CollegeBoard AP

AP® Calculus AB
2011 Scoring Guidelines
Form B

The College Board

The College Board is a not-for-profit membership association whose mission is to connect students to college success and opportunity. Founded in 1900, the College Board is composed of more than 5,700 schools, colleges, universities and other educational organizations. Each year, the College Board serves seven million students and their parents, 23,000 high schools, and 3,800 colleges through major programs and services in college readiness, college admission, guidance, assessment, financial aid and enrollment. Among its widely recognized programs are the SAT®, the PSAT/NMSQT®, the Advanced Placement Program® (AP®), SpringBoard® and ACCUPLACER®. The College Board is committed to the principles of excellence and equity, and that commitment is embodied in all of its programs, services, activities and concerns.

Visit the College Board on the Web: www.collegeboard.com.
AP Central is the official online home for the AP Program: apcentral.collegeboard.com.

Question 1

A cylindrical can of radius 10 millimeters is used to measure rainfall in Stormville. The can is initially empty, and rain enters the can during a 60-day period. The height of water in the can is modeled by the function S, where $S(t)$ is measured in millimeters and t is measured in days for $0 \le t \le 60$. The rate at which the height of the water is rising in the can is given by $S'(t) = 2\sin(0.03t) + 1.5$.

(a) According to the model, what is the height of the water in the can at the end of the 60-day period?

(b) According to the model, what is the average rate of change in the height of water in the can over the 60-day period? Show the computations that lead to your answer. Indicate units of measure.

(c) Assuming no evaporation occurs, at what rate is the volume of water in the can changing at time $t = 7$? Indicate units of measure.

(d) During the same 60-day period, rain on Monsoon Mountain accumulates in a can identical to the one in Stormville. The height of the water in the can on Monsoon Mountain is modeled by the function M, where $M(t) = \frac{1}{400}\left(3t^3 - 30t^2 + 330t\right)$. The height $M(t)$ is measured in millimeters, and t is measured in days for $0 \le t \le 60$. Let $D(t) = M'(t) - S'(t)$. Apply the Intermediate Value Theorem to the function D on the interval $0 \le t \le 60$ to justify that there exists a time t, $0 < t < 60$, at which the heights of water in the two cans are changing at the same rate.

(a) $S(60) = \int_0^{60} S'(t)\,dt = 171.813$ mm

$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) $\frac{S(60) - S(0)}{60} = 2.863$ or 2.864 mm/day

1 : answer

(c) $V(t) = 100\pi S(t)$

$V'(7) = 100\pi S'(7) = 602.218$

The volume of water in the can is increasing at a rate of 602.218 mm^3/day.

$2 : \begin{cases} 1 : \text{relationship between } V \text{ and } S \\ 1 : \text{answer} \end{cases}$

(d) $D(0) = -0.675 < 0$ and $D(60) = 69.37730 > 0$

Because D is continuous, the Intermediate Value Theorem implies that there is a time t, $0 < t < 60$, at which $D(t) = 0$. At this time, the heights of water in the two cans are changing at the same rate.

$2 : \begin{cases} 1 : \text{considers } D(0) \text{ and } D(60) \\ 1 : \text{justification} \end{cases}$

1 : units in (b) or (c)

AP® CALCULUS AB
2011 SCORING GUIDELINES (Form B)

Question 2

A 12,000-liter tank of water is filled to capacity. At time $t = 0$, water begins to drain out of the tank at a rate modeled by $r(t)$, measured in liters per hour, where r is given by the piecewise-defined function

$$r(t) = \begin{cases} \dfrac{600t}{t+3} & \text{for } 0 \le t \le 5 \\ 1000e^{-0.2t} & \text{for } t > 5 \end{cases}$$

(a) Is r continuous at $t = 5$? Show the work that leads to your answer.

(b) Find the average rate at which water is draining from the tank between time $t = 0$ and time $t = 8$ hours.

(c) Find $r'(3)$. Using correct units, explain the meaning of that value in the context of this problem.

(d) Write, but do not solve, an equation involving an integral to find the time A when the amount of water in the tank is 9000 liters.

(a) $\lim_{t\to 5^-} r(t) = \lim_{t\to 5^-}\left(\dfrac{600t}{t+3}\right) = 375 = r(5)$

$\lim_{t\to 5^+} r(t) = \lim_{t\to 5^+}\left(1000e^{-0.2t}\right) = 367.879$

Because the left-hand and right-hand limits are not equal, r is not continuous at $t = 5$.

