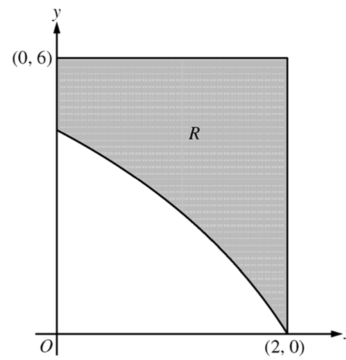
CollegeBoard AP

AP® Calculus BC
2010 Scoring Guidelines
Form B

The College Board

Question 1

In the figure above, R is the shaded region in the first quadrant bounded by the graph of $y = 4\ln(3-x)$, the horizontal line $y = 6$, and the vertical line $x = 2$.



(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the horizontal line $y = 8$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of the solid.

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1 : Correct limits in an integral in (a), (b), or (c)

(a) $\int_0^2 (6 - 4\ln(3-x))\,dx = 6.816$ or 6.817

$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) $\pi\int_0^2 \left((8 - 4\ln(3-x))^2 - (8-6)^2\right) dx$
$= 168.179$ or 168.180

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(c) $\int_0^2 (6 - 4\ln(3-x))^2\,dx = 26.266$ or 26.267

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

Question 2

The velocity vector of a particle moving in the plane has components given by

$$\frac{dx}{dt} = 14\cos\left(t^2\right)\sin\left(e^t\right) \text{ and } \frac{dy}{dt} = 1 + 2\sin\left(t^2\right), \text{ for } 0 \le t \le 1.5.$$

At time $t = 0$, the position of the particle is $(-2, 3)$.

(a) For $0 < t < 1.5$, find all values of t at which the line tangent to the path of the particle is vertical.

(b) Write an equation for the line tangent to the path of the particle at $t = 1$.

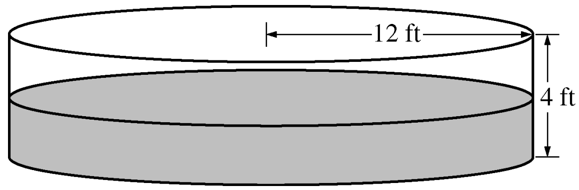
(c) Find the speed of the particle at $t = 1$.

(d) Find the acceleration vector of the particle at $t = 1$.

(a) The tangent line is vertical when $x'(t) = 0$ and $y'(t) \neq 0$. On $0 < t < 1.5$, this happens at $t = 1.253$ and $t = 1.144$ or 1.145.	$2 : \begin{cases} 1 : \text{sets } \dfrac{dx}{dy} = 0 \\ 1 : \text{answer} \end{cases}$
(b) $\left.\dfrac{dy}{dx}\right\rvert_{t=1} = \dfrac{y'(1)}{x'(1)} = 0.863447$ $x(1) = -2 + \int_0^1 x'(t)\,dt = 9.314695$ $y(1) = 3 + \int_0^1 y'(t)\,dt = 4.620537$ The line tangent to the path of the particle at $t = 1$ has equation $y = 4.621 + 0.863(x - 9.315)$.	$4 : \begin{cases} 1 : \left.\dfrac{dy}{dx}\right\rvert_{t=1} \\ 1 : x(1) \\ 1 : y(1) \\ 1 : \text{equation} \end{cases}$
(c) Speed $= \sqrt{(x'(1))^2 + (y'(1))^2} = 4.105$	1 : answer
(d) Acceleration vector: $\langle x''(1), y''(1)\rangle = \langle -28.425, 2.161\rangle$	$2 : \begin{cases} 1 : x''(1) \\ 1 : y''(1) \end{cases}$

Question 3

t	0	2	4	6	8	10	12
$P(t)$	0	46	53	57	60	62	63



The figure above shows an aboveground swimming pool in the shape of a cylinder with a radius of 12 feet and a height of 4 feet. The pool contains 1000 cubic feet of water at time $t = 0$. During the time interval $0 \le t \le 12$ hours, water is pumped into the pool at the rate $P(t)$ cubic feet per hour. The table above gives values of $P(t)$ for selected values of t. During the same time interval, water is leaking from the pool at the rate $R(t)$ cubic feet per hour, where $R(t) = 25e^{-0.05t}$. (Note: The volume V of a cylinder with radius r and height h is given by $V = \pi r^2 h$.)

(a) Use a midpoint Riemann sum with three subintervals of equal length to approximate the total amount of water that was pumped into the pool during the time interval $0 \le t \le 12$ hours. Show the computations that lead to your answer.

(b) Calculate the total amount of water that leaked out of the pool during the time interval $0 \le t \le 12$ hours.

(c) Use the results from parts (a) and (b) to approximate the volume of water in the pool at time $t = 12$ hours. Round your answer to the nearest cubic foot.

(d) Find the rate at which the volume of water in the pool is increasing at time $t = 8$ hours. How fast is the water level in the pool rising at $t = 8$ hours? Indicate units of measure in both answers.

