Question 6

Solutions to the differential equation $\frac{dy}{dx} = xy^3$ also satisfy $\frac{d^2y}{dx^2} = y^3\left(1 + 3x^2y^2\right)$. Let $y = f(x)$ be a particular solution to the differential equation $\frac{dy}{dx} = xy^3$ with $f(1) = 2$.

(a) Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$.

(b) Use the tangent line equation from part (a) to approximate $f(1.1)$. Given that $f(x) > 0$ for $1 < x < 1.1$, is the approximation for $f(1.1)$ greater than or less than $f(1.1)$? Explain your reasoning.

(c) Find the particular solution $y = f(x)$ with initial condition $f(1) = 2$.

(a) $f'(1) = \left.\frac{dy}{dx}\right|_{(1,\,2)} = 8$

An equation of the tangent line is $y = 2 + 8(x - 1)$.

$2 : \begin{cases} 1 : f'(1) \\ 1 : \text{answer} \end{cases}$

(b) $f(1.1) \approx 2.8$

Since $y = f(x) > 0$ on the interval $1 \le x < 1.1$,

$\frac{d^2y}{dx^2} = y^3\left(1 + 3x^2y^2\right) > 0$ on this interval.

Therefore on the interval $1 < x < 1.1$, the line tangent to the graph of $y = f(x)$ at $x = 1$ lies below the curve and the approximation 2.8 is less than $f(1.1)$.

$2 : \begin{cases} 1 : \text{approximation} \\ 1 : \text{conclusion with explanation} \end{cases}$

(c) $\frac{dy}{dx} = xy^3$

$$\int \frac{1}{y^3}\,dy = \int x\,dx$$

$$-\frac{1}{2y^2} = \frac{x^2}{2} + C$$

$$-\frac{1}{2 \cdot 2^2} = \frac{1^2}{2} + C \Rightarrow C = -\frac{5}{8}$$

$$y^2 = \frac{1}{\frac{5}{4} - x^2}$$

$$f(x) = \frac{2}{\sqrt{5 - 4x^2}}, \quad \frac{-\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

6A.

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Work for problem 6(a)

at $x=1 \quad y=2$

$\frac{dy}{dx} = (1)(2)^3 = 8 \qquad m=8$

$y = mx + b$

$2 = 8(1) + b$

$b = -6$

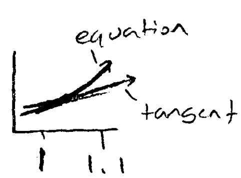
at $x=1$
tangent line
$\boxed{y = 8x - 6}$

Work for problem 6(b)

$y = 8(1.1) - 6$

$y = \boxed{2.8}$

$f(x) > 0$ means all points are positive from $1 < x < 1.1$. The second derivative is also positive from $1 < x < 1.1$ so the concavity is up. $\therefore$ the approximation is an $\boxed{\text{underestimate}}$



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6A2

Work for problem 6(c)

$$\frac{dy}{dx} = xy^3$$

$$\frac{dy}{y^3} = x\,dx \rightarrow y^{-3}dy = x\,dx$$

$$-\tfrac{1}{2}y^{-2} = \tfrac{1}{2}x^2 + C$$

$$-\tfrac{1}{2}(2)^{-2} = \tfrac{1}{2}(1)^2 + C$$

$$-\frac{1}{8} = \tfrac{1}{2} + C$$

$$\boxed{-5/8 = C}$$

$$-\tfrac{1}{2}y^{-2} = \tfrac{1}{2}x^2 - 5/8$$

$$y^{-2} = -x^2 + 5/4$$

$$\boxed{y = \frac{1}{\sqrt{-x^2 + 5/4}}}$$

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6B,

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Work for problem 6(a)

a) $x=1$, $y=2$ $\quad \frac{dy}{dx}=(1)(2)^3=8$

$y-2=8(x-1)$

$y=8x-6$

Work for problem 6(b)

b) $y=8(1.1)-6$

$y=8.8-6$

$y=2.8$

$2^3(1+3(1)^2(2)^2)=104$

$+$

It is an over approximation because the f'' at $x=1.1$ is $+$ which means $f(x)$ is concave up.

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6B$_2$

Work for problem 6(c)

c) $\frac{dy}{dx} = xy^3$

$\frac{dy}{y^3} = x\,dx$

$\frac{y^{-2}}{-2} = \frac{x^2}{2} + C$

$-\frac{1}{2y^2} = \frac{x^2}{2} + C$

$\frac{1}{y^2} = -x^2 + C$

$y^2 = \frac{1}{-x^2} + C$

$y = \sqrt{\frac{1}{-x^2}} + C$

$c = 1$

$y = \sqrt{\frac{1}{-x^2}} + 1$

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NO CALCULATOR ALLOWED

6C1

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Work for problem 6(a)

$\frac{dy}{dx} = xy^3 \quad f(1) = 2$

$1 \; 2^3 \quad \frac{dy}{dx} = 8$

$y - 2 = 8(x - 1)$

Work for problem 6(b)

$y - 2 = 8(1.1 - 1)$

$+2 \quad .8 \quad .1 \quad f(1.1) \approx 2.8$

$y = 2.8$

The approximation would be greater than because this approximation uses the equation of tangent line not the original equation.

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NO CALCULATOR ALLOWED

6C2

Work for problem 6(c)

$\frac{dy}{dx} = xy^3$

$\int xy^3\,dx$

$= \frac{1}{2}x^2 \frac{1}{4}y^4$

init cond $f(1)=2 = \frac{1}{2}(1)^2 \cdot \frac{1}{4}(2)^4$

$\frac{1}{2} \cdot \frac{4}{1} = 2$

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Question 6

Overview

This problem identified f as a particular solution to the differential equation $\frac{dy}{dx} = xy^3$ with $f(1) = 2$. It was also given that solutions to this differential equation satisfy $\frac{d^2y}{dx^2} = y^3\left(1 + 3x^2y^2\right)$. Part (a) asked for an equation of the line tangent to the graph of f at $x = 1$. Students needed to evaluate the given expression for $\frac{dy}{dx}$ at the point $(1, 2)$ to find the slope of this line. Part (b) asked for an approximation to $f(1.1)$ using the tangent line equation from part (a). Given that $f(x) > 0$ for $1 < x < 1.1$, students were asked to determine whether this approximation is greater than or less than $f(1.1)$. In order to make the determination, students needed to use the given second derivative together with the fact that f is positive on the interval to ascertain the relative position of the tangent line and the graph of $y = f(x)$ for $1 < x < 1.1$. Part (c) asked for the particular solution $y = f(x)$ with initial condition $f(1) = 2$. Students should have used the method of separation of variables.

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), and 3 points in part (c). In part (a) the student's work is correct. In part (b) the student earned the approximation point. A local argument for the explanation did not earn the second point. In part (c) the student earned the first 3 points. The student never uses the initial conditions, so no additional points were earned.

Sample: 6C
Score: 3

The student earned 3 points: 2 points in part (a), 1 point in part (b), and no points in part (c). In part (a) the student's work is correct. In part (b) the student earned the approximation point. The student's conclusion is not correct. In part (c) the student never separates the variables and was not eligible for any points.