Question 5

Let f and g be the functions defined by $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{4x}{1 + 4x^2}$, for all $x > 0$.

(a) Find the absolute maximum value of g on the open interval $(0, \infty)$ if the maximum exists. Find the absolute minimum value of g on the open interval $(0, \infty)$ if the minimum exists. Justify your answers.

(b) Find the area of the unbounded region in the first quadrant to the right of the vertical line $x = 1$, below the graph of f, and above the graph of g.

(a) $$g'(x) = \frac{4\left(1 + 4x^2\right) - 4x(8x)}{\left(1 + 4x^2\right)^2} = \frac{4\left(1 - 4x^2\right)}{\left(1 + 4x^2\right)^2}$$

For $x > 0$, $g'(x) = 0$ for $x = \dfrac{1}{2}$.

$g'(x) > 0$ for $0 < x < \dfrac{1}{2}$

$g'(x) < 0$ for $x > \dfrac{1}{2}$

$g\left(\dfrac{1}{2}\right) = 1$

Therefore g has a maximum value of 1 at $x = \dfrac{1}{2}$, and g has no minimum value on the open interval $(0, \infty)$.

$$5: \begin{cases} 2: g'(x) \\ 1: \text{critical point} \\ 1: \text{answers} \\ 1: \text{justification} \end{cases}$$

(b) $$\int_1^{\infty} (f(x) - g(x))\,dx = \lim_{b\to\infty} \int_1^b (f(x) - g(x))\,dx$$

$$= \lim_{b\to\infty} \left(\ln(x) - \frac{1}{2}\ln\left(1 + 4x^2\right)\right)\Bigg|_{x=1}^{x=b}$$

$$= \lim_{b\to\infty} \left(\ln(b) - \frac{1}{2}\ln\left(1 + 4b^2\right) + \frac{1}{2}\ln(5)\right)$$

$$= \lim_{b\to\infty} \ln\left(\frac{b\sqrt{5}}{\sqrt{1 + 4b^2}}\right)$$

$$= \lim_{b\to\infty} \ln\left(\frac{\sqrt{5b^2}}{\sqrt{1 + 4b^2}}\right)$$

$$= \frac{1}{2}\lim_{b\to\infty} \ln\left(\frac{5b^2}{1 + 4b^2}\right)$$

$$= \frac{1}{2}\ln\frac{5}{4}$$

$$4: \begin{cases} 1: \text{integral} \\ 2: \text{antidifferentiation} \\ 1: \text{answer} \end{cases}$$

Work for problem 5(a)

$g(x) = \frac{4x}{1+4x^2}$

$g'(x) = \frac{4 \cdot (4x^2+1) - 4x \cdot 8x}{(1+4x^2)^2}$

$= \frac{-16x^2+4}{(1+4x^2)^2} = -16x \frac{(x^2-\frac{1}{4})}{(1+4x^2)^2} = -16 \frac{(x-\frac{1}{2})(x+\frac{1}{2})}{(1+4x^3)^2}$

when $0<x<\frac{1}{2} \Rightarrow g'(x) > 0$.

when $x > \frac{1}{2} \Rightarrow g'(x) < 0$.

if $x \to \infty \Rightarrow g(x)$ converges to 0.

So there is no minimum value

but there is the maximum when $x = \frac{1}{2}$

$\hookrightarrow g(\frac{1}{2}) = \frac{2}{1+1} = 1$.

answer: no minimum

maximum = 1

(when $x = \frac{1}{2}$)

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NO CALCULATOR ALLOWED

Work for problem 5(b)

$$\text{area} = \int_1^\infty |g(x) - f(x)|\,dx = \int_1^\infty (f(x) - g(x))\,dx$$

$$= \int_1^\infty \left(\frac{1}{x} - \frac{4x}{1+4x^2}\right) dx.$$

$$= \left[\ln x - \frac{1}{2}\ln(1+4x^2)\right]_1^\infty$$

$$= \left[\ln \frac{x}{\sqrt{1+4x^2}}\right]_1^\infty$$

$$= \lim_{t\to\infty} \left[\ln \frac{x}{\sqrt{1+4x^2}}\right]_1^t$$

$$= \lim_{t\to\infty} \left(\ln \frac{t}{\sqrt{1+4t^2}} - \ln \frac{1}{\sqrt{5}}\right)$$