2 : conclusion with analysis

(b) $\dfrac{1}{8}\int_0^8 r(t)\,dt = \dfrac{1}{8}\left(\int_0^5 \dfrac{600t}{t+3}\,dt + \int_5^8 1000e^{-0.2t}\,dt\right)$

$= 258.052$ or 258.053

3 : { 1 : integrand; 1 : limits and constant; 1 : answer }

(c) $r'(3) = 50$

The rate at which water is draining out of the tank at time $t = 3$ hours is increasing at 50 liters/hour^2.

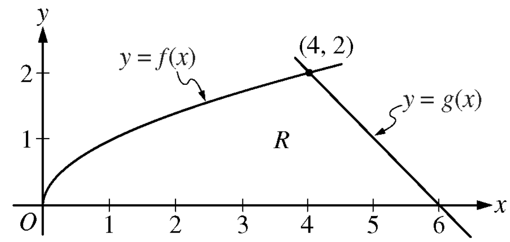
2 : { 1 : $r'(3)$; 1 : meaning of $r'(3)$ }

(d) $12{,}000 - \int_0^A r(t)\,dt = 9000$

2 : { 1 : integral; 1 : equation }

Question 3

The functions f and g are given by $f(x) = \sqrt{x}$ and $g(x) = 6 - x$. Let R be the region bounded by the x-axis and the graphs of f and g, as shown in the figure above.



(a) Find the area of R.

(b) The region R is the base of a solid. For each y, where $0 \le y \le 2$, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose base lies in R and whose height is $2y$. Write, but do not evaluate, an integral expression that gives the volume of the solid.

(c) There is a point P on the graph of f at which the line tangent to the graph of f is perpendicular to the graph of g. Find the coordinates of point P.

(a) $\text{Area} = \int_0^4 \sqrt{x}\,dx + \frac{1}{2}\cdot 2\cdot 2 = \frac{2}{3}x^{3/2}\Big|_{x=0}^{x=4} + 2 = \frac{22}{3}$

$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) $y = \sqrt{x} \;\Rightarrow\; x = y^2$
$y = 6 - x \;\Rightarrow\; x = 6 - y$

$\text{Width} = (6 - y) - y^2$

$\text{Volume} = \int_0^2 2y\left(6 - y - y^2\right)dy$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(c) $g'(x) = -1$

Thus a line perpendicular to the graph of g has slope 1.

$f'(x) = \frac{1}{2\sqrt{x}}$

$\frac{1}{2\sqrt{x}} = 1 \;\Rightarrow\; x = \frac{1}{4}$

The point P has coordinates $\left(\frac{1}{4}, \frac{1}{2}\right)$.

$3 : \begin{cases} 1 : f'(x) \\ 1 : \text{equation} \\ 1 : \text{answer} \end{cases}$

AP® CALCULUS AB
2011 SCORING GUIDELINES (Form B)

Question 4

Consider a differentiable function f having domain all positive real numbers, and for which it is known that $f'(x) = (4 - x)x^{-3}$ for $x > 0$.

(a) Find the x-coordinate of the critical point of f. Determine whether the point is a relative maximum, a relative minimum, or neither for the function f. Justify your answer.

(b) Find all intervals on which the graph of f is concave down. Justify your answer.

(c) Given that $f(1) = 2$, determine the function f.

(a) $f'(x) = 0$ at $x = 4$
$f'(x) > 0$ for $0 < x < 4$
$f'(x) < 0$ for $x > 4$
Therefore f has a relative maximum at $x = 4$.

$3 : \begin{cases} 1 : x = 4 \\ 1 : \text{relative maximum} \\ 1 : \text{justification} \end{cases}$

(b)
$$\begin{aligned} f''(x) &= -x^{-3} + (4 - x)\left(-3x^{-4}\right) \\ &= -x^{-3} - 12x^{-4} + 3x^{-3} \\ &= 2x^{-4}(x - 6) \\ &= \frac{2(x-6)}{x^4} \end{aligned}$$

$f''(x) < 0$ for $0 < x < 6$

The graph of f is concave down on the interval $0 < x < 6$.

$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with justification} \end{cases}$

(c)
$$\begin{aligned} f(x) &= 2 + \int_1^x \left(4t^{-3} - t^{-2}\right) dt \\ &= 2 + \left[-2t^{-2} + t^{-1}\right]_{t=1}^{t=x} \\ &= 3 - 2x^{-2} + x^{-1} \end{aligned}$$

$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

AP® CALCULUS AB
2011 SCORING GUIDELINES (Form B)

Question 5

t (seconds)	0	10	40	60
$B(t)$ (meters)	100	136	9	49
$v(t)$ (meters per second)	2.0	2.3	2.5	4.6

Ben rides a unicycle back and forth along a straight east-west track. The twice-differentiable function B models Ben's position on the track, measured in meters from the western end of the track, at time t, measured in seconds from the start of the ride. The table above gives values for $B(t)$ and Ben's velocity, $v(t)$, measured in meters per second, at selected times t.