(a) $\int_0^{12} P(t)\,dt \approx 46 \cdot 4 + 57 \cdot 4 + 62 \cdot 4 = 660 \text{ ft}^3$

$2 : \begin{cases} 1 : \text{midpoint sum} \\ 1 : \text{answer} \end{cases}$

(b) $\int_0^{12} R(t)\,dt = 225.594 \text{ ft}^3$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(c) $1000 + \int_0^{12} P(t)\,dt - \int_0^{12} R(t)\,dt = 1434.406$

At time $t = 12$ hours, the volume of water in the pool is approximately 1434 ft^3.

1 : answer

(d) $V'(t) = P(t) - R(t)$

$V'(8) = P(8) - R(8) = 60 - 25e^{-0.4} = 43.241 \text{ or } 43.242 \text{ ft}^3/\text{hr}$

$V = \pi(12)^2 h$

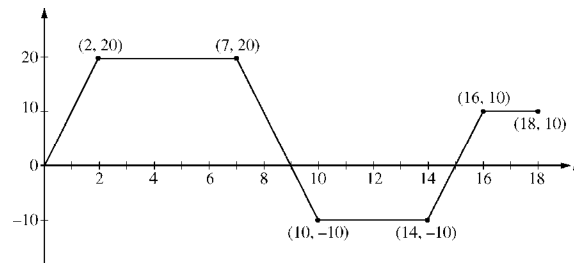
$\frac{dV}{dt} = 144\pi \frac{dh}{dt}$

$\left.\frac{dh}{dt}\right|_{t=8} = \frac{1}{144\pi} \cdot \left.\frac{dV}{dt}\right|_{t=8} = 0.095 \text{ or } 0.096 \text{ ft/hr}$

$4 : \begin{cases} 1 : V'(8) \\ 1 : \text{ equation relating } \frac{dV}{dt} \text{ and } \frac{dh}{dt} \\ 1 : \left.\frac{dh}{dt}\right|_{t=8} \\ 1 : \text{units of ft}^3/\text{hr and ft/hr} \end{cases}$

Question 4

A squirrel starts at building A at time $t = 0$ and travels along a straight wire connected to building B. For $0 \le t \le 18$, the squirrel's velocity is modeled by the piecewise-linear function defined by the graph above.



Graph of v

(a) At what times in the interval $0 < t < 18$, if any, does the squirrel change direction? Give a reason for your answer.

(b) At what time in the interval $0 \le t \le 18$ is the squirrel farthest from building A? How far from building A is the squirrel at this time?

(c) Find the total distance the squirrel travels during the time interval $0 \le t \le 18$.

(d) Write expressions for the squirrel's acceleration $a(t)$, velocity $v(t)$, and distance $x(t)$ from building A that are valid for the time interval $7 < t < 10$.

(a) The squirrel changes direction whenever its velocity changes sign. This occurs at $t = 9$ and $t = 15$.

$2 : \begin{cases} 1 : t\text{-values} \\ 1 : \text{explanation} \end{cases}$

(b) Velocity is 0 at $t = 0$, $t = 9$, and $t = 15$.

t	position at time t
0	0
9	$\frac{9+5}{2} \cdot 20 = 140$
15	$140 - \frac{6+4}{2} \cdot 10 = 90$
18	$90 + \frac{3+2}{2} \cdot 10 = 115$

The squirrel is farthest from building A at time $t = 9$; its greatest distance from the building is 140.

$2 : \begin{cases} 1 : \text{identifies candidates} \\ 1 : \text{answers} \end{cases}$

(c) The total distance traveled is $\int_0^{18} |v(t)|\, dt = 140 + 50 + 25 = 215$.

1 : answer

(d) For $7 < t < 10$, $a(t) = \dfrac{20 - (-10)}{7 - 10} = -10$

$$v(t) = 20 - 10(t - 7) = -10t + 90$$

$$x(7) = \frac{7+5}{2} \cdot 20 = 120$$

$$x(t) = x(7) + \int_7^t (-10u + 90)\, du$$

$$= 120 + \left(-5u^2 + 90u\right)\Big|_{u=7}^{u=t}$$

$$= -5t^2 + 90t - 265$$

$4 : \begin{cases} 1 : a(t) \\ 1 : v(t) \\ 2 : x(t) \end{cases}$

Question 5

Let f and g be the functions defined by $f(x) = \frac{1}{x}$ and $g(x) = \frac{4x}{1+4x^2}$, for all $x > 0$.

(a) Find the absolute maximum value of g on the open interval $(0, \infty)$ if the maximum exists. Find the absolute minimum value of g on the open interval $(0, \infty)$ if the minimum exists. Justify your answers.

(b) Find the area of the unbounded region in the first quadrant to the right of the vertical line $x = 1$, below the graph of f, and above the graph of g.

(a) $$g'(x) = \frac{4\left(1+4x^2\right) - 4x(8x)}{\left(1+4x^2\right)^2} = \frac{4\left(1-4x^2\right)}{\left(1+4x^2\right)^2}$$

For $x > 0$, $g'(x) = 0$ for $x = \frac{1}{2}$.

