AP® CALCULUS BC
2010 SCORING GUIDELINES

Question 5

Consider the differential equation $\frac{dy}{dx} = 1 - y$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(1) = 0$. For this particular solution, $f(x) < 1$ for all values of x.

(a) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

(b) Find $\lim_{x \to 1} \frac{f(x)}{x^3 - 1}$. Show the work that leads to your answer.

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = 1 - y$ with the initial condition $f(1) = 0$.

(a)
$$f\left(\frac{1}{2}\right) \approx f(1) + \left(\left.\frac{dy}{dx}\right|_{(1,\,0)}\right) \cdot \Delta x$$
$$= 0 + 1 \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2}$$
$$f(0) \approx f\left(\frac{1}{2}\right) + \left(\left.\frac{dy}{dx}\right|_{\left(\frac{1}{2},\,-\frac{1}{2}\right)}\right) \cdot \Delta x$$
$$\approx -\frac{1}{2} + \frac{3}{2} \cdot \left(-\frac{1}{2}\right) = -\frac{5}{4}$$

$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{answer} \end{cases}$

(b) Since f is differentiable at $x = 1$, f is continuous at $x = 1$. So, $\lim_{x \to 1} f(x) = 0 = \lim_{x \to 1}\left(x^3 - 1\right)$ and we may apply L'Hospital's Rule.
$$\lim_{x \to 1} \frac{f(x)}{x^3 - 1} = \lim_{x \to 1} \frac{f'(x)}{3x^2} = \frac{\lim_{x \to 1} f'(x)}{\lim_{x \to 1} 3x^2} = \frac{1}{3}$$

$2 : \begin{cases} 1 : \text{use of L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

(c)
$$\frac{dy}{dx} = 1 - y$$
$$\int \frac{1}{1 - y}\, dy = \int 1\, dx$$
$$-\ln|1 - y| = x + C$$
$$-\ln 1 = 1 + C \Rightarrow C = -1$$
$$\ln|1 - y| = 1 - x$$
$$|1 - y| = e^{1-x}$$
$$f(x) = 1 - e^{1-x}$$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

NO CALCULATOR ALLOWED

Work for problem 5(a)

Using this table, we calculate $\frac{dy}{dx}$, allowing us to approximate $\Delta y \approx \frac{dy}{dx} \cdot \Delta x$

x	y	dy/dx	Δy	Δx
1	0	$1-0=1$	-0.5	-0.5
0.5	-0.5	$1-(-0.5)=1.5$	-0.75	-0.5
0	-1.25	X	X	X

$$f(0) = y(0) \approx \boxed{-1.25}$$

Work for problem 5(b)

We are given $f(1)=0$, and $f'(x) = 1 - f(x)$

$$\lim_{x\to 1} \frac{f(x)}{x^3-1} = \frac{0}{0} = \text{Indeterminate}$$

Using L'Hopital's

$$\lim_{x\to 1} \frac{f(x)}{x^3-1} = \lim_{x\to 1} \frac{f'(x)}{3x^2} = \lim_{x\to 1} \frac{1-f(x)}{3x^2} = \boxed{\frac{1}{3}}$$

Continue problem 5 on page 13.

Work for problem 5(c)

$$\frac{dy}{dx} = 1 - y$$

$$\frac{dy}{1-y} = dx$$

$$\int \frac{dy}{1-y} = \int dx$$

$$-\ln|1-y| = x + c$$

$$\ln|1-y| = -x + c$$

$$e^{\ln|1-y|} = Ae^{-x}$$

$$1 - y = Ae^{-x}$$

$$y = 1 - Ae^{-x}$$

$$y(1) = 1 - \frac{A}{e} = 0$$

$$1 = \frac{A}{e} \Rightarrow A = e$$

$$y = 1 - e \cdot e^{-x} \Rightarrow \boxed{y = 1 - e^{-x+1}}$$

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Work for problem 5(a)

x	y	Δx	m	Δy
1	0	$-\frac{1}{2}$	1	$-\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{4}$
0	$-\frac{5}{4}$			

