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Question 1

An object moving along a curve in the xy-plane has position \((x(t), y(t))\) at time \(t\) with

\[
\frac{dx}{dt} = \cos(t^3) \quad \text{and} \quad \frac{dy}{dt} = 3 \sin(t^2)
\]

for \(0 \leq t \leq 3\). At time \(t = 2\), the object is at position (4,5).

(a) Write an equation for the line tangent to the curve at (4,5).
(b) Find the speed of the object at time \(t = 2\).
(c) Find the total distance traveled by the object over the time interval \(0 \leq t \leq 1\).
(d) Find the position of the object at time \(t = 3\).

(a) \[
\frac{dy}{dx} = \frac{3 \sin(t^2)}{\cos(t^3)}
\]

\[
\frac{dy}{dx}\bigg|_{t=2} = \frac{3 \sin(2^2)}{\cos(2^3)} = 15.604
\]

\[
y - 5 = 15.604(x - 4)
\]

1 : tangent line

1 : answer

(b) Speed = \[
\sqrt{\cos^2(8) + 9 \sin^2(4)} = 2.275
\]

1 : answer

(c) Distance = \[
\int_{0}^{1} \sqrt{\cos^2(t^3) + 9 \sin^2(t^2)} \, dt = 1.458
\]

2 : distance integral

\(< -1 > \) each integrand error

3 : \(< -1 > \) error in limits

1 : answer

(d) \[
x(3) = 4 + \int_{2}^{3} \cos(t^3) \, dt = 3.953 \text{ or } 3.954
\]

\[
y(3) = 5 + \int_{2}^{3} 3 \sin(t^2) \, dt = 4.906
\]

1 : definite integral for \(x\)

1 : answer for \(x(3)\)

4 : \1: definite integral for \(y\)

1 : answer for \(y(3)\)
Question 2

The temperature, in degrees Celsius (°C), of the water in a pond is a differentiable function \( W \) of time \( t \). The table above shows the water temperature as recorded every 3 days over a 15-day period.

(a) Use data from the table to find an approximation for \( W'(12) \). Show the computations that lead to your answer. Indicate units of measure.

(b) Approximate the average temperature, in degrees Celsius, of the water over the time interval \( 0 \leq t \leq 15 \) days by using a trapezoidal approximation with subintervals of length \( \Delta t = 3 \) days.

(c) A student proposes the function \( P \), given by \( P(t) = 20 + 10te^{-t/3} \), as a model for the temperature of the water in the pond at time \( t \), where \( t \) is measured in days and \( P(t) \) is measured in degrees Celsius. Find \( P'(12) \). Using appropriate units, explain the meaning of your answer in terms of water temperature.

(d) Use the function \( P \) defined in part (c) to find the average value, in degrees Celsius, of \( P(t) \) over the time interval \( 0 \leq t \leq 15 \) days.

\[
\begin{array}{|c|c|}
\hline
\text{(days)} & \text{\( W(t) \) (°C)} \\
\hline
0 & 20 \\
3 & 31 \\
6 & 28 \\
9 & 24 \\
12 & 22 \\
15 & 21 \\
\hline
\end{array}
\]

(a) Difference quotient; e.g.

\[
W'(12) \approx \frac{W(15) - W(12)}{15 - 12} = -\frac{1}{3} \text{ °C/day or}
\]

\[
W'(12) \approx \frac{W(12) - W(9)}{12 - 9} = -\frac{2}{3} \text{ °C/day or}
\]

\[
W'(12) \approx \frac{W(15) - W(9)}{15 - 9} = -\frac{1}{2} \text{ °C/day}
\]

(b) \( \frac{3}{2} (20 + 2(31) + 2(28) + 2(24) + 2(22) + 21) = 376.5 \)

Average temperature \( \approx \frac{1}{15} (376.5) = 25.1 \text{ °C} \)

(c) \( P'(12) = 10e^{-t/3} - \frac{10}{3}te^{-t/3} \bigg|_{t=12} \)

\[
= -30e^{-4} = -0.549 \text{ °C/day}
\]

This means that the temperature is decreasing at the rate of 0.549 °C/day when \( t = 12 \) days.

(d) \( \frac{1}{15} \int_{0}^{15} (20 + 10te^{-t/3}) \, dt = 25.757 \text{ °C} \)
A car is traveling on a straight road with velocity 55 ft/sec at time $t = 0$. For $0 \leq t \leq 18$ seconds, the car’s acceleration $a(t)$, in ft/sec$^2$, is the piecewise linear function defined by the graph above.

(a) Is the velocity of the car increasing at $t = 2$ seconds? Why or why not?

(b) At what time in the interval $0 \leq t \leq 18$, other than $t = 0$, is the velocity of the car 55 ft/sec? Why?

(c) On the time interval $0 \leq t \leq 18$, what is the car’s absolute maximum velocity, in ft/sec, and at what time does it occur? Justify your answer.

(d) At what times in the interval $0 \leq t \leq 18$, if any, is the car’s velocity equal to zero? Justify your answer.

(a) Since $v'(2) = a(2)$ and $a(2) = 15 > 0$, the velocity is increasing at $t = 2$.

(b) At time $t = 12$ because

$$v(12) - v(0) = \int_0^{12} a(t) \, dt = 0.$$  

(c) The absolute maximum velocity is 115 ft/sec at $t = 6$.

