Question 4

The function f is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \le x \le 5$.

(a) Find $f'(x)$.

(b) Write an equation for the line tangent to the graph of f at $x = -3$.

(c) Let g be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x + 7 & \text{for } -3 < x \le 5. \end{cases}$

Is g continuous at $x = -3$? Use the definition of continuity to explain your answer.

(d) Find the value of $\int_0^5 x\sqrt{25 - x^2}\, dx$.

(a) $f'(x) = \frac{1}{2}\left(25 - x^2\right)^{-1/2}(-2x) = \frac{-x}{\sqrt{25 - x^2}}, \quad -5 < x < 5$

2 : $f'(x)$

(b) $f'(-3) = \frac{3}{\sqrt{25 - 9}} = \frac{3}{4}$

$f(-3) = \sqrt{25 - 9} = 4$

An equation for the tangent line is $y = 4 + \frac{3}{4}(x + 3)$.

$2 : \begin{cases} 1 : f'(-3) \\ 1 : \text{answer} \end{cases}$

(c) $\lim\limits_{x \to -3^-} g(x) = \lim\limits_{x \to -3^-} f(x) = \lim\limits_{x \to -3^-} \sqrt{25 - x^2} = 4$

$\lim\limits_{x \to -3^+} g(x) = \lim\limits_{x \to -3^+} (x + 7) = 4$

Therefore, $\lim\limits_{x \to -3} g(x) = 4$.

$g(-3) = f(-3) = 4$

So, $\lim\limits_{x \to -3} g(x) = g(-3)$.

Therefore, g is continuous at $x = -3$.

$2 : \begin{cases} 1 : \text{considers one-sided limits} \\ 1 : \text{answer with explanation} \end{cases}$

(d) Let $u = 25 - x^2 \Rightarrow du = -2x\,dx$

$$\int_0^5 x\sqrt{25 - x^2}\, dx = -\frac{1}{2}\int_{25}^{0} \sqrt{u}\, du$$
$$= \left[-\frac{1}{2} \cdot \frac{2}{3} u^{3/2}\right]_{u=25}^{u=0}$$
$$= -\frac{1}{3}(0 - 125) = \frac{125}{3}$$

$3 : \begin{cases} 2 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

 4A₁

NO CALCULATOR ALLOWED

4. The function f is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \le x \le 5$.

(a) Find $f'(x)$. $= (25-x^2)^{1/2}$

$$f'(x) = \frac{1 \cdot -2x}{2\sqrt{25-x^2}} = \boxed{\frac{-x}{\sqrt{25-x^2}}}$$

(b) Write an equation for the line tangent to the graph of f at $x = -3$.

$$f'(-3) = \frac{3}{\sqrt{25-9}} = \frac{3}{\sqrt{16}} = \frac{3}{4} \qquad f(-3) = \sqrt{25-9} = \sqrt{16} = 4$$

$$\boxed{y = \frac{3}{4}(x+3) + 4}$$

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4A2

NO CALCULATOR ALLOWED

(c) Let g be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x+7 & \text{for } -3 < x \le 5. \end{cases}$

Is g continuous at $x = -3$? Use the definition of continuity to explain your answer.

$\lim_{x \to -3^-} g(x) = f(-3) = 4$

$\lim_{x \to -3^+} g(x) = -3+7 = 4$

therfore $\lim_{x \to -3} g(x) = 4$

$g(-3) = f(-3) = 4$

$\lim_{x \to -3} g(x)$ exists and is equal to $g(-3)$

$\boxed{g \text{ is continuous at } x = -3}$

(d) Find the value of $\int_0^5 x\sqrt{25 - x^2}\, dx$.

$u = 25 - x^2$

$-\frac{1}{2} du = \cancel{+2}x\, dx$

$-\frac{1}{2}\int_{25}^{0} u^{1/2} du = \frac{1}{2}\int_{0}^{25} u^{1/2} du = \frac{1}{2}\cdot\frac{2u^{3/2}}{3}\Big|_0^{25}$

$= \frac{1}{3}\cdot 25^{3/2} = \boxed{\frac{125}{3}}$

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4B,

4 4 4 4 4 4 4 4 4 4

NO CALCULATOR ALLOWED

4. The function f is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \le x \le 5$.

(a) Find $f'(x)$.

$f'(x) = \frac{1}{2}(25 - x^2)^{-\frac{1}{2}}(-2x)$

$f'(x) = \frac{-x}{\sqrt{25-x^2}}$

(b) Write an equation for the line tangent to the graph of f at $x = -3$.

