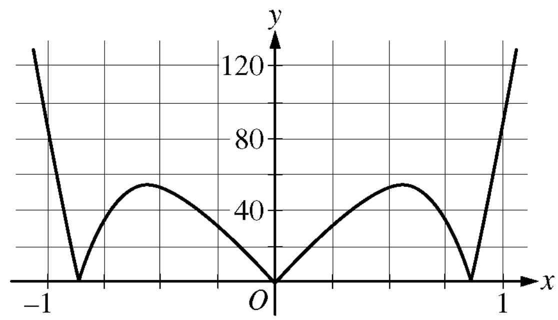
AP® CALCULUS BC
2011 SCORING GUIDELINES

Question 6

Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.



Graph of $y = \left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

(a) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$

$\sin\left(x^2\right) = x^2 - \dfrac{x^6}{3!} + \dfrac{x^{10}}{5!} - \dfrac{x^{14}}{7!} + \cdots$

$3 : \begin{cases} 1 : \text{series for } \sin x \\ 2 : \text{series for } \sin\left(x^2\right) \end{cases}$

(b) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$

$f(x) = 1 + \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \dfrac{121x^6}{6!} + \cdots$

$3 : \begin{cases} 1 : \text{series for } \cos x \\ 2 : \text{series for } f(x) \end{cases}$

(c) $\dfrac{f^{(6)}(0)}{6!}$ is the coefficient of x^6 in the Taylor series for f about $x = 0$. Therefore $f^{(6)}(0) = -121$.

1 : answer

(d) The graph of $y = \left|f^{(5)}(x)\right|$ indicates that $\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right| < 40$.

Therefore

$$\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| \le \frac{\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right|}{5!} \cdot \left(\frac{1}{4}\right)^5 < \frac{40}{120 \cdot 4^5} = \frac{1}{3072} < \frac{1}{3000}.$$

$2 : \begin{cases} 1 : \text{form of the error bound} \\ 1 : \text{analysis} \end{cases}$

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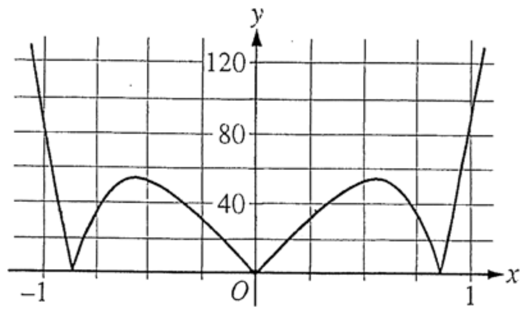
6A₁



Graph of $y = \left| f^{(5)}(x) \right|$

Work for problem 6(a)

$$\sin x \approx \boxed{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}}$$

$$\sin(x^2) \approx \boxed{x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!}}$$

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Work for problem 6(b)

$$\cos x \approx \boxed{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}}$$

$$\sin(x^2) + \cos(x) \approx 1 - \frac{x^2}{2!} + x^2 + \frac{x^4}{4!} - \frac{x^6}{6!} - \frac{x^6}{3!}$$

$$\approx \boxed{1 - x^2\left(\frac{1}{2} - 1\right) + \frac{x^4}{4!} - x^6\left(\frac{1}{6!} + \frac{1}{3!}\right)}$$

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6A2

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Work for problem 6(c)

By the definition of taylor series:

$$\frac{f^{(6)}(0)\,\cancel{(x)^6}}{6!} = -\cancel{x^6}\left(\frac{1}{6!}+\frac{1}{3!}\right)$$

$$\frac{f^{(6)}(0)}{\cancel{6!}} = -\left(\frac{1+4\cdot5\cdot6}{\cancel{6!}}\right)$$

$$f^{(6)}(0) = -(1+120)$$

$$= \boxed{-121}$$

Work for problem 6(d)

$\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right|$ is equivalent to the error of $P_4\left(\frac{1}{4}\right)$

Using the lagrangian error formula we have

$$E < \frac{f^{(n+1)}(x)\,x^{n+1}}{(n+1)!} = \frac{f^{(5)}\left(\frac{1}{4}\right)\cdot\left(\frac{1}{4}\right)^5}{5!}$$

$$= \frac{f^{(5)}\left(\frac{1}{4}\right)}{120\cdot1024}$$

Looking at the graph, we see that $f^{(5)}\left(\frac{1}{4}\right)$ is a number less that 40.

If it were 40, then the maximum error would be

$\frac{40}{120\cdot1024} = \frac{1}{3\cdot1024}$, which is less than $\frac{1}{3000}$.

Since $f^{(5)}\left(\frac{1}{4}\right)$ is less than 40, then the maximum error must be less than $\frac{1}{3000}$ Q.E.D.

