A) $\dfrac{dy}{dx} = 1 + x$ (B) $\dfrac{dy}{dx} = x^2$ (C) $\dfrac{dy}{dx} = x + y$ (D) $\dfrac{dy}{dx} = \dfrac{x}{y}$ (E) $\dfrac{dy}{dx} = \ln y$

Answers:

4. D

5. B

6. B

7. D

9. A

10. C

Directions for Slope Field Program
Enter the derivative as Y1 in the equation editor. Use [alpha] y (not Y1 from the VARS menu). Adjust the viewing window if necessary. Return to the HOME screen and run the slope field program. (This program will not run by pushing GRAPH because the [alpha] y is not recognized.)

Slope Field (TI-82/83)

```
ClrDraw
FnOff
7(Xmax-Xmin)/83→H
7(Ymax-Ymin)/55→K
1/(.4H)²→A
1/(.4K)²→B
Xmin+.5H→X
Ymin+.5K→Z
1→I
Lbl A1
1→J
Z→Y
Lbl A2
Y1→T
1/√((A+B*T²)→C
T*C→S
X→U
Y→V
Line(U-C,V-S,U+C,V+S)
V+K→Y
IS>(J,9)
Goto A2
U+H→X
IS>(I,12)
Goto A1
```

Winplot is available for *free* at http://math.exeter.edu/rparris. This program also does graphs (rectangular, polar, parametric, implicit plots) in 2D and 3D. It also draws solids of revolution and solids with regular cross sections. Instructions for Winplot are available at http://matcmadison.edu/alehnen/winptut/winpltut.htm.