AP® CALCULUS BC
2010 SCORING GUIDELINES

Question 6

$$f(x) = \begin{cases} \dfrac{\cos x - 1}{x^2} & \text{for } x \neq 0 \\ -\dfrac{1}{2} & \text{for } x = 0 \end{cases}$$

The function f, defined above, has derivatives of all orders. Let g be the function defined by $g(x) = 1 + \int_0^x f(t)\,dt.$

(a) Write the first three nonzero terms and the general term of the Taylor series for $\cos x$ about $x = 0$. Use this series to write the first three nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use the Taylor series for f about $x = 0$ found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at $x = 0$. Give a reason for your answer.

(c) Write the fifth-degree Taylor polynomial for g about $x = 0$.

(d) The Taylor series for g about $x = 0$, evaluated at $x = 1$, is an alternating series with individual terms that decrease in absolute value to 0. Use the third-degree Taylor polynomial for g about $x = 0$ to estimate the value of $g(1)$. Explain why this estimate differs from the actual value of $g(1)$ by less than $\frac{1}{6!}$.

(a) $\cos(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \cdots + (-1)^n \dfrac{x^{2n}}{(2n)!} + \cdots$

$f(x) = -\dfrac{1}{2} + \dfrac{x^2}{4!} - \dfrac{x^4}{6!} + \cdots + (-1)^{n+1} \dfrac{x^{2n}}{(2n+2)!} + \cdots$

$3 : \begin{cases} 1 : \text{terms for } \cos x \\ 2 : \text{terms for } f \\ \quad 1 : \text{first three terms} \\ \quad 1 : \text{general term} \end{cases}$

(b) $f'(0)$ is the coefficient of x in the Taylor series for f about $x = 0$, so $f'(0) = 0$.

$\dfrac{f''(0)}{2!} = \dfrac{1}{4!}$ is the coefficient of x^2 in the Taylor series for f about $x = 0$, so $f''(0) = \dfrac{1}{12}$.

Therefore, by the Second Derivative Test, f has a relative minimum at $x = 0$.

$2 : \begin{cases} 1 : \text{determines } f'(0) \\ 1 : \text{answer with reason} \end{cases}$

(c) $P_5(x) = 1 - \dfrac{x}{2} + \dfrac{x^3}{3 \cdot 4!} - \dfrac{x^5}{5 \cdot 6!}$

$2 : \begin{cases} 1 : \text{two correct terms} \\ 1 : \text{remaining terms} \end{cases}$

(d) $g(1) \approx 1 - \dfrac{1}{2} + \dfrac{1}{3 \cdot 4!} = \dfrac{37}{72}$

Since the Taylor series for g about $x = 0$ evaluated at $x = 1$ is alternating and the terms decrease in absolute value to 0, we know $\left| g(1) - \dfrac{37}{72} \right| < \dfrac{1}{5 \cdot 6!} < \dfrac{1}{6!}$.

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{explanation} \end{cases}$

6A,

Work for problem 6(a)

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{(2n)!}$

first three nonzero terms : $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

general term = $\frac{(-1)^n (x)^{2n}}{(2n)!}$

$f(x) = \frac{\cos x - 1}{x^2}, x \neq 0$

first nonzero terms = $-\frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!}$

general term = $\frac{(-1)^{n+1} (x)^{2n}}{(2n+2)!}$

Work for problem 6(b)

$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (x)^{2n}}{(2n+2)!} = -\frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} + \ldots$

$f'(x) = \frac{2x}{4!} - \frac{4x^3}{6!} + \ldots = 0$ $\quad f'(x) = x\left(\frac{2}{4!} - \frac{4x^2}{6!} + \ldots\right)$

$x = 0$

$f''(x) = \frac{2}{4!} - \frac{12x^2}{6!} + \ldots$

$f''(0) = \frac{2}{4!} > 0$

$\therefore \exists$ relative min at $x = 0$ by the second derivative test

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6A2

Work for problem 6(c)

$\sum_{n=0} \frac{f^n(0)}{n!}(x)^n$

$g(x) \approx P_5(x) = 1 - \frac{1}{2}x + \frac{2}{4!3!}x^3 - \frac{24}{6!5!}x^5$

$g(0) = 1$

$g'(0) = -\frac{1}{2}$ $\quad g'(x) = f(x)$

$g''(0) = 0$ $\quad g''(x) = f'(x)$

$g'''(0) = \frac{2}{4!}$ $\quad g'''(x) = f''(x)$

$g^4(0) = 0$ $\quad g^4(x) = f'''(x) = \frac{-24x}{6!} + \ldots$

$g^5(0) = -\frac{24}{6!}$ $\quad g^5(x) = f^4(x) = -\frac{24}{6!}$

Work for problem 6(d)

$g(x) \approx P_3(x) = 1 - \frac{1}{2}x + \frac{2}{4!3!}x^3$ $\qquad R_3(x) = \frac{24}{6!5!}x^5$

$g(1) \approx P_3(1) = 1 - \frac{1}{2}(1) + \frac{2}{4!3!}(1)^3$ $\qquad R_3(1) = \frac{24}{6!5!}$

