AP Calculus BC
2000 Scoring Guidelines

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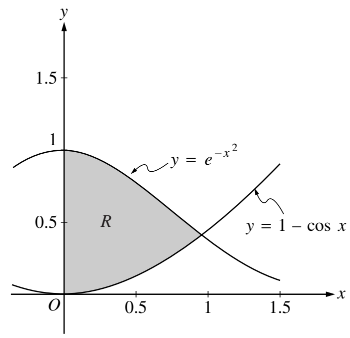
AP Calculus AB–1 / BC–1 2000



Let R be the shaded region in the first quadrant enclosed by the graphs of $y = e^{-x^2}$, $y = 1 - \cos x$, and the y-axis, as shown in the figure above.

(a) Find the area of the region R.

(b) Find the volume of the solid generated when the region R is revolved about the x-axis.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

Region R

$e^{-x^2} = 1 - \cos x$ at $x = 0.941944 = A$

1 : Correct limits in an integral in (a), (b), or (c).

(a) Area $= \displaystyle\int_0^A \left(e^{-x^2} - (1 - \cos x)\right) dx$

$= 0.590$ or 0.591

$$2 \begin{cases} 1: \text{ integrand} \\ 1: \text{ answer} \end{cases}$$

(b) Volume $= \pi \displaystyle\int_0^A \left(\left(e^{-x^2}\right)^2 - (1 - \cos x)^2\right) dx$

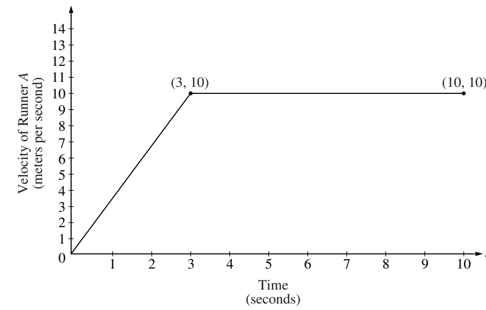
$= 0.55596\pi = 1.746$ or 1.747

$$3 \begin{cases} 2: \text{ integrand and constant} \\ \quad <-1> \text{ each error} \\ 1: \text{ answer} \end{cases}$$

(c) Volume $= \displaystyle\int_0^A \left(e^{-x^2} - (1 - \cos x)\right)^2 dx$

$= 0.461$

$$3 \begin{cases} 2: \text{ integrand} \\ \quad <-1> \text{ each error} \\ \quad \text{Note: 0/2 if not of the form} \\ \qquad k\displaystyle\int_c^d (f(x) - g(x))^2\, dx \\ 1: \text{ answer} \end{cases}$$



Two runners, A and B, run on a straight racetrack for $0 \le t \le 10$ seconds. The graph above, which consists of two line segments, shows the velocity, in meters per second, of Runner A. The velocity, in meters per second, of Runner B is given by the function v defined by $v(t) = \dfrac{24t}{2t+3}$.

(a) Find the velocity of Runner A and the velocity of Runner B at time $t = 2$ seconds. Indicate units of measure.

(b) Find the acceleration of Runner A and the acceleration of Runner B at time $t = 2$ seconds. Indicate units of measure.

(c) Find the total distance run by Runner A and the total distance run by Runner B over the time interval $0 \le t \le 10$ seconds. Indicate units of measure.

(a) Runner A: velocity $= \dfrac{10}{3} \cdot 2 = \dfrac{20}{3}$

$= 6.666$ or 6.667 meters/sec

Runner B: $v(2) = \dfrac{48}{7} = 6.857$ meters/sec

$2 \begin{cases} 1: \text{ velocity for Runner } A \\ 1: \text{ velocity for Runner } B \end{cases}$

(b) Runner A: acceleration $= \dfrac{10}{3} = 3.333$ meters/sec^2

Runner B: $a(2) = v'(2) = \left.\dfrac{72}{(2t+3)^2}\right|_{t=2}$

$= \dfrac{72}{49} = 1.469$ meters/sec^2

$2 \begin{cases} 1: \text{ acceleration for Runner } A \\ 1: \text{ acceleration for Runner } B \end{cases}$

(c) Runner A: distance $= \dfrac{1}{2}(3)(10) + 7(10) = 85$ meters

Runner B: distance $= \displaystyle\int_0^{10} \frac{24t}{2t+3}\,dt = 83.336$ meters

$4 \begin{cases} 2: \text{ distance for Runner } A \\ \quad 1: \text{ method} \\ \quad 1: \text{ answer} \\ 2: \text{ distance for Runner } B \\ \quad 1: \text{ integral} \\ \quad 1: \text{ answer} \end{cases}$

(units) meters/sec in part (a), meters/sec^2 in part (b), and meters in part (c), or equivalent.

1: units

The Taylor series about $x = 5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 5$ is given by $f^{(n)}(5) = \dfrac{(-1)^n n!}{2^n (n+2)}$, and $f(5) = \dfrac{1}{2}$.

