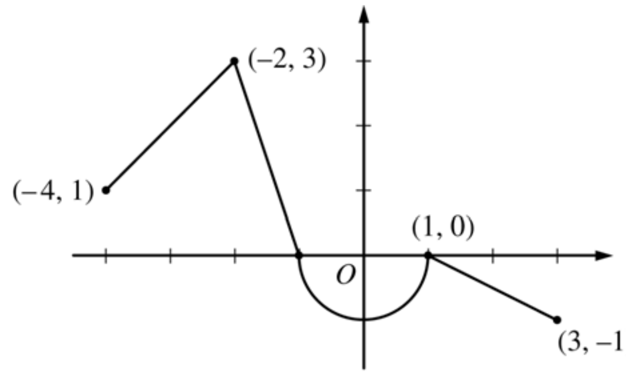
## Question 3

Let $f$ be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let $g$ be the function given by $g(x) = \int_1^x f(t)\,dt$.




Graph of $f$

(a) Find the values of $g(2)$ and $g(-2)$.

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

(c) Find the $x$-coordinate of each point at which the graph of $g$ has a horizontal tangent line. For each of these points, determine whether $g$ has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

(d) For $-4 < x < 3$, find all values of $x$ for which the graph of $g$ has a point of inflection. Explain your reasoning.

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(a) $g(2) = \int_1^2 f(t)\,dt = -\frac{1}{2}(1)\left(\frac{1}{2}\right) = -\frac{1}{4}$

$g(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^{1} f(t)\,dt$

$= -\left(\frac{3}{2} - \frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{3}{2}$

$2 : \begin{cases} 1 : g(2) \\ 1 : g(-2) \end{cases}$

(b) $g'(x) = f(x) \;\Rightarrow\; g'(-3) = f(-3) = 2$

$g''(x) = f'(x) \;\Rightarrow\; g''(-3) = f'(-3) = 1$

$2 : \begin{cases} 1 : g'(-3) \\ 1 : g''(-3) \end{cases}$

(c) The graph of $g$ has a horizontal tangent line where $g'(x) = f(x) = 0$. This occurs at $x = -1$ and $x = 1$.

$g'(x)$ changes sign from positive to negative at $x = -1$. Therefore, $g$ has a relative maximum at $x = -1$.

$g'(x)$ does not change sign at $x = 1$. Therefore, $g$ has neither a relative maximum nor a relative minimum at $x = 1$.

$3 : \begin{cases} 1 : \text{considers } g'(x) = 0 \\ 1 : x = -1 \text{ and } x = 1 \\ 1 : \text{answers with justifications} \end{cases}$

(d) The graph of $g$ has a point of inflection at each of $x = -2$, $x = 0$, and $x = 1$ because $g''(x) = f'(x)$ changes sign at each of these values.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{explanation} \end{cases}$

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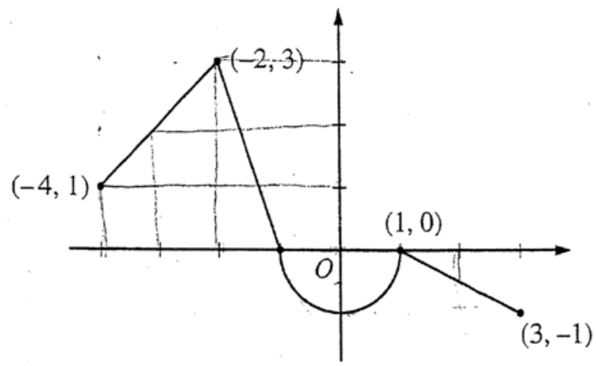


Graph of $f$

3. Let $f$ be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let $g$ be the function given by $g(x) = \int_1^x f(t)\,dt$.

(a) Find the values of $g(2)$ and $g(-2)$.

$g(2) = \int_1^2 f(t)\,dt$

$g(2) = -\frac{1}{2}(1)(\frac{1}{2})$

$\boxed{g(2) = -\frac{1}{4}}$

$g(-2) = \int_1^{-2} f(t)\,dt$

$g(-2) = \frac{1}{2}\pi(1)^2 - (\frac{1}{2}(1)(3))$

$\boxed{g(-2) = \frac{\pi - 3}{2}}$

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

$g'(x) = f(x)$

$\boxed{g'(-3) = 2}$

$g''(x) = f'(x)$

$\boxed{g''(-3) = 1}$

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Continue problem 3 on page 15.

