AP® CALCULUS BC
2011 SCORING GUIDELINES (Form B)

Question 6

Let $f(x) = \ln\left(1 + x^3\right)$.

(a) The Maclaurin series for $\ln(1 + x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^n}{n} + \cdots$. Use the series to write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) The radius of convergence of the Maclaurin series for f is 1. Determine the interval of convergence. Show the work that leads to your answer.

(c) Write the first four nonzero terms of the Maclaurin series for $f'\left(t^2\right)$. If $g(x) = \int_0^x f'\left(t^2\right)\,dt$, use the first two nonzero terms of the Maclaurin series for g to approximate $g(1)$.

(d) The Maclaurin series for g, evaluated at $x = 1$, is a convergent alternating series with individual terms that decrease in absolute value to 0. Show that your approximation in part (c) must differ from $g(1)$ by less than $\frac{1}{5}$.

(a) $x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^{3n}}{n} + \cdots$

$2 : \begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

(b) The interval of convergence is centered at $x = 0$.
At $x = -1$, the series is $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{n} - \cdots$, which diverges because the harmonic series diverges.
At $x = 1$, the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1} \cdot \frac{1}{n} + \cdots$, the alternating harmonic series, which converges.

Therefore the interval of convergence is $-1 < x \le 1$.

2 : answer with analysis

(c) The Maclaurin series for $f'(x)$, $f'\left(t^2\right)$, and $g(x)$ are

$$f'(x) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot 3x^{3n-1} = 3x^2 - 3x^5 + 3x^8 - 3x^{11} + \cdots$$

$$f'\left(t^2\right) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot 3t^{6n-2} = 3t^4 - 3t^{10} + 3t^{16} - 3t^{22} + \cdots$$

$$g(x) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{3x^{6n-1}}{6n-1} = \frac{3x^5}{5} - \frac{3x^{11}}{11} + \frac{3x^{17}}{17} - \frac{3x^{23}}{23} + \cdots$$

Thus $g(1) \approx \frac{3}{5} - \frac{3}{11} = \frac{18}{55}$.

$4 : \begin{cases} 1 : \text{two terms for } f'\left(t^2\right) \\ 1 : \text{other terms for } f'\left(t^2\right) \\ 1 : \text{first two terms for } g(x) \\ 1 : \text{approximation} \end{cases}$

(d) The Maclaurin series for g evaluated at $x = 1$ is alternating, and the terms decrease in absolute value to 0.
Thus $\left|g(1) - \frac{18}{55}\right| < \frac{3 \cdot 1^{17}}{17} = \frac{3}{17} < \frac{1}{5}$.

1 : analysis

NO CALCULATOR ALLOWED

Work for problem 6(a)

$g(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^n}{n} + \cdots$

$f(x) = \ln(1+x^3) = g(x^3) = x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots + (-1)^{n+1} \frac{x^{3n}}{n} + \cdots$

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Work for problem 6(b)

The series is centered around $x=0$. The interval is $-1 < x < 1$. If we check the boundaries,

$x=-1 \to -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} \cdots$, which diverges (harmonic)

$x=1 \to 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$, which converges (alternating harmonic)

So, the interval of convergence is $-1 < x \leq 1$

Continue problem 6 on page 15.

Work for problem 6(c)

$f'(x) = 3x^2 - 3x^5 + 3x^8 - 3x^{11}$

$f'(t^2) = 3t^4 - 3t^{10} + 3t^{16} - 3t^{22}$

$g(x) = \int_0^x f'(t^2)\,dt$

$g(1) = \int_0^1 f'(t^2)\,dt = \int_0^1 (3t^4 - 3t^{10} + 3t^{16})\,dt$

$g(1) \simeq \left(\frac{3}{5}t^5 - \frac{3}{11}t^{11}\right)\Big|_0^1 = \frac{3}{5} - \frac{3}{11} = \frac{33-15}{55} = \frac{18}{55}$

Work for problem 6(d)

$g(x) = \left(\frac{3}{5}t^5 - \frac{3}{11}t^{11} + \frac{3}{17}t^{17}\right)\Big|_0^x$