(a) Write the third-degree Taylor polynomial for f about $x = 5$.

(b) Find the radius of convergence of the Taylor series for f about $x = 5$.

(c) Show that the sixth-degree Taylor polynomial for f about $x = 5$ approximates $f(6)$ with error less than $\dfrac{1}{1000}$.

(a) $f'(5) = \dfrac{-1!}{2(3)}$, $f''(5) = \dfrac{2!}{4(4)}$, $f'''(5) = \dfrac{-3!}{8(5)}$

$$P_3(f,5)(x) = \frac{1}{2} - \frac{1}{6}(x-5) + \frac{1}{16}(x-5)^2 - \frac{1}{40}(x-5)^3$$

3 : $P_3(f,5)(x)$
<–1> each error or missing term

Note: <–1> max for improper use of extra terms, equality or +…

(b) $a_n = \dfrac{f^{(n)}(5)}{n!} = \dfrac{(-1)^n}{2^n(n+2)}$

$$\lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}(x-5)^{n+1}}{2^{n+1}(n+3)}}{\dfrac{(-1)^n(x-5)^n}{2^n(n+2)}}\right| = \lim_{n\to\infty}\frac{1}{2}\left(\frac{n+2}{n+3}\right)|x-5|$$

$$= \frac{|x-5|}{2} < 1$$

The radius of convergence is 2.

4 $\begin{cases} 1: \text{ general term} \\ 1: \text{ sets up ratio test} \\ 1: \text{ computes the limit} \\ 1: \text{ applies ratio test to get radius of convergence} \end{cases}$

(c) The Taylor series about $x = 5$ for the function f, when evaluated at $x = 6$, is an alternating series with absolute value of terms decreasing to 0. The error in approximating $f(6)$ with the 6th degree Taylor polynomial at $x = 6$ is less than the first omitted term in the series.

$$\left|f(6) - P_6(f,5)(6)\right| \le \frac{1}{2^7(9)} = \frac{1}{1152} < \frac{1}{1000}$$

2 $\begin{cases} 1: \text{ error bound} < \dfrac{1}{1000} \\ 1: \text{ refers to an alternating series and indicates the error bound is found from the next term} \end{cases}$

A moving particle has position $(x(t), y(t))$ at time t. The position of the particle at time $t = 1$ is (2,6) and the velocity vector at any time $t > 0$ is given by $\left(1 - \frac{1}{t^2}, 2 + \frac{1}{t^2}\right)$.

(a) Find the acceleration vector at time $t = 3$.

(b) Find the position of the particle at time $t = 3$.

(c) For what time $t > 0$ does the line tangent to the path of the particle at $(x(t), y(t))$ have a slope of 8?

(d) The particle approaches a line as $t \to \infty$. Find the slope of this line. Show the work that leads to your conclusion.

(a) acceleration vector $= (x''(t), y''(t)) = \left(\frac{2}{t^3}, -\frac{2}{t^3}\right)$

$(x''(3), y''(3)) = \left(\frac{2}{27}, -\frac{2}{27}\right)$

$$2\begin{cases} 1: \text{ components of acceleration vector as a function of } t \\ 1: \text{ acceleration vector at } t = 3 \end{cases}$$

(b) $(x(t), y(t)) = \left(t + \frac{1}{t} + C_1,\ 2t - \frac{1}{t} + C_2\right)$

$(2,6) = (x(1), y(1)) = (2 + C_1, 1 + C_2)$

$C_1 = 0$, $C_2 = 5$

$(x(3), y(3)) = \left(3 + \frac{1}{3}, 6 - \frac{1}{3} + 5\right) = \left(\frac{10}{3}, \frac{32}{3}\right)$

$$3\begin{cases} 1: \text{ antidifferentiation} \\ 1: \text{ uses initial condition at } t = 1 \\ 1: \text{ position at } t = 3 \end{cases}$$

Note: max 1/3 [1–0–0] if no constants of integration

(c) $\frac{dy}{dx} = \dfrac{2 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = 8$

$2 + \frac{1}{t^2} = 8\left(1 - \frac{1}{t^2}\right)$; $\quad t^2 = \frac{9}{6}$

$t = \sqrt{\frac{3}{2}}$

$$2\begin{cases} 1: \ \frac{dy}{dx} = 8 \text{ as equation in } t \\ 1: \text{ solution for } t \end{cases}$$

(d) $\lim\limits_{t\to\infty} \frac{dy}{dx} = \lim\limits_{t\to\infty} \dfrac{2 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = 2$

– or –

Since $x(t) \to \infty$ as $t \to \infty$, the slope of the line is

$\lim\limits_{t\to\infty} \frac{y(t)}{x(t)} = \lim\limits_{t\to\infty} \dfrac{2t - \frac{1}{t} + 5}{t + \frac{1}{t}} = 2$

$$2\begin{cases} 1: \text{ considers limit of } \frac{dy}{dx} \text{ or } \frac{y(t)}{x(t)} \\ 1: \text{ answer} \end{cases}$$

Note: 0/2 if no consideration of limit

Consider the curve given by $xy^2 - x^3y = 6$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{3x^2y - y^2}{2xy - x^3}$.

