AP Calculus BC
1999 Sample Student Responses

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4. The function $f$ has derivatives of all orders for all real numbers $x$. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

(a) Write the third-degree Taylor polynomial for $f$ about $x = 2$ and use it to approximate $f(1.5)$.

$$T_3(x) = -3 + 5(x-2) + \frac{3}{2} (x-2)^2 - \frac{4}{3} (x-2)^3$$

$$f(1.5) \approx -3 + 5(1.5-2) + \frac{3}{2} (1.5-2)^2 - \frac{4}{3} (1.5-2)^3$$

$$= -4.958$$

(b) The fourth derivative of $f$ satisfies the inequality $|f^{(4)}(x)| \leq 3$ for all $x$ in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \neq -5$.

$$R_3(1.5) = \frac{f^{(4)}(z)}{4!} (1.5-2)^4 \quad \text{for some } 1.5 \leq z \leq 2.$$  

Thus,  

$$R_3(1.5) \leq \frac{3}{4!} (1.5-2)^4 = 0.0078125$$

Thus,  

$$-4.958 - 0.0078125 \leq f(1.5) \leq -4.958 + 0.0078125$$

$$-4.966 \leq f(1.5) \leq -4.950$$

Thus, $f(1.5) \neq -5$.

Continue problem 4 on page 11.
(c) Write the fourth-degree Taylor polynomial, \(P(x)\), for \(g(x) = f(x^2 + 2)\) about \(x = 0\). Use \(P\) to explain why \(g\) must have a relative minimum at \(x = 0\).

\[ P(x) = -3 + 5x^2 + \frac{3}{2}x^4 \]

Since the coefficient of \(x\) is 0, \(\frac{g'(0)}{1!} = 0\), so \(g'(0) = 0\).
Since the coefficient of \(x^2\) is 5, \(\frac{g''(0)}{2!} = 5\), so \(g''(0) = 10\).

Thus, since \(g''(0)\) is positive and \(g'(0) = 0\), \(P(x)\) must have a relative minimum at \(x = 0\) by the second derivative test.
4. The function \( f \) has derivatives of all orders for all real numbers \( x \). Assume \( f(2) = -3 \), \( f'(2) = 5 \), \( f''(2) = 3 \), and \( f'''(2) = -8 \).

(a) Write the third-degree Taylor polynomial for \( f \) about \( x = 2 \) and use it to approximate \( f(1.5) \).

\[
f(x) = f(2) + f'(2)(x - 2) + \frac{f''(2)}{2!}(x - 2)^2 + \frac{f'''(2)}{3!}(x - 2)^3
\]

\[
= -3 + 5(x - 2) + \frac{3}{2}(x - 2)^2 - \frac{3}{3}(x - 2)^3
\]

\[
f(1.5) = -4.958
\]

(b) The fourth derivative of \( f \) satisfies the inequality \( |f^{(4)}(x)| \leq 3 \) for all \( x \) in the closed interval \([1.5, 2]\). Use the Lagrange error bound on the approximation to \( f(1.5) \) found in part (a) to explain why \( f(1.5) \neq -5 \).

\[
|\text{error}| \leq a_{n+1}
\]

\[
|\text{error}| \leq a_4
\]

\[
|\text{error}| \leq \frac{3}{4!}(x - 2)^4
\]

\[
|\text{error}| \leq \frac{3}{24}(1.5 - 2)^4
\]

\[
|\text{error}| \leq .0078
\]

The truncation error is no greater than .0078

\[
-4.958 - .0078 < f(1.5) \leq -4.958 + .0078
\]

\[
-4.9658 < f(1.5) \leq -4.9502
\]

\[
\therefore f(1.5) \neq -5
\]

Continue problem 4 on page 11.
(c) Write the fourth-degree Taylor polynomial, \( P(x) \), for \( g(x) = f(x^2 + 2) \) about \( x = 0 \). Use \( P \) to explain why \( g \) must have a relative minimum at \( x = 0 \).

\[
P(x) = -3 + 5(x^2 + 2 - 2) + \frac{3}{2!} (x^2 + 2 - 2)^2
\]
\[
= -3 + 5x^2 + \frac{3}{2} x^4
\]

\[
P'(x) = 10x + 6x^3
\]
\[
0 = 2x (5 + 3x^2)
\]
\[
x = 0
\]

\[
\frac{-1}{0} + \quad P'(x) = g'(x)
\]

\[
\therefore \quad g(x) \text{ has a relative minimum at } x = 0
\]
4. The function \( f \) has derivatives of all orders for all real numbers \( x \). Assume \( f(2) = -3, f'(2) = 5, f''(2) = 3, \) and \( f'''(2) = -8 \).

(a) Write the third-degree Taylor polynomial for \( f \) about \( x = 2 \) and use it to approximate \( f(1.5) \).

\[
\begin{align*}
\frac{f(x)}{f(x)} &= -3 + 5(x-2) + \frac{3}{2!}(x-2)^2 - \frac{8}{3!}(x-2)^3 \\
\frac{f(1.5)}{f(1.5)} &= -3 + 5(1.5-2) + \frac{3}{2!}(1.5-2)^2 - \frac{8}{3!}(1.5-2)^3 \\
\frac{f(1.5)}{f(1.5)} &= -4.958
\end{align*}
\]

(b) The fourth derivative of \( f \) satisfies the inequality \( |f^{(4)}(x)| \leq 3 \) for all \( x \) in the closed interval \([1.5, 2]\). Use the Lagrange error bound on the approximation to \( f(1.5) \) found in part (a) to explain why \( f(1.5) \neq -5 \).

\[
\begin{align*}
\frac{f^3(x)}{f^3(x)} &> \frac{f^4(x)}{4!} \\
\frac{-3 \cdot 1.5^3}{3!} &> \frac{f^4(1.5)}{4!} \\
1.667 &> 0.002604(f^4)
\end{align*}
\]

Since there is an error of at least \( 0.002604 \), \( f(1.5) \neq -5 \).

Continue problem 4 on page 11.
(c) Write the fourth-degree Taylor polynomial, $P(x)$, for $g(x) = f(x^2 + 2)$ about $x = 0$. Use $P$ to explain why $g$ must have a relative minimum at $x = 0$.

$$g(x) = f(x^2 + 2)$$

$$P(x) = -3 + 5(x^2 + 2 - 2) + \frac{3(x^2 + 2 - 2)^2}{2!}$$

$$\boxed{P(x) = -3 + 5x^2 + \frac{3x^4}{2!}}$$

$$P'(x) = 10x + \frac{12x^3}{2}$$

$$P'(x) = 10x + 6x^3$$

$$0 = 2x(5 + 3x^2)$$

$$x = 0 \text{ other 2 roots are complex}$$

$$\begin{array}{c}
\phantom{-}0 \\
\end{array}$$

Because $P'(x)$ has a relative min at $x = 0$, so does $g(x)$.