Question 3

A particle is moving along a curve so that its position at time t is $(x(t), y(t))$, where $x(t) = t^2 - 4t + 8$ and $y(t)$ is not explicitly given. Both x and y are measured in meters, and t is measured in seconds. It is known that $\frac{dy}{dt} = te^{t-3} - 1$.

(a) Find the speed of the particle at time $t = 3$ seconds.

(b) Find the total distance traveled by the particle for $0 \le t \le 4$ seconds.

(c) Find the time t, $0 \le t \le 4$, when the line tangent to the path of the particle is horizontal. Is the direction of motion of the particle toward the left or toward the right at that time? Give a reason for your answer.

(d) There is a point with x-coordinate 5 through which the particle passes twice. Find each of the following.

(i) The two values of t when that occurs

(ii) The slopes of the lines tangent to the particle's path at that point

(iii) The y-coordinate of that point, given $y(2) = 3 + \frac{1}{e}$

(a) Speed $= \sqrt{(x'(3))^2 + (y'(3))^2} = 2.828$ meters per second	1 : answer
(b) $x'(t) = 2t - 4$ Distance $= \int_0^4 \sqrt{(2t-4)^2 + \left(te^{t-3} - 1\right)^2}\, dt = 11.587$ or 11.588 meters	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(c) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 0$ when $te^{t-3} - 1 = 0$ and $2t - 4 \ne 0$ This occurs at $t = 2.20794$. Since $x'(2.20794) > 0$, the particle is moving toward the right at time $t = 2.207$ or 2.208.	$3 : \begin{cases} 1 : \text{considers } \frac{dy}{dx} = 0 \\ 1 : t = 2.207 \text{ or } 2.208 \\ 1 : \text{direction of motion with reason} \end{cases}$
(d) $x(t) = 5$ at $t = 1$ and $t = 3$ At time $t = 1$, the slope is $\left.\frac{dy}{dx}\right\|_{t=1} = \left.\frac{dy/dt}{dx/dt}\right\|_{t=1} = 0.432$. At time $t = 3$, the slope is $\left.\frac{dy}{dx}\right\|_{t=3} = \left.\frac{dy/dt}{dx/dt}\right\|_{t=3} = 1$. $y(1) = y(3) = 3 + \frac{1}{e} + \int_2^3 \frac{dy}{dt}\, dt = 4$	$3 : \begin{cases} 1 : t = 1 \text{ and } t = 3 \\ 1 : \text{slopes} \\ 1 : y\text{-coordinate} \end{cases}$

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Work for problem 3(a)

a. Speed $= \cancel{\int} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ $\quad \frac{dx}{dt} = 2t - 4$

Speed $= \sqrt{(2t-4)^2 + (te^{t-3} - 1)^2} \; \Big|_{t=3} \rightarrow$ 2.828 m/s

Work for problem 3(b)

total distance $= \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int_0^4 \sqrt{(2t-4)^2 + (te^{t-3} - 1)^2}\, dt$

$=$ 11.588 meters

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Continue problem 3 on page 9.

Work for problem 3(c)

Find where $\frac{dy}{dx} = 0$.

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{te^{t-3}-1}{2t-4}$

$0 = \frac{te^{t-3}-1}{2t-4}$, $t = 2.208$

If $\frac{dx}{dt}$ is positive, motion is to right, if negative, motion is to left.

$\frac{dx}{dt} = 2t-4$, $|t = 2.208 = 0.416$, therefore the motion is to the right.

Work for problem 3(d)

i. $x(t) = 5$, $5 = t^2 - 4t + 8$, $t = 1$ and $t = 3$

ii. at $t=1$: $\frac{dy/dt}{dx/dt} = dy/dx = \frac{te^{t-3}-1}{2t-4}$, $|t = 1 \rightarrow$ $\frac{dy}{dx}$ at $t=1$ is 0.432

at $t=3$: $\frac{te^{t-3}-1}{2t-4}$, $|t = 3 \rightarrow$ $\frac{dy}{dx}$ at $t=3$ is 1

iii. $y(3) = y(2) + \int_2^3 \frac{dy}{dt}$

$y(3) = \left(3+\frac{1}{e}\right) + \int_2^3 te^{t-3} - 1\, dt$

$y(3) = 4 = y(1)$

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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3B1

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Work for problem 3(a)

speed $= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$\frac{dx}{dt} = 2t - 4 \qquad \frac{dy}{dt} = te^{t-3} - 1, \ t = 3$

speed $= \sqrt{(2)^2 + (3e^0 - 1)^2} =$

$\sqrt{8} = \boxed{2.8284 \ \text{units}/s}$

Work for problem 3(b)

distance $= \int |v(t)| =$

$v(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

distance $= \int_0^4 \sqrt{(2t-4)^2 + (te^{t-3} - 1)} =$

$\boxed{11.5877 \ \text{units}}$

Continue problem 3 on page 9.

Work for problem 3(c)

$\frac{dy}{dx} = 0$ when horizontal

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{te^{t-3} - 1}{2t-4} = 0$

$\boxed{t = 2.20794}$

The particle is moving to the left because $\frac{dy}{dx}$ changes from positive to negative.