(b) Find all points on the curve whose x-coordinate is 1, and write an equation for the tangent line at each of these points.

(c) Find the x-coordinate of each point on the curve where the tangent line is vertical.

(a) $y^2 + 2xy\dfrac{dy}{dx} - 3x^2y - x^3\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx}\left(2xy - x^3\right) = 3x^2y - y^2$

$\dfrac{dy}{dx} = \dfrac{3x^2y - y^2}{2xy - x^3}$

$$2\begin{cases} 1: \text{implicit differentiation} \\ 1: \text{verifies expression for } \dfrac{dy}{dx} \end{cases}$$

(b) When $x = 1$, $y^2 - y = 6$

$y^2 - y - 6 = 0$

$(y-3)(y+2) = 0$

$y = 3,\ y = -2$

At $(1,3)$, $\dfrac{dy}{dx} = \dfrac{9-9}{6-1} = 0$

Tangent line equation is $y = 3$

At $(1,-2)$, $\dfrac{dy}{dx} = \dfrac{-6-4}{-4-1} = \dfrac{-10}{-5} = 2$

Tangent line equation is $y + 2 = 2(x-1)$

$$4\begin{cases} 1: & y^2 - y = 6 \\ 1: & \text{solves for } y \\ 2: & \text{tangent lines} \end{cases}$$

Note: 0/4 if not solving an equation of the form $y^2 - y = k$

(c) Tangent line is vertical when $2xy - x^3 = 0$

$x\left(2y - x^2\right) = 0$ gives $x = 0$ or $y = \dfrac{1}{2}x^2$

There is no point on the curve with x-coordinate 0.

When $y = \dfrac{1}{2}x^2$, $\dfrac{1}{4}x^5 - \dfrac{1}{2}x^5 = 6$

$-\dfrac{1}{4}x^5 = 6$

$x = \sqrt[5]{-24}$

$$3\begin{cases} 1: & \text{sets denominator of } \dfrac{dy}{dx} \text{ equal to 0} \\ 1: & \text{substitutes } y = \dfrac{1}{2}x^2 \text{ or } x = \pm\sqrt{2y} \\ & \text{into the equation for the curve} \\ 1: & \text{solves for } x\text{-coordinate} \end{cases}$$

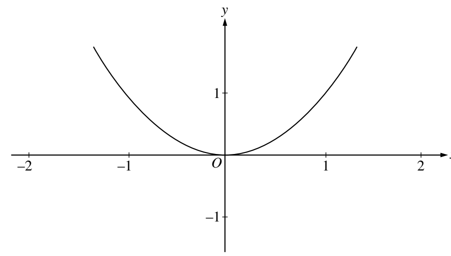
Consider the differential equation given by $\frac{dy}{dx} = x(y-1)^2$.

(a) On the axes provided, sketch a slope field for the given differential equation at the eleven points indicated.

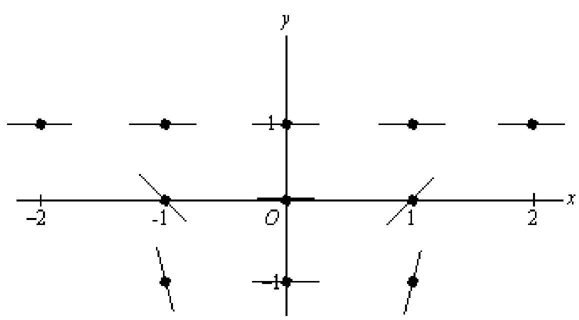
(b) Use the slope field for the given differential equation to explain why a solution could not have the graph shown below.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -1$.

(d) Find the range of the solution found in part (c).



(a)



2 { 1 : zero slope at 7 points with $y = 1$ and $x = 0$
1 : negative slope at $(-1, 0)$ and $(-1, -1)$ positive slope at $(1, 0)$ and $(1, -1)$ steeper slope at $y = -1$ than $y = 0$ }

(b) The graph does not have slope 0 where $y = 1$.

– or –

The slope field shown suggests that solutions are asymptotic to $y = 1$ from below, but the graph does not exhibit this behavior.

1 : reason

(c) $\frac{1}{(y-1)^2}\,dy = x\,dx$

$-\frac{1}{y-1} = \frac{1}{2}x^2 + C$

$\frac{1}{2} = 0 + C; \quad C = \frac{1}{2}$

$-\frac{1}{y-1} = \frac{1}{2}\left(x^2 + 1\right); \quad y = 1 - \frac{2}{x^2+1}$

5 { 1 : separates variables
1 : antiderivatives
1 : constant of integration
1 : uses initial condition $f(0) = -1$
1 : solves for y
0/1 if y is linear }

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

(d) range is $-1 \le y < 1$

1 : answer

0/1 if –1 not in range