$1 - \frac{1}{2} + \frac{2}{4!3!}$ $\qquad \begin{array}{r} 24 \\ \times 6 \\ \hline 144 \end{array}$ $\qquad = \frac{4!}{6!5!}$ $\qquad \begin{array}{r} 720 \\ \times 120 \\ \hline 144 \\ 720 \\ \hline 86400 \end{array}$

$\frac{1}{2} + \frac{2}{144}$ $\qquad = \frac{1}{(6!)5} < \frac{1}{6!}$

$\boxed{\frac{74}{144}}$

6B,

Work for problem 6(a)

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

$f(x) = -\frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!}$

Work for problem 6(b)

$f'(x) = \frac{2x}{4!} - \frac{4x^3}{6!}$

$f'(0) = 0$

$f''(x) = \frac{2}{4!} - \frac{12x^2}{6!}$

$f''(0) = \frac{1}{12} > 0$

at $x=0$ $f(x)$ is at a minimum by second derivative test.

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6B2

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Work for problem 6(c)

$g(x) = 1 + \int_0^x f(t)\,dt$

$= 1 - \frac{x}{2!} + \frac{x^3}{(3)(4!)} - \frac{x^5}{(6!)(5)}$

Work for problem 6(d)

$g(1) \approx 1 - \frac{1}{2} + \frac{1}{(3)(4!)}$

→ to get error take next term and plug in for $x = 1$

→ $\left| -\frac{1^5}{(6!)(5)} \right| < \frac{1}{6!}$

this method of evaluating next term to find error works because g about $x=0$ and evaluated at $x=1$ is an alternating series w/ terms that decrease in absolute value to 0.

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6C,

Work for problem 6(a)

$\cos x \Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad : \quad 1 - \frac{x^2}{2!} + \frac{x^3}{4!} - \frac{x^6}{6!} \cdots (-1)^n \frac{x^{2n}}{(2n)!}$

$f(x) = \frac{\cos x - 1}{x^2} \quad : \quad \frac{\left(1 - \frac{\cancel{x^2}}{2!} + \frac{x^{\cancel{3}\,2}}{4!} - \frac{x^{\cancel{6}\,4}}{6!} \cdots \frac{x^{2n}}{(2n)!}\right)}{x^2}$

$= -\frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} \cdots \frac{(-1)^{n+1} x^{2n}}{(2n)!}$

Work for problem 6(b)

$f'(0) = 0$

The Taylor series for f about $x = 0$

$f''(0) = \frac{(-1)^{4+1} 6^{2n}}{(2 \cdot 4)!}$

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6C2

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Work for problem 6(c)

Taylor for $g(x) = 1 + \int_0^x -\frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} + \frac{x^6}{8!} - \frac{x^8}{10!}$

$= 1 - \frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \frac{x^7}{8!} - \frac{x^9}{10!}$

Work for problem 6(d)

g about $x=0$, evaluated @ $x=1$

Lagrange Error using third degree of $-\frac{x^5}{6!}$

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AP® CALCULUS BC
2010 SCORING COMMENTARY

Question 6

Overview

This problem provided the function f defined by $f(x) = \frac{\cos x - 1}{x^2}$ for $x \neq 0$ and $f(0) = -\frac{1}{2}$. It was given that f has derivatives of all orders, and the function g is defined by $g(x) = 1 + \int_0^x f(t)\,dt$. Part (a) asked for the first three nonzero terms and the general term of the Taylor series for $\cos x$ about $x = 0$. Students were to use this with algebraic manipulation to find the first three nonzero terms and the general term of the Taylor series for f about $x = 0$. In part (b) students were asked to use the Taylor series for f about $x = 0$ to determine whether f has a relative maximum, relative minimum or neither at $x = 0$. From the series for f students can establish that $f'(0) = 0$ and $f''(0) = \frac{1}{4 \cdot 3}$ and resolve this issue using the Second Derivative Test. Part (c) asked for the fifth-degree Taylor polynomial for g about $x = 0$. In part (d) it was given that the Taylor series for g about $x = 0$ is an alternating series whose terms decrease in absolute value to 0. Students were asked to use the third-degree Taylor polynomial for g about $x = 0$ to estimate $g(1)$ and to explain why this estimate is within $\frac{1}{6!}$ of the actual value. The properties of the series for $g(1)$ allow us to bound the error in this approximation by the absolute value of the next term in the series.

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 1 point in part (a), 1 point in part (b), 2 points in part (c), and 2 points in part (d). In part (a) the student earned the second point. Although the first three terms for the series for $\cos x$ are correct, the student omits the general term. The first three terms of the series for f are correct, but the student also omits the general term. The student did not lose any points for the use of "=" in both cases. In part (b) the student incorrectly declares $f'(x)$ and $f''(x)$ to be polynomials and provides a correct solution without dealing with a series. The student earned just 1 point as a result. In parts (c) and (d), the student's work is correct. The student did not lose any points for an incorrect use of the equals sign in part (c).

Sample: 6C
Score: 4

The student earned 4 points: 2 points in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d). In part (a) the student earned the first point for the first three terms and the general term of the series for $\cos x$. The first three terms of the series for f are correct, but the general term is incorrect. The student earned 1 of the possible points for the series for f. In part (b) the student earned the first point for determining $f'(0)$. In part (c) the student earned 1 point for the correct first two terms. In part (d) the student's work is incorrect.