AP® CALCULUS AB
2011 SCORING GUIDELINES (Form B)

Question 2

A 12,000-liter tank of water is filled to capacity. At time $t = 0$, water begins to drain out of the tank at a rate modeled by $r(t)$, measured in liters per hour, where r is given by the piecewise-defined function

$$r(t) = \begin{cases} \dfrac{600t}{t+3} & \text{for } 0 \le t \le 5 \\ 1000e^{-0.2t} & \text{for } t > 5 \end{cases}$$

(a) Is r continuous at $t = 5$? Show the work that leads to your answer.

(b) Find the average rate at which water is draining from the tank between time $t = 0$ and time $t = 8$ hours.

(c) Find $r'(3)$. Using correct units, explain the meaning of that value in the context of this problem.

(d) Write, but do not solve, an equation involving an integral to find the time A when the amount of water in the tank is 9000 liters.

(a) $\lim\limits_{t \to 5^-} r(t) = \lim\limits_{t \to 5^-} \left(\dfrac{600t}{t+3} \right) = 375 = r(5)$

$\lim\limits_{t \to 5^+} r(t) = \lim\limits_{t \to 5^+} \left(1000e^{-0.2t} \right) = 367.879$

Because the left-hand and right-hand limits are not equal, r is not continuous at $t = 5$.

2 : conclusion with analysis

(b) $\dfrac{1}{8}\displaystyle\int_0^8 r(t)\,dt = \dfrac{1}{8}\left(\int_0^5 \dfrac{600t}{t+3}\,dt + \int_5^8 1000e^{-0.2t}\,dt \right)$

$= 258.052$ or 258.053

3 : { 1 : integrand; 1 : limits and constant; 1 : answer }

(c) $r'(3) = 50$

The rate at which water is draining out of the tank at time $t = 3$ hours is increasing at 50 liters / hour^2.

2 : { 1 : $r'(3)$; 1 : meaning of $r'(3)$ }

(d) $12{,}000 - \displaystyle\int_0^A r(t)\,dt = 9000$

2 : { 1 : integral; 1 : equation }

Work for problem 2(a)

$r(5) = \frac{600t}{t+3} = 375$

$\lim_{x \to 5} 1000e^{-0.2t} = 1000e^{-0.2 \times 5} = 367.879$

no

Work for problem 2(b)

$\left(\int_0^5 \left(\frac{600t}{t+3}\right) dt + \int_5^8 (1000e^{-0.2t}) dt \right) \div 8$

$= (1234.507 + 829.915) \div 8$

$= 258.053$ liters/hour.

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Continue problem 2 on page 7.

Work for problem 2(c)

$$r'(3) = \frac{d}{dt}\left(\frac{600t}{t+3}\right)$$

$$= \frac{(t+3)(600) - (600t)(1)}{(t+3)^2}$$

$$= \frac{3600 - 1800}{36}$$

$$= 50$$

The rate of at which water is draining is increasing at 50 liters/h^2 at $t=3$.

Work for problem 2(d)

$$9000 = 12000 - \int_0^x r(t)\,dt$$

END OF PART A OF SECTION II
IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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Work for problem 2(a)

function is continuous only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

$\lim_{x \to 5^-} \frac{600t}{t+3} = \left\{ \frac{600 \cdot 5}{8} \right\} = 375$

$\lim_{x \to 5^+} 1000e^{-0.2 \cdot 5} = 367.879$

$\lim_{x \to 5^-} f(t) \neq \lim_{x \to 5^+} f(t)$ => function is not continuous at $t = 5^-$

Work for problem 2(b)

Average rate: $\frac{1}{b-a} \int_a^b f(x)\,dx$

$\frac{1}{8-0}\left[\int_0^5 \frac{600t}{t+3}\,dt + \int_5^8 1000e^{-0.2t}\,dt\right] = 258.0257$ liters per hour

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Continue problem 2 on page 7.

Work for problem 2(c)

$$r'(t)\big|_{0<t\le 5} = \frac{600(t+3)-600t}{(t+3)^2}$$

$$r'(t) = \frac{600t+1800-600t}{t^2+6t+9} = \frac{1800}{9+18+9} =$$

$$= \frac{1800}{36} = 50 \ \frac{lit}{hour^2}$$

The rate of change of water at time period $t=3$

Work for problem 2(d)

$$\int_0^A r(t)\,dt = 9000$$

END OF PART A OF SECTION II

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Work for problem 2(a)

r will be continuous if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$, in our case

$\lim_{t \to 5^-} r(t) = \lim_{t \to 5^+} r(t)$

$\lim_{t \to 5^-} \frac{600t}{t+3} = 375^-$

$\lim_{t \to 5^+} 1000e^{-0,2t} = 367,8794$

$\lim_{t \to 5^-} \neq \lim_{t \to 5^+}$, so r is discontinuous.

Work for problem 2(b)

$\int_0^5 \frac{600t}{t+3} + \int_5^8 1000e^{-0,2t} =$ [redacted]

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Continue problem 2 on page 7.

Work for problem 2(c)

$0 \le 3 \le 5$, so we use $r(t) = \frac{600t}{t+3}$

$r'(t) = \frac{600(t+3) - 600t}{(t+3)^2}$

$r'(3) = \frac{600 \cdot 6 - 600 \cdot 3}{6^2} = 50$ liters in quadrant / hour
(liters^2 / hour)

$r(t)$ is a rate
$r'(t)$ is acceleration

Answer: $r'(3) = 50$ liters^2 / hour

Work for problem 2(d)

$1000e^{-0.2t} = 2000$

$e^{-0.2t} = 9$

$-0.2t = \ln 9$

END OF PART A OF SECTION II
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AP® CALCULUS AB
2011 SCORING COMMENTARY (Form B)

Question 2

Sample: 2A
Score: 9

The student earned all 9 points. In part (a) it is sufficient to show that $r(5) \neq \lim_{t \to 5^+} r(t)$. In part (d) an ideal solution would present an A as the upper limit of integration.

Sample: 2B
Score: 6

The student earned 6 points: 2 points in part (a), 2 points in part (b), 1 point in part (c), and 1 point in part (d). In part (a) the student's work is correct. Note that the last digit in the student's calculation looks like an 8, but when compared with the student's work in part (c), this digit is clearly intended to be a 9. In part (b) the student earned the points for the integrand and the limits and constant. The student's answer is not accurate to three decimal places, so the answer point was not earned. In part (c) the student computes the value of $r'(3)$. The student's interpretation of the meaning is vague. In part (d) the student earned the integral point.

Sample: 2C
Score: 4

The student earned 4 points: 2 points in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student earned the integrand point. The response is missing the $\frac{1}{8}$, so no other points were earned in part (b). In part (c) the student computes the value of $r'(3)$. The student's interpretation of the meaning is insufficient. In part (d) the student's work is incorrect.