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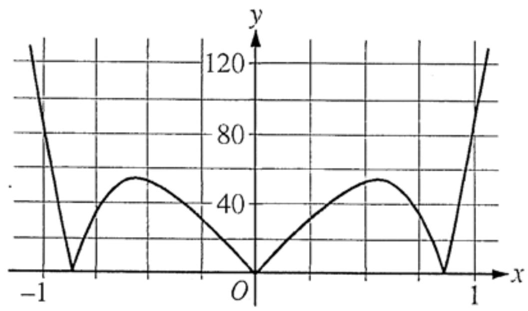
6B1



Graph of $y = \left| f^{(5)}(x) \right|$

Work for problem 6(a)

$\sin 0 = 0$

$\sin x = \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \ldots$

$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots$

$\frac{1\,1\,2\,6}{72}$

Work for problem 6(b)

$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$\frac{x^{2n}}{2n!}$

$-\frac{120x^6}{7\,20} + \frac{x^6}{720}$

$-\frac{119x^6}{720}$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

$\cos 0 = 1$

$f(x) = 1 + \frac{2x^2}{2} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{3!} + \frac{x^6}{6!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!}$

$f(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{119x^6}{6!} + \cancel{\frac{x^{10}}{5!}} - \cancel{\frac{x^{14}}{7!}} + \ldots$

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Continue problem 6 on page 15.

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6B2

Work for problem 6(c)

$$\frac{f^{4}(0)}{4!} = \frac{-119}{4!}$$

$$f^{4}(0) = -\frac{119}{\cancel{4!}} \cdot \cancel{4!}$$

$$\boxed{f^{4}(0) = -119}$$

Work for problem 6(d)

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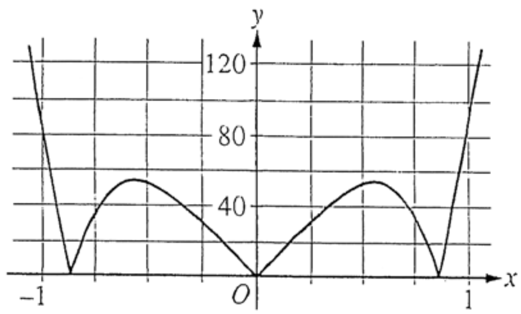
6C1



Graph of $y = \left|f^{(5)}(x)\right|$

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Work for problem 6(a)

$\cancel{1+\frac{x^2}{2}+\frac{x^4}{3!}}$

$\cancel{1+\frac{x}{1}+\frac{x^3}{2!}+\frac{x^5}{3!}}$

$P_4 = x - \frac{x^3}{4!} + \frac{x^5}{6!} + \frac{x^7}{8!}$

$P_4 = x^2 - \frac{x^6}{4!} + \frac{x^{10}}{6!} - \frac{x^{14}}{8!}$

Work for problem 6(b)

$P_4 = 1 - \frac{x^2}{2} + \frac{x^4}{3!} + \frac{x^6}{4!}$

$P_4 = x^2 + 1 - \frac{x^6}{4!} + \frac{x^2}{2} + \frac{x^{10}}{6!} + \frac{x^4}{3!} - \frac{x^{14}}{8!} + \frac{x^6}{4!}$

Continue problem 6 on page 15

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6C2

Work for problem 6(c)

$P_6 = 0 + 1 - 0 + 0 + \cdots = 1$

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Work for problem 6(d)

Using Lagrange error calculation, the error of $P_4 - f(x)$ at $x = 1/4$ is less than $1/3000$ or 0.033%

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Question 6

Overview

The series problem defined $f(x) = \sin\left(x^2\right) + \cos x$ and provided a graph of $y = \left|f^{(5)}(x)\right|$. Parts (a) and (b) concerned series manipulations. Part (a) asked for the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$ and also for the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$. Part (b) asked for the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$ and also for the first four nonzero terms of the Taylor series for $f(x)$ about $x = 0$. Part (c) asked for the value of $f^{(6)}(0)$. Although an energetic student could have started by computing the sixth derivative of *f*, it was expected that students would have recognized that the coefficient of x^6 in the Taylor series for $f(x)$ about $x = 0$ is $\frac{f^{(6)}(0)}{6!}$. Part (d) tested the Lagrange error bound for $P_4(x)$, the fourth-degree Taylor polynomial for *f* about $x = 0$. Students needed to acquire a correct and sufficient bound on $\left|f^{(5)}(x)\right|$ for $0 \le x \le \frac{1}{4}$ from the supplied graph, and use this bound to verify that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 3 points in part (a), 2 points in part (b), 1 point in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student gives the correct series for cosine. There is evidence of adding the correct two series but the addition is incorrect, so only 2 of the possible 3 points were earned. In part (c) the student's value is consistent with the work in part (b), so the point was earned.

Sample: 6C
Score: 3

The student earned 3 points: 2 points in part (a), 1 point in part (b), no point in part (c), and no points in part (d). In part (a) the student gives an incorrect series for sine. The student correctly doubles all of the exponents and so earned the last 2 points. In part (b) the student gives an incorrect series for cosine. There is evidence of adding the appropriate series, but the student does not combine the appropriate terms, and so earned only 1 of 2 possible points. In part (c) the student's answer is both incorrect and inconsistent with the work in part (b). In part (d) the student does not present any form of the error bound.