$$= \lim_{t\to\infty} \left(\ln \frac{1}{\sqrt{\frac{1}{t^2}+4}}\right) - \ln\frac{1}{\sqrt{5}}$$

$$= \ln\frac{1}{2} - \ln\frac{1}{\sqrt{5}} = \ln\frac{\sqrt{5}}{2}.$$

Answer: $\ln \frac{\sqrt{5}}{2}$.

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5B, 1

NO CALCULATOR ALLOWED

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Work for problem 5(a)

$$g'(x) = \frac{4-16x^2}{(1+4x^2)^2}$$

when $g'(x) = 0$, $g(x)$ reach extremas

$$4-16x^2 = 0$$

$$x = \pm\frac{1}{2}$$

$g(\frac{1}{2}) = 1$ $\quad$ $g(-\frac{1}{2}) = -1$

since $g(\frac{1}{2}) > g(-\frac{1}{2})$,

$g(\frac{1}{2}) = 1$ is the absolute maximum

$g(-\frac{1}{2}) = -1$ is the absolute minimum

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5B₂

NO CALCULATOR ALLOWED

Work for problem 5(b)

$f(x) = g(x)$

$\frac{1}{x} = \frac{4x}{1+4x^2}$

$4x^2 = 1+4x^2$

x does not exist

therefore there is no intersections between f(x) and g(x)

the area is equal to $\int_1^\infty f(x) - g(x)\, dx$

$\int_1^\infty f(x) - g(x)\, dx$

$= \int_1^\infty \frac{1}{x} - \frac{4x}{1+4x^2}\, dx$

$= \lim_{k\to\infty} \ln x - \frac{1}{2}\ln(1+4x^2) \Big|_1^k$

$=$

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NO CALCULATOR ALLOWED

Work for problem 5(a)

$g'(x) = \frac{8x \cdot 4x - (1+4x^2)\cdot 4}{(1+4x^2)^2 \cancel{4x^2\,8x}} = \frac{32x^2 - 16x^2 - 4}{(\cancel{8x}\,1+4x^2)^2} = \frac{16x^2-4}{(1+4x^2)^2} = \frac{4(4x^2-1)}{(1+4x^2)^2}$

$\cancel{\lim_{x\to 0} 16x^2 - 4 = -4}$

$\lim_{x\to 0} 1+4x^2 = 1$

$\therefore \cancel{\lim_{x\to 0} g'}$ when $g'(x)=0$ $x=\frac{1}{2}$

the maximum of $g(x)$ is $g(\frac{1}{2}) = 1$

$\cancel{g'(x)}\cdot$ $1+4x^2 \neq 0$

$\therefore$ the minimum doesn't exist.

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 5C₂

NO CALCULATOR ALLOWED

Work for problem 5(b)

the area = ~~is~~ . $\cancel{\int_1^{\infty} \left(\frac{1}{x} \quad \frac{4x}{1+4x^2} \right) dx}$

$$\int_1^{\infty} \frac{1}{x}\, dx + \int_1^{\infty} \frac{-4x}{1+4x^2}\, dx \ -$$

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Question 5

Sample: 5A
Score: 9

The student earned all 9 points.

Sample: 5B
Score: 6

The student earned 6 points: 3 points in part (a) and 3 points in part (b). In part (a) the student finds $g'(x)$ and the critical point, but the analysis and conclusion did not earn any points. In part (b) the student sets up the correct improper integral and antidifferentiates correctly. Since the evaluation is not completed, the answer point was not earned.

Sample: 5C
Score: 4

The student earned 4 points: 3 points in part (a) and 1 point in part (b). In part (a) the student finds $g'(x)$ with a reversal of terms in the numerator, so 1 point was earned. The student finds the critical number and the maximum value and also asserts that there is no minimum. The third and fourth points were earned. In part (b) the student sets up the improper integral but does not do any additional work.