$g'(x) > 0$ for $0 < x < \frac{1}{2}$

$g'(x) < 0$ for $x > \frac{1}{2}$

$g\left(\frac{1}{2}\right) = 1$

Therefore g has a maximum value of 1 at $x = \frac{1}{2}$, and g has no minimum value on the open interval $(0, \infty)$.

$$5 : \begin{cases} 2 : g'(x) \\ 1 : \text{critical point} \\ 1 : \text{answers} \\ 1 : \text{justification} \end{cases}$$

(b) $$\int_1^\infty (f(x) - g(x))\,dx = \lim_{b\to\infty}\int_1^b (f(x) - g(x))\,dx$$

$$= \lim_{b\to\infty}\left(\ln(x) - \frac{1}{2}\ln\left(1+4x^2\right)\right)\Big|_{x=1}^{x=b}$$

$$= \lim_{b\to\infty}\left(\ln(b) - \frac{1}{2}\ln\left(1+4b^2\right) + \frac{1}{2}\ln(5)\right)$$

$$= \lim_{b\to\infty}\ln\left(\frac{b\sqrt{5}}{\sqrt{1+4b^2}}\right)$$

$$= \lim_{b\to\infty}\ln\left(\frac{\sqrt{5b^2}}{\sqrt{1+4b^2}}\right)$$

$$= \frac{1}{2}\lim_{b\to\infty}\ln\left(\frac{5b^2}{1+4b^2}\right)$$

$$= \frac{1}{2}\ln\frac{5}{4}$$

$$4 : \begin{cases} 1 : \text{integral} \\ 2 : \text{antidifferentiation} \\ 1 : \text{answer} \end{cases}$$

Question 6

The Maclaurin series for the function f is given by $f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1}$ on its interval of convergence.

(a) Find the interval of convergence for the Maclaurin series of f. Justify your answer.

(b) Show that $y = f(x)$ is a solution to the differential equation $xy' - y = \frac{4x^2}{1+2x}$ for $|x| < R$, where R is the radius of convergence from part (a).

(a) $$\lim_{n\to\infty}\left|\frac{\frac{(2x)^{n+1}}{(n+1)-1}}{\frac{(2x)^n}{n-1}}\right| = \lim_{n\to\infty}\left|2x\cdot\frac{n-1}{n}\right| = \lim_{n\to\infty}\left|2x\cdot\frac{n-1}{n}\right| = |2x|$$

$|2x| < 1$ for $|x| < \frac{1}{2}$

Therefore the radius of convergence is $\frac{1}{2}$.

When $x = -\frac{1}{2}$, the series is $\sum_{n=2}^{\infty} \frac{(-1)^n(-1)^n}{n-1} = \sum_{n=2}^{\infty} \frac{1}{n-1}$.

This is the harmonic series, which diverges.

When $x = \frac{1}{2}$, the series is $\sum_{n=2}^{\infty} \frac{(-1)^n 1^n}{n-1} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1}$.

This is the alternating harmonic series, which converges.

The interval of convergence for the Maclaurin series of f is $\left(-\frac{1}{2}, \frac{1}{2}\right]$.

$$5 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{limit evaluation} \\ 1 : \text{radius of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$$

(b) $$y = \frac{(2x)^2}{1} - \frac{(2x)^3}{2} + \frac{(2x)^4}{3} - \cdots + \frac{(-1)^n(2x)^n}{n-1} + \cdots$$

$$= 4x^2 - 4x^3 + \frac{16}{3}x^4 - \cdots + \frac{(-1)^n(2x)^n}{n-1} + \cdots$$

$$y' = 8x - 12x^2 + \frac{64}{3}x^3 - \cdots + \frac{(-1)^n n(2x)^{n-1}\cdot 2}{n-1} + \cdots$$

$$xy' = 8x^2 - 12x^3 + \frac{64}{3}x^4 - \cdots + \frac{(-1)^n n(2x)^n}{n-1} + \cdots$$

$$xy' - y = 4x^2 - 8x^3 + 16x^4 - \cdots + (-1)^n(2x)^n + \cdots$$

$$= 4x^2\left(1 - 2x + 4x^2 - \cdots + (-1)^n(2x)^{n-2} + \cdots\right)$$

The series $1 - 2x + 4x^2 - \cdots + (-1)^n(2x)^{n-2} + \cdots = \sum_{n=0}^{\infty}(-2x)^n$ is a geometric series that converges to $\frac{1}{1+2x}$ for $|x| < \frac{1}{2}$. Therefore

$$xy' - y = 4x^2\cdot\frac{1}{1+2x} \text{ for } |x| < \frac{1}{2}.$$

$$4 : \begin{cases} 1 : \text{series for } y' \\ 1 : \text{series for } xy' \\ 1 : \text{series for } xy' - y \\ 1 : \text{analysis with geometric series} \end{cases}$$