$f(0) \approx -\frac{5}{4}$

Work for problem 5(b)

$$\lim_{x\to 1} \frac{f(x)}{x^3-1} \to \lim_{x\to 1} \frac{f'(x)}{3x^2} \to \lim_{x\to 1} \frac{f''(x)}{6x}$$

$$\to \frac{-1}{6x} = \frac{-1}{6(1)} = -\frac{1}{6}$$

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Continue problem 5 on page 1

Work for problem 5(c)

$f(1)=0$

$\frac{dy}{dx} = 1-y$

$\frac{dy}{1-y} = dx$

$\int \frac{dy}{1-y} = \int dx$

$\ln|1-y| = x + c$

$e^{\ln|1-y|} = e^x + e^c$

$1-y = Ae^x$

$y = 1-Ae^x$

$0 = 1-Ae^1$

$1 = Ae$

$A = \frac{1}{e}$

$y = 1-(\frac{1}{e})(e^x)$

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 5C1

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Work for problem 5(a)

$\frac{dy}{dx} = 1 - y \quad f(1) = 0 \quad f(x) < 1$

$x = 1 \rightarrow f(1) = 0$

$f(.5) \approx f(1) + (-.5)(f'(1))$

$= 0 + (-.5)(1-0)$

$= -.5$

$f(0) \approx f(.5) + (-.5)(f'(.5))$

$= -.5 + (-.5)(1.5)$

$= -.5 + -.75$

$\approx \boxed{-1.25}$

Work for problem 5(b)

$\lim_{x \to 1} \frac{f(x)}{x^3 - 1}$

l'hospital to the rescue!

$\lim_{x \to 1} \frac{1-y}{3x^2} = \frac{1 - f(1)}{3(1)^2}$

$= \frac{1-0}{3}$

$= \boxed{1/3}$

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NO CALCULATOR ALLOWED

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Work for problem 5(c)

dont forget +C

$$\frac{dy}{dx} = 1 - y$$

$$\cancel{\int dy = \int dx - y\,dx}$$

$$\cancel{y = x - \int y\,dx}$$

$$\cancel{(xy - \int x\,dy)}$$

$$\int \frac{dy}{1-y} = \int dx$$

$$\ln|1-y| = x$$

$$1 - y = e^{x} + C$$

$$y = -e^{x} + C$$

$$f(1) = 0 \Rightarrow 0 = -e^{1} + C$$

$$C = e$$

$$\boxed{y = -e^{x} + e}$$

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Question 5

Overview

This problem presented the differential equation $\frac{dy}{dx} = 1 - y$ with a particular solution $y = f(x)$ satisfying $f(1) = 0.$ It was also given that $f(x) < 1$ for all values of x. Part (a) asked the students to use Euler's method with two steps of equal size to approximate $f(0).$ Part (b) asked for the evaluation of $\lim_{x \to 1} \frac{f(x)}{x^3 - 1},$ anticipating that students would recognize an invitation to apply L'Hospital's Rule. Part (c) asked for the particular solution $y = f(x)$ with initial condition $f(1) = 0.$ Students should have used the method of separation of variables.

Sample: 5A
Score: 9

The student earned all 9 points.

Sample: 5B
Score: 6

The student earned 6 points: 2 points in part (a), no points in part (b), and 4 points in part (c). In part (a) the student's work is correct. In part (b) the student does not justify the use of L'Hospital's Rule and applies it too many times. In this particular case, the student moves beyond the first derivative and declares an incorrect answer. In part (c) the student earned the separation, constant of integration, and initial condition points. The final answer for $y = f(x)$ is consistent with the student's antiderivative error (missing a factor of -1) and earned the point for solving for y.

Sample: 5C
Score: 4

The student earned 4 points: 2 points in part (a), 1 point in part (b), and 1 point in part (c). In part (a) the student's work is correct. In part (b) the student earned the answer point but does not justify the use of L'Hospital's Rule. In part (c) the student earned the separation point. The student has an incorrect antiderivative and incorrectly applies the constant of integration. As a result, no additional points were earned.