(a) Use the data in the table to approximate Ben's acceleration at time $t = 5$ seconds. Indicate units of measure.

(b) Using correct units, interpret the meaning of $\int_0^{60} |v(t)|\, dt$ in the context of this problem. Approximate $\int_0^{60} |v(t)|\, dt$ using a left Riemann sum with the subintervals indicated by the data in the table.

(c) For $40 \le t \le 60$, must there be a time t when Ben's velocity is 2 meters per second? Justify your answer.

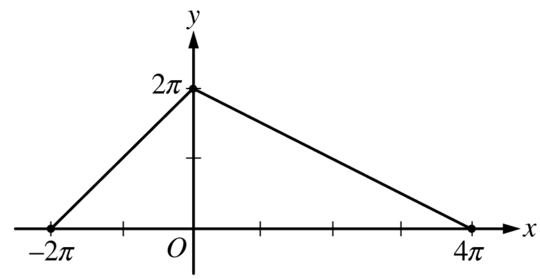
(d) A light is directly above the western end of the track. Ben rides so that at time t, the distance $L(t)$ between Ben and the light satisfies $(L(t))^2 = 12^2 + (B(t))^2$. At what rate is the distance between Ben and the light changing at time $t = 40$?

Solution	Scoring
(a) $a(5) \approx \dfrac{v(10) - v(0)}{10 - 0} = \dfrac{0.3}{10} = 0.03 \text{ meters}/\text{sec}^2$	1 : answer
(b) $\int_0^{60} \lvert v(t)\rvert\, dt$ is the total distance, in meters, that Ben rides over the 60-second interval $t = 0$ to $t = 60$. $\int_0^{60} \lvert v(t)\rvert\, dt \approx 2.0 \cdot 10 + 2.3(40 - 10) + 2.5(60 - 40) = 139$ meters	2 : { 1 : meaning of integral 1 : approximation
(c) Because $\dfrac{B(60) - B(40)}{60 - 40} = \dfrac{49 - 9}{20} = 2$, the Mean Value Theorem implies there is a time t, $40 < t < 60$, such that $v(t) = 2$.	2 : { 1 : difference quotient 1 : conclusion with justification
(d) $2L(t)L'(t) = 2B(t)B'(t)$ $L'(40) = \dfrac{B(40)v(40)}{L(40)} = \dfrac{9 \cdot 2.5}{\sqrt{144 + 81}} = \dfrac{3}{2}$ meters/sec	3 : { 1 : derivatives 1 : uses $B'(t) = v(t)$ 1 : answer
	1 : units in (a) or (b)

AP® CALCULUS AB
2011 SCORING GUIDELINES (Form B)

Question 6

Let g be the piecewise-linear function defined on $[-2\pi, 4\pi]$ whose graph is given above, and let $f(x) = g(x) - \cos\left(\frac{x}{2}\right)$.



Graph of g

(a) Find $\int_{-2\pi}^{4\pi} f(x)\,dx$. Show the computations that lead to your answer.

(b) Find all x-values in the open interval $(-2\pi, 4\pi)$ for which f has a critical point.

(c) Let $h(x) = \int_0^{3x} g(t)\,dt$. Find $h'\left(-\frac{\pi}{3}\right)$.

(a) $$\int_{-2\pi}^{4\pi} f(x)\,dx = \int_{-2\pi}^{4\pi}\left(g(x) - \cos\left(\frac{x}{2}\right)\right)dx$$
$$= 6\pi^2 - \left[2\sin\left(\frac{x}{2}\right)\right]_{x=-2\pi}^{x=4\pi}$$
$$= 6\pi^2$$

$2 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) $$f'(x) = g'(x) + \frac{1}{2}\sin\left(\frac{x}{2}\right) = \begin{cases} 1 + \frac{1}{2}\sin\left(\frac{x}{2}\right) & \text{for } -2\pi < x < 0 \\ -\frac{1}{2} + \frac{1}{2}\sin\left(\frac{x}{2}\right) & \text{for } 0 < x < 4\pi \end{cases}$$

$f'(x)$ does not exist at $x = 0$.
For $-2\pi < x < 0$, $f'(x) \neq 0$.
For $0 < x < 4\pi$, $f'(x) = 0$ when $x = \pi$.

f has critical points at $x = 0$ and $x = \pi$.

$4 : \begin{cases} 1 : \frac{d}{dx}\left(\cos\left(\frac{x}{2}\right)\right) \\ 1 : g'(x) \\ 1 : x = 0 \\ 1 : x = \pi \end{cases}$

(c) $h'(x) = g(3x) \cdot 3$
$h'\left(-\frac{\pi}{3}\right) = 3g(-\pi) = 3\pi$

$3 : \begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$