**NO CALCULATOR ALLOWED**

(c) Find the $x$-coordinate of each point at which the graph of $g$ has a horizontal tangent line. For each of these points, determine whether $g$ has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

$g'(x) = f(x) = 0$

$x = -1, 1$

At $x=-1$ $g$ has a relative maximum because $g'(x)=f(x)$ changes from positive to negative

At $x=1$ $g$ has neither because $g'(x)=f(x)$ does not change sign

(d) For $-4 < x < 3$, find all values of $x$ for which the graph of $g$ has a point of inflection. Explain your reasoning.

$g$ has inflection points where $g''(x) = f'(x)$ changes sign. This occurs at $x = -2, 0, 1$

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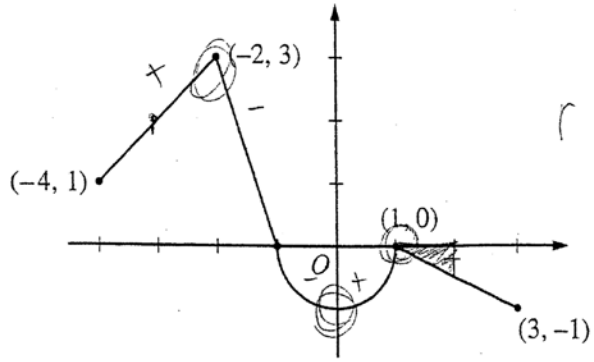


Graph of $f$

3. Let $f$ be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let $g$ be the function given by $g(x) = \int_1^x f(t)\,dt$.

(a) Find the values of $g(2)$ and $g(-2)$.

$A = \frac{bh}{2}$

$A = \frac{1(-\frac{1}{2})}{2}$

$A = -\frac{1}{4}$

$g(x) = \int_1^x f(t)\,dt$

$g(2) = \int_1^2 f(t)\,dt$

$g(2) = \frac{1(-.5)}{2}$

$\boxed{g(2) = -\frac{1}{4}}$

$g(-2) = -\int_{-2}^{1} f(t)\,dt$

$A = \frac{bh}{2}$ $\quad A = \pi r^2$

$-\left(\frac{1(3)}{2} - \frac{1}{2}\pi\right)$

$-\frac{3}{2} + \frac{1}{2}\pi$

$\boxed{g(-2) = \frac{1}{2}\pi - \frac{3}{2}}$

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

$m = \frac{\Delta y}{\Delta x}$

$-\frac{2}{2} = -1$

$(-3, 3)$

$g'(x) = f(x)$

$g'(-3) = f(-3)$

$\boxed{g'(-3) = 3}$

$g''(x) = f'(x)$

$g''(-3) = f'(-3)$

$\boxed{g''(-3) = -1}$

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Continue problem 3 on page 15.

**NO CALCULATOR ALLOWED**

(c) Find the $x$-coordinate of each point at which the graph of $g$ has a horizontal tangent line. For each of these points, determine whether $g$ has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

$g'(x) = 0 \leftarrow$ horizontal tangent line

$g'(x) = f(x) = 0$

at $x = -1, x = 1$

| | | | | | |
|---|---|---|---|---|---|
| $g'$ | + | | − | | − |
| $g$ | inc | −1 | dec | 1 | dec |

at $x=-1$, there is a rel. max.
$f'$ changes from + to −

at $x=1$, there is neither a maximum nor a minimum because the slope remains negative

(d) For $-4 < x < 3$, find all values of $x$ for which the graph of $g$ has a point of inflection. Explain your reasoning.

$g''(x) = f'(x)$

g will have a POI where the slope of f changes signs

at $\boxed{x=-2,\ x=0,\ x=1}$

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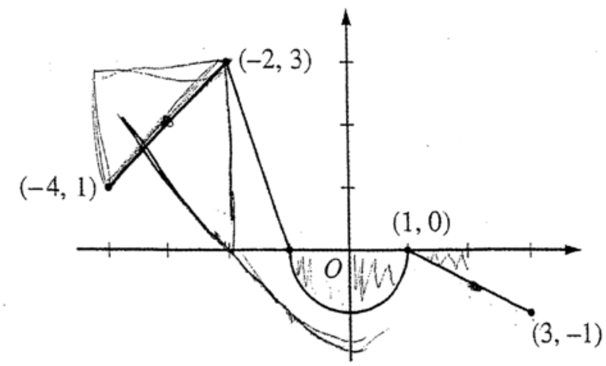
3C1




Graph of $f$

3. Let $f$ be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let $g$ be the function given by $g(x) = \int_1^x f(t)\,dt$.