The absolute maximum must occur at $t = 6$ or at an endpoint.

$$v(6) = 55 + \int_0^6 a(t) \, dt$$

$$= 55 + 2(15) + \frac{1}{2}(4)(15) = 115 > v(0)$$

$$\int_6^{18} a(t) \, dt < 0 \text{ so } v(18) < v(6)$$

(d) The car’s velocity is never equal to 0. The absolute minimum occurs at $t = 16$ where

$$v(16) = 115 + \int_6^{16} a(t) \, dt = 115 - 105 = 10 > 0.$$
Let $h$ be a function defined for all $x \neq 0$ such that $h(4) = -3$ and the derivative of $h$ is given by $h'(x) = \frac{x^2 - 2}{x}$ for all $x \neq 0$.

(a) Find all values of $x$ for which the graph of $h$ has a horizontal tangent, and determine whether $h$ has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

(b) On what intervals, if any, is the graph of $h$ concave up? Justify your answer.

(c) Write an equation for the line tangent to the graph of $h$ at $x = 4$.

(d) Does the line tangent to the graph of $h$ at $x = 4$ lie above or below the graph of $h$ for $x > 4$? Why?

(a) $h'(x) = 0$ at $x = \pm \sqrt{2}$

\[
\begin{array}{c|cccc}
  & - & 0 & + & - \\
\hline
x & -\sqrt{2} & 0 & \sqrt{2} & \\
\end{array}
\]

Local minima at $x = -\sqrt{2}$ and at $x = \sqrt{2}$

(b) $h''(x) = 1 + \frac{2}{x^2} > 0$ for all $x \neq 0$. Therefore, the graph of $h$ is concave up for all $x \neq 0$.

(c) $h'(4) = \frac{16 - 2}{4} = \frac{7}{2}$

$$y + 3 = \frac{7}{2}(x - 4)$$

(d) The tangent line is below the graph because the graph of $h$ is concave up for $x > 4$. 
Let \( f \) be the function satisfying \( f'(x) = -3xf(x) \), for all real numbers \( x \), with \( f(1) = 4 \) and \( \lim_{x \to \infty} f(x) = 0 \).

(a) Evaluate \( \int_{1}^{\infty} -3xf(x) \, dx \). Show the work that leads to your answer.

(b) Use Euler’s method, starting at \( x = 1 \) with a step size of 0.5, to approximate \( f(2) \).

(c) Write an expression for \( y = f(x) \) by solving the differential equation \( \frac{dy}{dx} = -3xy \) with the initial condition \( f(1) = 4 \).

(a) \( \int_{1}^{\infty} -3xf(x) \, dx \)

\[
\begin{align*}
= \int_{1}^{\infty} f'(x) \, dx & \quad \text{lim}_{b \to \infty} \int_{1}^{b} f'(x) \, dx = \lim_{b \to \infty} f(x) \\
= \lim_{b \to \infty} f(b) - f(1) & = 0 - 4 = -4
\end{align*}
\]

(b) \( f(1.5) \approx f(1) + f'(1)(0.5) \)

\[
= 4 - 3(1)(4)(0.5) = -2
\]

\( f(2) \approx -2 + f'(1.5)(0.5) \)

\[
\approx -2 - 3(1.5)(-2)(0.5) = 2.5
\]

(c) \[ \frac{1}{y} \, dy = -3x \, dx \]

\[
\ln y = -\frac{3}{2} x^2 + k
\]

\[
y = Ce^{-\frac{3}{2} x^2}
\]

\[
4 = Ce^{-\frac{3}{2}} \quad ; \quad C = 4e^{\frac{3}{2}}
\]

\[
y = 4e^{\frac{3}{2}} e^{-\frac{3}{2} x^2}
\]
A function \( f \) is defined by

\[
f(x) = \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots
\]

for all \( x \) in the interval of convergence of the given power series.

(a) Find the interval of convergence for this power series. Show the work that leads to your answer.

(b) Find \( \lim_{x \to 0} \frac{f(x) - \frac{1}{3}}{x} \).

(c) Write the first three nonzero terms and the general term for an infinite series that represents \( \int_0^1 f(x) \, dx \).

(d) Find the sum of the series determined in part (c).

(a) \[
\lim_{n \to \infty} \left| \frac{(n+2)x^{n+1}}{3^{n+2}} \right| = \lim_{n \to \infty} \left| (n+2) \frac{x}{3} \right| = \left| \frac{x}{3} \right| < 1
\]

At \( x = -3 \), the series is \( \sum_{n=0}^{\infty} (-1)^n \frac{n+1}{3} \), which diverges.

At \( x = 3 \), the series is \( \sum_{n=0}^{\infty} \frac{n+1}{3} \), which diverges.

Therefore, the interval of convergence is \(-3 < x < 3\).

(b) \[
\lim_{x \to 0} \frac{f(x) - \frac{1}{3}}{x} = \lim_{x \to 0} \left( \frac{2}{3^2} + \frac{3}{3^3}x + \frac{4}{3^4}x^2 + \cdots \right) = \frac{2}{9}
\]

(c) \[
\int_0^1 f(x) \, dx = \int_0^1 \left( \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots \right) \, dx
\]

\[
= \left. \left( \frac{1}{3}x + \frac{1}{3^2}x^2 + \frac{1}{3^3}x^3 + \cdots + \frac{1}{3^{n+1}}x^{n+1} + \cdots \right) \right|_{x=0}^{x=1}
\]

\[
= \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n+1}} + \cdots
\]

(d) The series representing \( \int_0^1 f(x) \, dx \) is a geometric series.

Therefore, \( \int_0^1 f(x) \, dx = \frac{1}{3 - \frac{1}{3}} = \frac{1}{2} \).