$f(-3) = \sqrt{16} = 4$

$f'(-3) = \frac{3}{4}$

$y - 4 = \frac{3}{4}(x + 3)$

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Continue problem 4 on page 17.

(c) Let g be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x+7 & \text{for } -3 < x \le 5. \end{cases}$

Is g continuous at $x = -3$? Use the definition of continuity to explain your answer.

$f(-3) = 4$ yes, $g(x)$ is continuous at

$-3+7=4$ $x=-3$

In order for a fct to be continuous, the $\lim\limits_{x\to h^+}$ must equal $\lim\limits_{x\to h^-}$ and the actual value of the fct must be equivalent to the value that the fct approaches from both the left and the right at $y=h$

(d) Find the value of $\int_0^5 x\sqrt{25-x^2}\,dx$.

$u = 25 - x^2$

$dx = \dfrac{du}{2x}$

$\frac{1}{2}\int_{25}^{0} \sqrt{u}\,du = -\frac{1}{2}\int_{0}^{25} \sqrt{u}\,du = -\frac{1}{2}\left(\frac{2}{3}\right)(u)^{\frac{3}{2}}\Big|_0^{25}$

$= -\frac{1}{3}(25)^{\frac{3}{2}} + \frac{1}{3}(0)^{\frac{3}{2}} = -\frac{125}{3}$

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NO CALCULATOR ALLOWED

4. The function f is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \le x \le 5$.

(a) Find $f'(x)$.

$f'(x) = (25 - x^2)^{1/2}$

$= 1/2(25 - x^2)^{-1/2}(-2x)$

$-x(25 - x^2)^{-1/2}$

$\frac{-x}{\sqrt{25 - x^2}}$

(b) Write an equation for the line tangent to the graph of f at $x = -3$.

$\frac{-3}{\sqrt{25 - (-3)^2}} = \frac{-3}{\sqrt{25 - 9}} = \frac{-3}{4}$

$y - 4 = \frac{-3}{4}(x + 3)$

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Continue problem 4 on page 17.

NO CALCULATOR ALLOWED

(c) Let g be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x+7 & \text{for } -3 < x \le 5. \end{cases}$

Is g continuous at $x = -3$? Use the definition of continuity to explain your answer.

$f(x) = x + 7$

$f(x) = 3 + 7$

$f(x) = 10$

No g is not continuous at x = 3.

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(d) Find the value of $\int_0^5 x\sqrt{25 - x^2}\, dx$.

$u = 25 - x^2$

$du = -2x$

$\cdot \frac{1}{2} \int_0^5 x\sqrt{u}\, du$

$\frac{1}{2} x^2 \left(\frac{2}{3} u^{3/2}\right)\Big|_0^5$

$\left(-\frac{1}{2}\right) \frac{1}{2} x^2 \left(\frac{2}{3}(25 - x^2)^{3/2}\right)\Big|_0^5 =$

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AP® CALCULUS AB
2012 SCORING COMMENTARY

Question 4

Overview

This problem presented a function f defined by $f(x) = \sqrt{25 - x^2}$ on the interval $-5 \le x \le 5$. In part (a) students were asked to find the derivative $f'(x)$. This involved correctly applying the chain rule to determine the symbolic derivative of f. Part (b) asked for an equation of the line tangent to the graph of f at the point where $x = -3$. Students needed to find the derivative at this point to determine the slope of the tangent line, the y-coordinate of the graph of f at this point, and then combine this information to provide an equation for the line. Part (c) presented a piecewise-defined function g that is equal to f on the interval $-5 \le x \le -3$ and to $x + 7$ on the interval $-3 < x \le 5$. Students were asked to use the definition of continuity to determine whether g is continuous at $x = -3$. Students should have evaluated the left-hand and right-hand limits as x approaches -3, and observed that these are the same and equal to the function value at that point. Part (d) asked students to evaluate the definite integral $\int_0^5 x\sqrt{25 - x^2}\,dx$, which can be done using the substitution $u = 25 - x^2$.

Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: 2 points in part (a), 2 points in part (b), no points in part (c), and 2 points in part (d). In parts (a) and (b) the student's work is correct. In part (c) the student's work is not sufficient for any points. In part (d) the student earned 1 of the 2 antiderivative points owing to a sign error. The student evaluates the definite integral in a manner consistent with the sign error and earned the answer point.

Sample: 4C
Score: 3

The student earned 3 points: 2 points in part (a), 1 point in part (b), no points in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student evaluates f' incorrectly but uses this value as the slope along with the point $(-3, 4)$ to write an equation of the tangent line. In parts (c) and (d) the student's work is incorrect.