Question 3

Intent of Question

The primary goals of this question were to assess students' ability to (1) recognize binomial distribution scenarios and calculate relevant binomial probabilities; (2) calculate expected values based on the binomial distribution and properties of expectation.

Solution

Part (a):

Let X denote the number of correct guesses, assuming that a student guesses randomly among the five options on all 25 questions. Then X has a binomial probability distribution with $n = 25$ and $p = \frac{1}{5} = 0.20$.

Part (b):

Let Y denote the number of correct responses on the seven questions for which the student guesses randomly from among the five options. Then Y has a binomial probability distribution with $n = 7$ and $p = 0.20$. Then the expected value of Y, $\text{E}(Y) = np = 7(0.20) = 1.4$ correct responses.

Next, using the scoring formula provided,
Score $= (18 + Y) - 0.25(7 - Y) + 0(0) = 16.25 + 1.25Y$.

The expected exam score is therefore:
$\text{E(Score)} = \text{E}(16.25 + 1.25Y) = 16.25 + 1.25\text{E}(Y) = 16.25 + 1.25(1.4) = 16.25 + 1.75 = 18$ correct responses.

Part (c):

Let Y be defined as in part (b). The student passes when Score ≥ 20, which means that $16.25 + 1.25Y \geq 20$, which means that $Y \geq \frac{20 - 16.25}{1.25} = 3$. In other words, in order to pass, the student must get three or more correct from the seven questions on which the student guesses.

Y has a binomial probability distribution with $n = 7$ and $p = 0.20$, so

$$P(Y \geq 3) = 1 - P(Y \leq 2) = 1 - \left[\binom{7}{0}(.2)^0(.8)^7 + \binom{7}{1}(.2)^1(.8)^6 + \binom{7}{2}(.2)^2(.8)^5\right] = 1 - 0.852 = 0.148.$$

Scoring

Parts (a), (b) and (c) are each scored as essentially correct (E), partially correct (P) or incorrect (I).

Question 3 (continued)

Part (a) is scored as follows:

Essentially correct (E) if the response identifies the correct type of probability distribution (binomial) *AND* identifies the two parameter values, $n = 25$ and $p = \frac{1}{5} = 0.20$, correctly. There are two components: naming the probability distribution and identifying the two parameter values.

Partially correct (P) if the response correctly identifies only one of the two components (either the name of the probability distribution or the parameter values).

Incorrect (I) if the response neither correctly names the distribution nor identifies both parameter values correctly.

Note: Notation such as $B(25, 0.2)$ will be scored as essentially correct (E) for this part.

Part (b) is scored as follows:

Essentially correct (E) if the number of trials, n, and the binomial probability of a success p for Y (the number of correct guesses) are used to find the expected value of Y, $\mathrm{E}(Y)$, and if the correct expected exam score is calculated using $\mathrm{E}(Y)$. There are two components: calculating $\mathrm{E}(Y)$ and calculating the expected exam score.

Partially correct (P) if only one of the two components is correct.

Incorrect (I) if neither component is correct.

Part (c) is scored as follows:

Essentially correct (E) if the response specifies that three or more correct guesses are needed and the binomial probability is calculated correctly. There are two components: correctly identifying the required probability and correct calculation of the probability.

Partially correct (P) if only one of the two components is correct.

Incorrect (I) if neither component is correct.

Question 3 (continued)

4 Complete Response

All three parts essentially correct

3 Substantial Response

Two parts essentially correct and one part partially correct

2 Developing Response

Two parts essentially correct and one part incorrect

OR

One part essentially correct and one or two parts partially correct

OR

Three parts partially correct

1 Minimal Response

One part essentially correct and two parts incorrect

OR

Two parts partially correct and one part incorrect

3A1

3. A test consisting of 25 multiple-choice questions with 5 answer choices for each question is administered. For each question, there is only 1 correct answer.

(a) Let X be the number of correct answers if a student guesses randomly from the 5 choices for each of the 25 questions. What is the probability distribution of X ?

The probability distribution is a binomial distribution with a sample size of 25 and a $\frac{1}{5}$ probability of success.

This test, like many multiple-choice tests, is scored using a penalty for guessing. The test score is determined by awarding 1 point for each question answered correctly, deducting 0.25 point for each question answered incorrectly, and ignoring any question that is omitted. That is, the test score is calculated using the following formula.

Score = (1 × number of correct answers) – (0.25 × number of incorrect answers) + (0 × number of omits)

For example, the score for a student who answers 17 questions correctly, answers 3 questions incorrectly, and omits 5 questions is

$$\text{Score} = (1 \times 17) - (0.25 \times 3) + (0 \times 5) = 16.25.$$

(b) Suppose a student knows the correct answers for 18 questions, answers those 18 questions correctly, and chooses randomly from the 5 choices for each of the other 7 questions. Show that the expected value of the student's score is 18 when using the scoring formula above.