$g(x) = \frac{3}{5}x^5 - \frac{3}{11}x^{11} + \frac{3}{17}x^{17} \ldots$

$e = \left|g(1) - \frac{18}{55}\right|$ must be smaller then the next term in the series, which is $\frac{3}{17}$, so

$\left|g(1) - \frac{18}{55}\right| < \frac{3}{17} < \frac{1}{5}$

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NO CALCULATOR ALLOWED

Work for problem 6(a)

f(x) puts x^3 instead of x on the Maclaurin series for $\ln(x+1)$

So, $x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^{3n}}{n} + \cdots$

Work for problem 6(b)

If we use ratio test

$$\frac{\frac{(-1)^{n+2} \cdot x^{3n+3}}{n+1}}{\frac{(-1)^{n+1} \cdot x^{3n}}{n}} = \left| \frac{n x^{3n+3}}{(n+1) x^{3n}} \right| = |x^3| < 1$$

$\therefore -1 < x^3 < 1 \quad \therefore -1 < x < 1$

$-1 < x < 1$

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Continue problem 6 on page 15.

NO CALCULATOR ALLOWED

Work for problem 6(c)

$f(t): t^3 - \frac{t^6}{2} + \frac{t^9}{3} - \frac{t^{12}}{4} + \cdots$ $\quad f'(t): 3t^2 - 3t^5 + 3t^8 - 3t^{11} \cdots$

$f'(t^2): 3t^4 - 3t^{10} + 3t^{16} - 3t^{22} + \cdots$

$g(1) = \int_0^1 f'(t^2)\,dt = \left(\frac{3t^5}{5} - \frac{3t^{11}}{11} + \cdots\right)\Big|_0^1$

$= \frac{3}{5} - \frac{3}{11} = \frac{18}{55}$ $\qquad \therefore \frac{18}{55}$

Work for problem 6(d)

I predicted g(1) by using first two nonzero terms.
However the third term is $\frac{3t^{17}}{17} \to \frac{3}{17}$

$\frac{3}{17} = 0.176 \cdots < \frac{1}{5}$

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NO CALCULATOR ALLOWED

Work for problem 6(a)

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

$\therefore f(x) = \ln(1+x^3) = x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots + (-1)^{n+1}\frac{x^{3n}}{n} + \cdots$

Work for problem 6(b)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{(-1)^{n+2}\frac{x^{3(n+1)}}{n+1}}{(-1)^{n+1}\frac{x^{3n}}{n}} = (-1)\cdot x^3 = -x^3$$

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NO CALCULATOR ALLOWED

Work for problem 6(c)

$$f'(x) = 3x^2 - \frac{6x^5}{2} + \frac{9x^8}{3} - \frac{12\cdot x^{11}}{4} + \cdots + (-1)^{n+1}\frac{3n\cdot x^{3n-1}}{n} + \cdots$$

$$f'(t^2) = 3t^4 - \frac{6\times t^{10}}{2} + \frac{9\times x^{16}}{3} - \frac{12\cdot x^{22}}{4} + \cdots$$

$$g(1) = 3\times 1 - \frac{6\times 1}{2}$$

$$= 3 - 3 = 0.$$

Work for problem 6(d)

when $x = 1$.

$$f(x) = 3 - 3\cdot 1 + 3\cdot 1 - 3\cdot 1 + \cdots + (-1)^2\cdot 3$$

$\therefore$ $0 < \frac{1}{5}$ $\therefore$ it abolute value is to 0.

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Question 6

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 2 points in part (a), no points in part (b), 4 points in part (c), and no point in part (d). In parts (a) and (c) the student's work is correct. In part (b) the student's work is incorrect. In part (d) the student uses the correct approach and has correct calculations, but the student's argument is incomplete in that it does not indicate that the error (the difference between $g(1)$ and the approximation) is what is less than $\frac{3}{17}$.

Sample: 6C
Score: 4

The student earned 4 points: 2 points in part (a), no points in part (b), 2 points in part (c), and no point in part (d). In part (a) the student's work is correct. In parts (b) and (d) the student's work is incorrect. In part (c) the student earned the first 2 points.