Work for problem 3(d)

$(5, y)$

a) $x(t) = t^2 - 4t + 8 = 5$

$t^2 - 4t + 3 = 0$ $\quad (t-1)(t-3)$

$\boxed{t = 1, t = 3}$

b) $\frac{dy}{dx} = \frac{te^{t-3} - 1}{2t - 4}$, $x = 5$ $\quad 3$

$= \boxed{\frac{5e^2 - 1}{6}}$

$\frac{dy}{dx} = \cancel{\frac{1}{2e^2}}$

$\frac{dy}{dt} = te^{t-3} - 1$

c) $y(2) = 3 + \frac{1}{e}$

$x = 5$ $\int_2^5 \frac{dy}{dt} = \boxed{\int_2^5 te^{t-3} - 1 + 3 + \frac{7}{e}}$

END OF PART ~~A OF SECTION II~~

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Work for problem 3(a)

$\int dy = \int_0^3 (te^{t-3} - 1)\,dt$

$y = -.95$

$x'(t) = 2t - 4$

$\text{speed} = \sqrt{x'(t)^2 + y'(t)^2}$

$t=3$ $\sqrt{(2t-4)^2 + (te^{t-3} - 1)^2} =$

$= 4 + 4 = 8 \text{ m/s}$

Work for problem 3(b)

$x(4) = 16 - 16 + 8 = 8 \text{ m}$

$x'(t) = 2t - 4$

$v(t) = \frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{te^{t-3} - 1}{2t - 4}$

$\int_0^4 \frac{te^{t-3} - 1}{2t - 4}\,dt$

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Continue problem 3 on page 9.

3C2

Work for problem 3(c)

$$\frac{dy}{dx} = \frac{te^{t-3}-1}{2t-4} = 0$$

$t=0$ right b/c x is positive

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Work for problem 3(d)

$t^2-4t+8=5$

$t^2-4t+3=0$

$(t-1)(t-3)$

$t=1,3$

$t=1$ $m = \frac{dy}{dx} = \frac{te^{t-3}-1}{2t-4} = \frac{e^{-2}-1}{-2}$

$t=3$ $m = \frac{2}{2} = 1$ $y(2) = 3 + \frac{1}{e}$

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END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

Question 3

Overview

This problem described the path of a particle whose motion is described by $(x(t),\ y(t))$, where $x(t) = t^2 - 4t + 8$ and $y(t)$ satisfies $\frac{dy}{dt} = te^{t-3} - 1.$ Part (a) asked for the speed of the particle at time $t = 3$ seconds. Part (b) asked for the total distance traveled by the particle for $0 \le t \le 4$ seconds. This is found by integrating $\sqrt{(x'(t))^2 + (y'(t))^2}$ over the interval $0 \le t \le 4.$ Part (c) asked for the time $t,\ 0 \le t \le 4,$ at which the line tangent to the particle's path is horizontal and whether the particle's direction of motion is toward the left or toward the right at that time. Students needed to solve $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 0$ for t and determine the sign of $\frac{dx}{dt}$ at this time to establish the left-to-right direction of motion. In part (d) it was given that there is a point with x-coordinate 5 through which the particle passes twice. Students were asked for (i) the two values of t when that occurs, (ii) the slopes of the lines tangent to the particle's path at that point and (iii) the y-coordinate of that point, given that $y(2) = 3 + \frac{1}{e}.$ After solving $x(t) = 5$ for $t = 1$ and $t = 3,$ the slopes can be found by evaluating $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ at each value of t and the y-coordinate by evaluating $y(3) = y(2) + \int_2^3 \frac{dy}{dt}\,dt$ (or the corresponding expression for $t = 1$).

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

The student earned 6 points: 1 point in part (a), 2 points in part (b), 2 points in part (c), and 1 point in part (d). In parts (a) and (b) the student's work is correct. In part (c) the student considers $\frac{dy}{dx} = 0$ and earned the first point. The student correctly solves the equation to find the time at which the line tangent to the path of the particle is horizontal and earned the second point. The student incorrectly reasons that the motion of the particle at the t-value presented can be determined from $\frac{dy}{dx}$ and did not earn the third point. In part (i) of part (d), the student correctly solves $x(t) = 5$ for the two values $t = 1$ and $t = 3$ where the x-coordinate is 5 and so earned the point. In part (ii) of part (d), the student does not evaluate $\frac{dy}{dx}$ at $t = 1$ and $t = 3$ and did not earn the point. In part (iii) of part (d), the student presents an expression and numerical evaluation for the y-coordinate at an incorrect t-value and did not earn the point.

Question 3 (continued)

Sample: 3C
Score: 3

The student earned 3 points: no points in part (a), no points in part (b), 1 point in part (c), and 2 points in part (d). In part (a) the student has an incorrect evaluation of the expression for the speed at $t = 3$ and did not earn the point. In part (b) the student presents an incorrect integrand in the definite integral and did not earn any points. In part (c) the student considers $\frac{dy}{dx} = 0$ and earned the first point. The student does not solve the equation and did not earn additional points in part (c). In part (i) of part (d), the student correctly solves $x(t) = 5$ for the two values $t = 1$ and $t = 3$ where the *x*-coordinate is 5 and so earned the point. In part (ii) of part (d), the student uses the chain rule to correctly evaluate $\frac{dy}{dx}$ to find the slopes of the lines tangent to the path of the particle at $t = 1$ and $t = 3$ and so earned the point. In part (iii) of part (d), the student does not present any work.