(a) Find the values of $g(2)$ and $g(-2)$.

$g'(x) = f(x)$

$g(2) = \pi(1)^2 \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \frac{1}{4}\pi + \frac{1}{4}$

$g(-2) = -\frac{1}{4}(1)^2\pi + \frac{1}{2} \cdot 3 \cdot 1 = -\frac{1}{4}\pi + \frac{3}{2}$

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

$g'(x) = f(x)$

$g'(-3) = f(3) = 2$

$g''(x) = f'(x)$

$g''(-3) = f'(-3) = 1$





Continue problem 3 on page 15.



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(c) Find the $x$-coordinate of each point at which the graph of $g$ has a horizontal tangent line. For each of these points, determine whether $g$ has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

$g'(x)=0 \quad g'(x)=f(x)=0$

$g'(x)=f(x)=0$ at $x=-1$

The point at $x=-1$ is a maximum because the graph of $f(x)|g'(x)$ transitions from positive to negative at this point. This means that on the original graph $(g(x))$ the graph is moving from increasing to decreasing at this point, which is representative of a maximum.

(d) For $-4 < x < 3$, find all values of $x$ for which the graph of $g$ has a point of inflection. Explain your reasoning.

$g''(x)=f'(x)$

There is a point of inflection at $x=-2,3$ because on the $f(x)|g'(x)$ graph they are presented as extrema. Extrema on a first derivative graph represents a point of inflection on the original graph. Therefore the extrema on the graph given, $x=-2,3$, are points of inflection on $g(x)$.





GO ON TO THE NEXT PAGE.

## Question 3

### Overview

This problem described a function $f$ that is defined and continuous on the interval $[-4, 3]$. The graph of $f$ on $[-4, 3]$ is given and consists of three line segments and a semicircle. The function $g$ is defined by $g(x) = \int_1^x f(t)\,dt$. Part (a) asked for the values of $g(2)$ and $g(-2)$. These values are given by $\int_1^2 f(t)\,dt$ and $\int_1^{-2} f(t)\,dt$, respectively, and are computed using geometry and a property of definite integrals. Part (b) asked for the values of $g'(-3)$ and $g''(-3)$, provided they exist. Students should have applied the Fundamental Theorem of Calculus to determine that $g'(-3) = f(-3)$ and $g''(-3) = f'(-3)$. Students should have used the graph provided to determine the value of $f$ and the slope of $f$ at the point where $x = -3$. Part (c) asked for the $x$-coordinate of each point where the graph of $g$ has a horizontal tangent line. Students were then asked to classify each of these points as the location of a relative minimum, relative maximum, or neither, with justification. Students should have recognized that horizontal tangent lines for $g$ occur where the derivative of $g$ takes on the value 0. These values can be read from the graph. Students should have applied a sign analysis to $f$ in order to classify these critical points. Part (d) asked for the $x$-coordinates of points of inflection for the graph of $g$ on the interval $-4 < x < 3$. Students should have reasoned graphically that these occur where $f$ changes from increasing to decreasing, or vice versa.

**Sample: 3A**
**Score: 9**

The student earned all 9 points.

**Sample: 3B**
**Score: 6**

The student earned 6 points: 2 points in part (a), no points in part (b), 2 points in part (c), and 2 points in part (d). In parts (a) and (d) the student's work is correct. In part (b) the student does not supply the correct values. In part (c) the student correctly considers $g'(x) = 0$ and identifies the correct $x$-values. The student does not give a correct justification for $x = -1$, confusing $f'$ with $g'$, and does not specify which function's slope is intended in the justification for $x = 1$.

**Sample: 3C**
**Score: 3**

The student earned 3 points: no points in part (a), 2 points in part (b), 1 point in part (c), and no points in part (d). In part (a) the student does not supply the correct values. In part (b) the student's work is correct. In part (c) the student considers $g'(x) = 0$, so the first point was earned. The student identifies only one of the $x$-values, so the second point was not earned. The student is not eligible for the third point. In part (d) the student does not identify the correct $x$-values.