Expected number of correct questions $= 18 + \frac{1}{5} \times 7 = 19.4$

Expected number of incorrect questions $= \frac{4}{5} \times 7 = 5.6$

Expected score $= (1 \times 19.4) - (0.25 \times 5.6) + (0 \times 0) = 19.4 - 1.4 = 18$

Therefore, the student's expected score $= 18$

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(c) A score of at least 20 is needed to pass the test. Suppose a student knows the correct answers for 18 questions, answers those 18 questions correctly, and chooses randomly from the 5 choices for each of the other 7 questions. What is the probability that the student will pass the test?

* Out of the 7 questions guessed, the student must get at least 3 correct (assuming this student did not skip any questions) in order to get a score of 20 − 18 = 2 from those 7 questions.

$$P(X \geq 3) = {}_7C_3 \left(\tfrac{1}{5}\right)^3 \left(\tfrac{4}{5}\right)^4 + {}_7C_4 \left(\tfrac{1}{5}\right)^4 \left(\tfrac{4}{3}\right)^3 + {}_7C_5 \left(\tfrac{1}{5}\right)^5 \left(\tfrac{4}{5}\right)^2 + {}_7C_6 \left(\tfrac{1}{5}\right)^6 \left(\tfrac{4}{5}\right)^1 + {}_7C_7 \left(\tfrac{1}{5}\right)^7$$

$$= 0.148032$$

The probability of the student passing the test

= 0.148032

* If the student got:

0 correct, 7 wrong ⇒ Score = (0×1) − (0.25×7) + (0×0) = −1.75

1 correct, 6 wrong ⇒ Score = (1×1) − (0.25×6) + (0×0) = −0.5

2 " , 5 " ⇒ Score = (2×1) − (0.25×5) + (0×0) = 0.75

3 " , 4 " ⇒ Score = (3×1) − (0.25×4) + (0×0) = 2

4 " , 3 " ⇒ Score = (4×1) − (0.25×3) + (0×0) = 3.25

5 " , 2 " ⇒ Score = (5×1) − (0.25×2) + (0×0) = 4.5

6 correct, 1 wrong ⇒ Score = (6×1) − (0.25×2) + (0×0) = 5.75

7 " , 0 wrong ⇒ Score = (7×1) − (0.25×0) + (0×0) = 7

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3B1

3. A test consisting of 25 multiple-choice questions with 5 answer choices for each question is administered. For each question, there is only 1 correct answer.

(a) Let X be the number of correct answers if a student guesses randomly from the 5 choices for each of the 25 questions. What is the probability distribution of X?

It is a binomial distribution since there are two choices (Correct/Incorrect) each problem is independent of each other and there's a limited amount of choices.

$$P = \binom{25}{X}\left(\frac{1}{5}\right)^X\left(\frac{4}{5}\right)^{25-X}$$

This test, like many multiple-choice tests, is scored using a penalty for guessing. The test score is determined by awarding 1 point for each question answered correctly, deducting 0.25 point for each question answered incorrectly, and ignoring any question that is omitted. That is, the test score is calculated using the following formula.

Score = (1 × number of correct answers) – (0.25 × number of incorrect answers) + (0 × number of omits)

For example, the score for a student who answers 17 questions correctly, answers 3 questions incorrectly, and omits 5 questions is

$$\text{Score} = (1 \times 17) - (0.25 \times 3) + (0 \times 5) = 16.25.$$

(b) Suppose a student knows the correct answers for 18 questions, answers those 18 questions correctly, and chooses randomly from the 5 choices for each of the other 7 questions. Show that the expected value of the student's score is 18 when using the scoring formula above.

~~The probability~~ The expected value for an incorrect answer would be $\left(\frac{4}{5}\right)(5)$ while the expected value for a correct answer would be $\left(\frac{1}{5}\right)(5)$. Thus, the expected value for incorrect answers would be 4 and that of correct answers would be 1. The other 2 are omitted.

After placing our expected numbers in, we get the following result:

$(1 \times 19) - (0.25 \times 4) + (0 \times 2) = 19 - 1 + 0 = 18.$

Thus, the expected value of the student's score is indeed 18

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(c) A score of at least 20 is needed to pass the test. Suppose a student knows the correct answers for 18 questions, answers those 18 questions correctly, and chooses randomly from the 5 choices for each of the other 7 questions. What is the probability that the student will pass the test?

The probability that the student passes the test is the same as 1 minus the probability the students doesn't pass the test.

If the student gets 1 correct, he will not pass.
" " " " 2 correct, he will also not pass [(2 − .25(3)) is less than 2]
Once the student gets 3 correct or more, he will pass.

Hence the probability of a student getting 0, 1 or 2 correct is

$P(0) + P(1) + P(2) = \binom{5}{0}\left(\frac{4}{5}\right)^5 + \binom{5}{1}\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^4 + \binom{5}{2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^3$

$= \left(\frac{4}{5}\right)^5 + \left(\frac{4}{5}\right)^4 + 10\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^3 = \left(\frac{4}{5}\right)^5 + \left(\frac{4}{5}\right)^4 + \frac{2}{5}\left(\frac{4}{5}\right)^3$

≈ 0.94208

Therefore, the chances the students doesn't pass is 0.94208

For the probability that the student will pass, we just take 1 minus that probability we calculated:

$1 - 0.94208 = 0.05792$

Thus, there's only a 5.792% that the student will pass.

