AP® CALCULUS AB
2012 SCORING GUIDELINES

Question 5

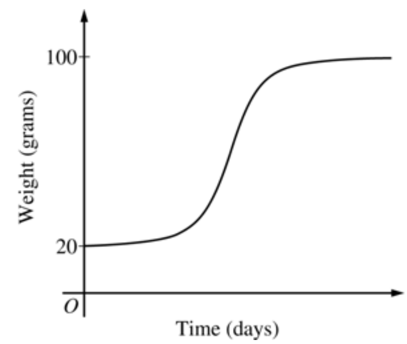
The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.



(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

(a) $\left.\frac{dB}{dt}\right|_{B=40} = \frac{1}{5}(60) = 12$

$\left.\frac{dB}{dt}\right|_{B=70} = \frac{1}{5}(30) = 6$

Because $\left.\frac{dB}{dt}\right|_{B=40} > \left.\frac{dB}{dt}\right|_{B=70}$, the bird is gaining weight faster when it weighs 40 grams.

$$2 : \begin{cases} 1 : \text{uses } \frac{dB}{dt} \\ 1 : \text{answer with reason} \end{cases}$$

(b) $\frac{d^2B}{dt^2} = -\frac{1}{5}\frac{dB}{dt} = -\frac{1}{5}\cdot\frac{1}{5}(100 - B) = -\frac{1}{25}(100 - B)$

Therefore, the graph of B is concave down for $20 \le B < 100$. A portion of the given graph is concave up.

$$2 : \begin{cases} 1 : \frac{d^2B}{dt^2} \text{ in terms of } B \\ 1 : \text{explanation} \end{cases}$$

(c) $\frac{dB}{dt} = \frac{1}{5}(100 - B)$

$\int \frac{1}{100 - B}\, dB = \int \frac{1}{5}\, dt$

$-\ln|100 - B| = \frac{1}{5}t + C$

Because $20 \le B < 100$, $|100 - B| = 100 - B$.

$-\ln(100 - 20) = \frac{1}{5}(0) + C \Rightarrow -\ln(80) = C$

$100 - B = 80e^{-t/5}$

$B(t) = 100 - 80e^{-t/5}, \quad t \ge 0$

$$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } B \end{cases}$$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

NO CALCULATOR ALLOWED

5A₁

5. The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

when it is 40 grams: $\frac{dB}{dt} = \frac{1}{5}(100-40) = 12$ g/day

when it is 70 grams: $\frac{dB}{dt} = \frac{1}{5}(100-70) = 6$ g/day

so the bird is gaining weight faster when it weighs 40 grams.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.

$\frac{dB}{dt} = 20 - \frac{1}{5}B$

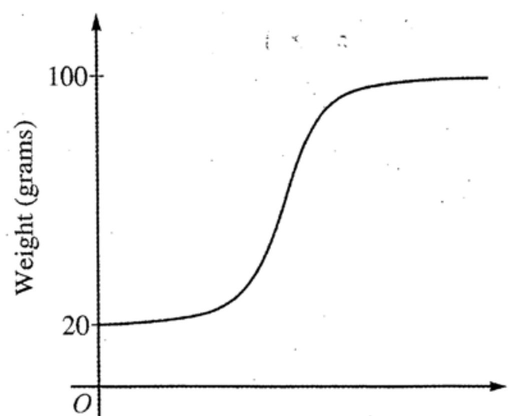
$\frac{d^2B}{dt^2} = -\frac{1}{5} \cdot \frac{dB}{dt}$

$= -\frac{1}{5}(20 - \frac{1}{5}B)$

$= \frac{1}{25}B - 4$

$\frac{1}{25}B - 4 > 0$

$B > 100$



so, the graph cannot be concave up when its weight is below 100 g.

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NO CALCULATOR ALLOWED

5A2

(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

$$\frac{dB}{dt} = \frac{1}{5}(100 - B)$$

$$\frac{1}{\frac{1}{5}(100-B)}\, dB = dt$$

$$\frac{5}{100-B}\, dB = dt$$

$$\int \frac{5}{100-B}\, dB = \int 1\, dt$$

$$-5\ln[100-B] = t + C$$

$$\ln(100-B) = -\frac{1}{5}(t+C)$$

$$100 - B = e^{-\frac{1}{5}(t+C)}$$

$$B = 100 - e^{-\frac{1}{5}(t+C)}$$

$$20 = 100 - e^{-\frac{1}{5}C}$$

$$e^{-\frac{1}{5}C} = 80$$

$$-\frac{1}{5}C = \ln 80$$

$$C = -5\ln 80$$

$$\therefore B = 100 - e^{-\frac{1}{5}(t - 5\ln 80)}$$

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NO CALCULATOR ALLOWED

5B₁

5. The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

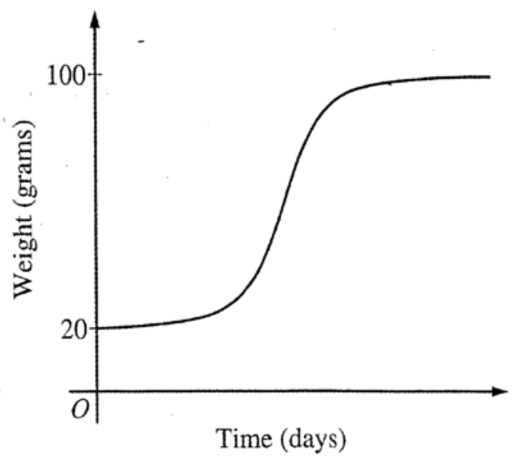
(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

Weight = 40 $\Rightarrow \frac{dB}{dt} = \frac{1}{5}(100-40) = \frac{60}{5}$ gram/day

Weight = 70 $\Rightarrow \frac{dB}{dt} = \frac{1}{5}(100-70) = \frac{30}{5}$ gram/dy

$\frac{60}{5} > \frac{30}{5} \Rightarrow$ at weight = 40gram, the rate of change of Bird weight is faster.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.



