The polar curve $r$ is given by $r(\theta) = 3\theta + \sin \theta$, where $0 \leq \theta \leq 2\pi$.

(a) Find the area in the second quadrant enclosed by the coordinate axes and the graph of $r$.

(b) For $\frac{\pi}{2} \leq \theta \leq \pi$, there is one point $P$ on the polar curve $r$ with $x$-coordinate $-3$. Find the angle $\theta$ that corresponds to point $P$. Find the $y$-coordinate of point $P$. Show the work that leads to your answers.

(c) A particle is traveling along the polar curve $r$ so that its position at time $t$ is $(x(t), y(t))$ and such that $\frac{d\theta}{dt} = 2$. Find $\frac{dy}{dt}$ at the instant that $\theta = \frac{2\pi}{3}$, and interpret the meaning of your answer in the context of the problem.

---

(a) Area $= \frac{1}{2} \int_{\pi/2}^{\pi} (r(\theta))^2 \, d\theta = 47.513$

(b) $-3 = r(\theta)\cos \theta = (3\theta + \sin \theta)\cos \theta$

$\theta = 2.01692$

$y = r(\theta)\sin (\theta) = 6.272$

(c) $y = r(\theta)\sin \theta = (3\theta + \sin \theta)\sin \theta$

$$\frac{dy}{dt} \bigg|_{\theta=2\pi/3} = \left[ \frac{dy}{d\theta} \cdot \frac{d\theta}{dt} \right]_{\theta=2\pi/3} = -2.819$$

The $y$-coordinate of the particle is decreasing at a rate of 2.819.
Work for problem 2(a)

\[ 
\text{Area} = \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} r^2 \, d\theta \\
= \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} (3\theta + \sin\theta)^2 \, d\theta \\
= 47.513 
\]

Work for problem 2(b)

\[ 
\begin{align*}
\rho &= r \cos \theta \\
\rho x &= (3\theta + \sin\theta) \cos \theta \\
x &= 3\theta \\
\theta &= 2.017 \text{ radians} \\
\end{align*} 
\]

The \( \phi \)-coordinate of point \( \rho = r \sin \theta \) is
\[ 
\begin{align*}
\rho &= (3\theta + \sin\theta) \cos \theta \\
&= 6.272 
\end{align*} 
\]

-6-

Continue problem 2 on page 7.
Work for problem 2(c)

\[
\frac{dy}{dt} = \frac{d}{dt} (r \sin \theta) = \frac{d}{d\theta} (r \sin \theta) \cdot \frac{d\theta}{dt} = \frac{d}{d\theta} (r \sin \theta) \cdot \frac{d\theta}{dt} = \frac{d}{d\theta} (3 \theta \sin \theta + \sin^2 \theta) \cdot \frac{d\theta}{dt} = (3 \sin \theta + 3 \theta \cos \theta + \sin 2\theta) \cdot \frac{d\theta}{dt}.
\]

\[
\frac{dy}{dt} \bigg|_{\theta = \frac{2\pi}{3}} = (3 \sin \frac{2\pi}{3} + 3(\frac{2\pi}{3}) \cos \frac{2\pi}{3} + \sin \frac{4\pi}{3}) \cdot 2 = -2.819.
\]

\(y\) is positive at the instant \(\theta = \frac{2\pi}{3}\) and \(\frac{dy}{dt}\) is negative at the instant \(\theta = \frac{2\pi}{3}\).

The particle is travelling towards the \(x\)-axis at the instant \(\theta = \frac{2\pi}{3}\).

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.
Work for problem 2(a)

When the graph is in the II quadrant, \( \theta \in \left( \frac{\pi}{2}, \pi \right) \)

So area = \( \int_{\frac{\pi}{2}}^{\pi} r(\theta)^2 d\theta = \int_{\frac{\pi}{2}}^{\pi} \left( 3\theta + 5\sin\theta \right)^2 d\theta \)

= 49.513

---

Work for problem 2(b)

\( r(\theta) = 3\theta + 5\sin\theta \).

Thus \( x(\theta) = (3\theta + 5\sin\theta)\cos\theta \).

\( y(\theta) = (3\theta + 5\sin\theta)\sin\theta \).

When \( x(\theta) = 3 \), \( \theta \in \left[ \frac{\pi}{2}, \pi \right] \), \( \theta = 2.017 \).

\( y(\theta) = x(\theta) - \tan\theta = -3 - \tan\theta = 6.271 \).

So: \( \theta = 2.017 \).

\( y(\theta) = 6.271 \)

---

Continue problem 2 on page 7.
Work for problem 2(c)

\[
\frac{dy}{dt} = \frac{d}{d\theta} \left( \frac{d\theta}{dt} \right)
\]

\[
\frac{dy}{d\theta} = \frac{d}{d\theta} \left( (3\theta + 5\cos\theta - 6\sin\theta) \right)
= 3\theta \cdot \cos\theta + 3\sin\theta + 2\sin\theta - (6\cos\theta)
\]

so when \( \theta = \frac{2\pi}{3} \)

\[
\frac{dy}{dt} = 2 \cdot \left[ 3 \cdot \frac{2\pi}{3} \cdot \left( -\frac{1}{2} \right) + 2 \cdot \frac{\sqrt{3}}{2} \cdot \left( \frac{1}{2} \right) \right]
= 2\sqrt{3} - 2\pi
\]

\( \square \) So when \( \theta = \frac{2\pi}{3} \)

\[
\frac{dy}{dt} = 2\sqrt{3} - 2\pi
\]

\( \square \) It means, when \( \theta = \frac{2\pi}{3} \), the speed of the particle on the direction \( y \) is \( (2\sqrt{3} - 2\pi) \).

END OF PART A OF SECTION II
IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.
Work for problem 2(a)

\[ A = \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} (9\theta^2 + 9\sin^2 \theta + 6\theta \sin \theta) \, d\theta \]

\[ = \frac{1}{2} \left( 3\theta^3 + \right. \]

Work for problem 2(b)

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]
\[ x = -3, \quad r \cos \theta = -3 \]
\[ r = 3 \theta + \sin \theta \]
\[ 3 \theta \cos \theta + \sin \theta \cos \theta = -3 \]

\[ y = \theta \tan \theta \]
Work for problem 2(c)

\[
\frac{dy}{dt} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dt}
\]

\[
\theta = \frac{2\pi}{3}
\]

\[
= (\sin \theta + r \cos \theta) \times 2
\]

\[
= (\frac{\sqrt{3}}{2} \cdot 0 - \frac{1}{2} \cdot r) \times 2
\]

\[
= \sqrt{3} - r
\]
Question 2

Sample: 2A
Score: 9

The student earned all 9 points. Because the particle is above the $x$-axis, it is sufficient that the student states “the particle is travelling towards the $x$-axis” in part (c).

Sample: 2B
Score: 6

The student earned 6 points: 2 points in part (a), 2 points in part (b), and 2 points in part (c). In part (a) the student’s integral is correct, so the first 2 points were earned. The answer is incorrect. In part (b) the student earned the equation point implicitly and earned the point for the value of $\theta$. The student’s answer is incorrect, possibly as a result of intermediate rounding. In part (c) the student earned the first 2 points. The student does not indicate that the $y$-coordinate of the particle is decreasing.

Sample: 2C
Score: 4

The student earned 4 points: 2 points in part (a), 1 point in part (b), and 1 point in part (c). In part (a) the student’s integral is correct, so the first 2 points were earned. In part (b) the fourth line of the student’s solution earned the first point. In part (c) the student earned the chain-rule point.