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3. A test consisting of 25 multiple-choice questions with 5 answer choices for each question is administered. For each question, there is only 1 correct answer.

   (a) Let X be the number of correct answers if a student guesses randomly from the 5 choices for each of the 25 questions. What is the probability distribution of X?

correct / incorrect → Binomial Distribution.

$P(x) = \binom{n}{r}(p)^r(1-p)^{n-r} = {}_nC_r(p)^r(1-p)^{n-r}$ → $= {}_5C_1\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^4 = 0.4096$

P(X) in percentile is 40.96%.

This test, like many multiple-choice tests, is scored using a penalty for guessing. The test score is determined by awarding 1 point for each question answered correctly, deducting 0.25 point for each question answered incorrectly, and ignoring any question that is omitted. That is, the test score is calculated using the following formula.

Score = (1 × number of correct answers) – (0.25 × number of incorrect answers) + (0 × number of omits)

For example, the score for a student who answers 17 questions correctly, answers 3 questions incorrectly, and omits 5 questions is

$$\text{Score} = (1 \times 17) - (0.25 \times 3) + (0 \times 5) = 16.25.$$

   (b) Suppose a student knows the correct answers for 18 questions, answers those 18 questions correctly, and chooses randomly from the 5 choices for each of the other 7 questions. Show that the expected value of the student's score is 18 when using the scoring formula above.

With the given formula, you need to get 19 questions correct, get 4 questions wrong, and omit 2 questions in order to get 18.

So, the formula would be

Score = (1×19) − (0.25×4) + (0×2) = 18

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3C2

(c) A score of at least 20 is needed to pass the test. Suppose a student knows the correct answers for 18 questions, answers those 18 questions correctly, and chooses randomly from the 5 choices for each of the other 7 questions. What is the probability that the student will pass the test?

In order to pass, a student should score at least 20.

With 18 problems correct, a student needs to get at least 3 questions right ($\because (1\times3)-(0.25\times4)=2$)

If we put the probability of getting 3 or more questions correct as a "P," it is easier to get "1−p", chances of getting 0, 1, or 2 questions right. So, with binomial probability, ${}_nC_r(p)^r(1-p)^{n-r}$.

$$1-p = {}_7C_0\left(\tfrac{1}{5}\right)^0\left(\tfrac{4}{5}\right)^7 + {}_7C_1\left(\tfrac{1}{5}\right)^1\left(\tfrac{4}{5}\right)^6 + {}_7C_2\left(\tfrac{1}{5}\right)^2\left(\tfrac{4}{5}\right)^5$$

$$= 0.2097152 + 0.3670016 + 0.2752512$$

$$= 0.851968.$$

So, ~~take~~ subtract "1−p" from 1, it will give us the value of p.

which is 0.148032.

The probability of the student passing a test is 0.148032.

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Question 3

Sample: 3A
Score: 4

Parts (a) and (b) of this question are answered clearly, correctly and concisely. In part (c) the response enumerates clearly the possible scores that can be obtained when guessing on seven questions to justify the conclusion that at least three questions must be guessed correctly to obtain a score of at least 20. The probability required in part (c) is calculated clearly and correctly. Parts (a), (b) and (c) were all scored as essentially correct. The entire answer, based on all three parts, was judged a complete response and earned a score of 4.

Sample: 3B
Score: 3

Part (a) of this response was scored as essentially correct. The probability distribution is identified by name and formula, and the parameter values are correctly identified by the formula. In part (b) the expected number of correct guesses is calculated for the situation where five questions are guessed and two are omitted, instead of for the described situation in which seven questions are guessed and none omitted. The resulting incorrect value for the expected number of correct guesses is then used correctly to calculate the expected score. Because only one of the two components of part (b) was answered correctly, part (b) was scored as partially correct. In part (c) the student continues to assume that five questions are guessed randomly, rather than seven. Aside from this error, the response to part (c) correctly determines that at least three correct guesses are needed and correctly calculates that probability by using the probability of the complement. Because the error of considering five guesses instead of seven was penalized in part (b), it was not penalized again in part (c), so this part was scored as essentially correct. The entire answer, based on all three parts, was judged a substantial response and earned a score of 3.

Sample: 3C
Score: 2

In part (a) the binomial distribution is identified by name. Although the value of p is correctly identified by formula, the value of n is given incorrectly as 5. Part (a) was therefore scored as partially correct. In part (b) the response describes one scenario in which a score of 18 is obtained. However, the response does not contain an expected value calculation, and no notion of averaging over possible outcomes is present, so part (b) was scored as incorrect. The response to part (c) includes a justification of why at least three correct guesses are needed, as well as a correct probability calculation. Part (c) was scored as essentially correct. The entire answer, based on all three parts, was judged a developing response and earned a score of 2.