$\frac{d^2B}{dt^2} = \frac{1}{5}\left(0 - \frac{dB}{dt}\right) = -\frac{1}{5}\frac{dB}{dt}$, $\frac{d^2B}{dt^2}$ is negative

$\Rightarrow$ the graph of B has to be concave down all the times

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NO CALCULATOR ALLOWED

5B2

(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

$$\frac{dB}{dt} = \frac{1}{5}(100 - B)$$

$$\int \frac{dB}{100-B} = \int \frac{1}{5}\,dt$$

$$\ln(100 - B) = \frac{t}{5} + C$$

$$100 - B = Ce^{t/5}$$

$$B(0) = 20 \Rightarrow 100 - 20 = Ce^{0} \Rightarrow C = 80$$

$\Rightarrow$ partialar solution: $\boxed{100 - B = 80e^{t/5}}$

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5C1

5. The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

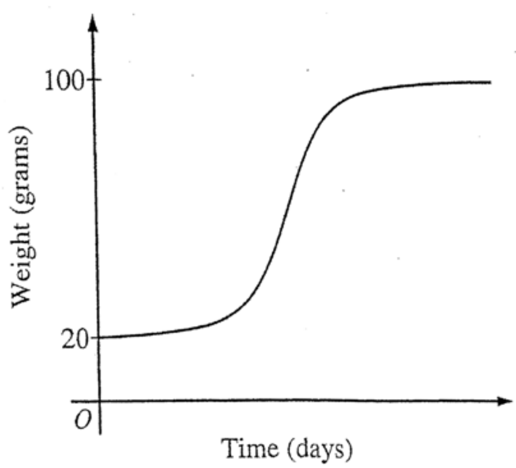
(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

$\frac{1}{5}(100-40)$ $\frac{60}{5}$ 12

$\frac{1}{5}(100-70)$ $\frac{30}{5}$ 6

gains weight faster when it weighs 40 grams because its growing at twice the rate it is when its 70 grams.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.



$20 - \frac{B}{5}$

$\frac{d^2B}{dt^2} \frac{4}{5} = -\frac{1}{5}$

Because $\frac{d^2B}{dt^2}$ is $-\frac{1}{5}$, this can't resemble the following because the concavity isn't negative.

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5C2

(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

$$\int \frac{1}{5}\,dt = \int \frac{dB}{100-B}$$

$$C + \frac{1}{5}t = -\frac{1}{2}(100-B)^{-2}$$

$$\frac{1}{5}t + C = \frac{-1}{2(100-B)^2}$$

$$-2\left(\frac{1}{5}t+C\right) = \frac{1}{100-B^2}$$

$$100-B^2 = \frac{1}{-2\left(\frac{1}{5}t+C\right)}$$

$$100 + \frac{1}{+2\left(\frac{1}{5}t+C\right)} = B^2$$

$$\sqrt{100+\frac{1}{2\left(\frac{1}{5}t+C\right)}} = B$$

$$\sqrt{100+\frac{1}{2(C)}} = 20$$

$$\begin{array}{r} 20 \\ 20 \\ \hline 400 \end{array}$$

$$\frac{1}{2C} = 300$$

$$\frac{\frac{1}{300} = 2C}{2}$$

$$C = \frac{2}{300}$$

$$B = \sqrt{100+\frac{1}{2\left(\frac{1}{5}t+\frac{2}{300}\right)}}$$

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Question 5

Overview

The context of this problem is weight gain of a baby bird. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. A function B modeling the weight of the bird satisfies $\frac{dB}{dt} = \frac{1}{5}(100 - B)$, where t is measured in days since the bird was first weighed. Part (a) asked whether the bird is gaining weight faster when it weighs 40 grams or when it weighs 70 grams. Students had to evaluate and compare $\frac{dB}{dt}$ for these two values of B. Part (b) asked for $\frac{d^2B}{dt^2}$ in terms of B. Students should have used a sign analysis of the second derivative to explain why the graph of B cannot resemble the given graph. Part (c) asked students to use separation of variables to solve the initial value problem $\frac{dB}{dt} = \frac{1}{5}(100 - B)$ with $B(0) = 20$ to find $B(t)$.

Sample: 5A
Score: 9

The student earned all 9 points. Note that in part (c) the student does not need absolute value on the fifth line because $B(0) = 20$.

Sample: 5B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), and 3 points in part (c). In part (a) the student's work is correct. In part (b) the first point was not earned because the student does not present $\frac{d^2B}{dt^2}$ in terms of B. The student's correct appeal to the chain rule and correct explanation earned the second point. In part (c) the student earned the first point with a correct separation on the second line. The second point was not earned because the student's antiderivative on the left-hand side on the third line is incorrect. (The antiderivative should be $-\ln(100 - B)$, with no absolute value needed.) A student who did not earn the second point is not eligible for the fifth point. The student earned the third point on the third line and the fourth point on the fifth line for correctly substituting 0 for t and 20 for B.

Sample: 5C
Score: 3

The student earned 3 points: 2 points in part (a), no points in part (b), and 1 point in part (c). In part (a) the student's work is correct. In part (b) the student makes a chain rule error and did not earn the first point. The student is not eligible for the second point in part (b). In part (c) the student presents a correct separation on the first line and earned the first point. The student's incorrect B-antiderivative makes the student ineligible for any additional points in part (c).