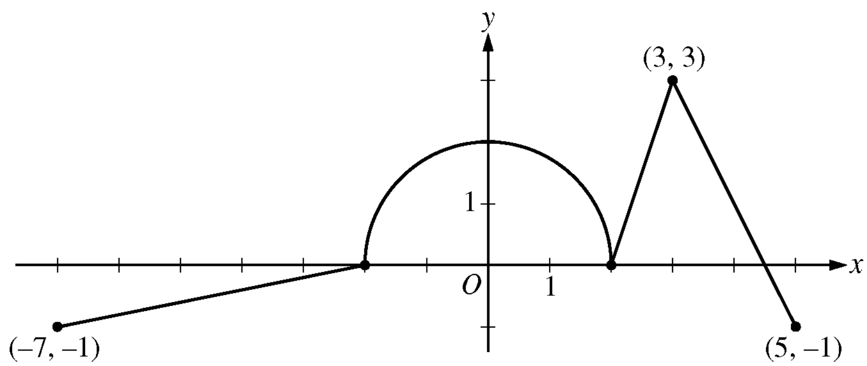
Question 5



Graph of g'

The function g is defined and differentiable on the closed interval $[-7, 5]$ and satisfies $g(0) = 5$. The graph of $y = g'(x)$, the derivative of g, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find $g(3)$ and $g(-2)$.

(b) Find the x-coordinate of each point of inflection of the graph of $y = g(x)$ on the interval $-7 < x < 5$. Explain your reasoning.

(c) The function h is defined by $h(x) = g(x) - \frac{1}{2}x^2$. Find the x-coordinate of each critical point of h, where $-7 < x < 5$, and classify each critical point as the location of a relative minimum, relative maximum, or neither a minimum nor a maximum. Explain your reasoning.

(a) $g(3) = 5 + \int_0^3 g'(x)\,dx = 5 + \frac{\pi \cdot 2^2}{4} + \frac{3}{2} = \frac{13}{2} + \pi$ $g(-2) = 5 + \int_0^{-2} g'(x)\,dx = 5 - \pi$	$3 : \begin{cases} 1 : \text{uses } g(0) = 5 \\ 1 : g(3) \\ 1 : g(-2) \end{cases}$
(b) The graph of $y = g(x)$ has points of inflection at $x = 0$, $x = 2$, and $x = 3$ because g' changes from increasing to decreasing at $x = 0$ and $x = 3$, and g' changes from decreasing to increasing at $x = 2$.	$2 : \begin{cases} 1 : \text{identifies } x = 0, 2, 3 \\ 1 : \text{explanation} \end{cases}$
(c) $h'(x) = g'(x) - x = 0 \Rightarrow g'(x) = x$ On the interval $-2 \le x \le 2$, $g'(x) = \sqrt{4 - x^2}$. On this interval, $g'(x) = x$ when $x = \sqrt{2}$. The only other solution to $g'(x) = x$ is $x = 3$. $h'(x) = g'(x) - x > 0$ for $0 \le x < \sqrt{2}$ $h'(x) = g'(x) - x \le 0$ for $\sqrt{2} < x \le 5$ Therefore h has a relative maximum at $x = \sqrt{2}$, and h has neither a minimum nor a maximum at $x = 3$.	$4 : \begin{cases} 1 : h'(x) \\ 1 : \text{identifies } x = \sqrt{2}, 3 \\ 1 : \text{answer for } \sqrt{2} \text{ with analysis} \\ 1 : \text{answer for 3 with analysis} \end{cases}$

 5A,

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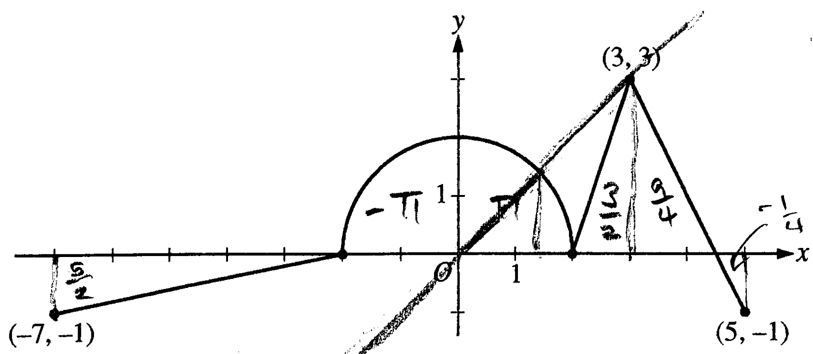

Graph of g'

Work for problem 5(a)

$\boxed{g(0)=5}$

$$g(3)-g(0)=\int_0^3 g'(x)\,dx$$

$$g(3)=5+\int_0^3 g'(x)\,dx$$

$$g(3)=5+\left(\pi+\frac{3}{2}\right)$$

$$\boxed{g(3)=\frac{13}{2}+\pi}$$

$$g(-2)-g(0)=\int_0^{-2} g'(x)\,dx$$

$$g(-2)=5+\int_0^{-2} g'(x)\,dx$$

$$g(-2)=5+(-\pi)$$

$$\boxed{g(-2)=5-\pi}$$

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Work for problem 5(b)

$\boxed{x = 0, 2, 3}$

g'(x) switches from increasing to decreasing or from decreasing to increasing at these three x-values.

Work for problem 5(c)

$h(x) = g(x) - \frac{1}{2}x^2$

$h'(x) = g'(x) - x$

Where $h'(x) = 0 \ldots$

$g'(x) - x = 0$

$g'(x) = x$

$\boxed{x = \sqrt{2}, 3}$

$x = \sqrt{2}$ is a relative maximum for $h(x)$ because $h'(x)$ switches from positive to negative at this point.

$x = 3$ is not a relative extremum for $h(x)$ because $h'(x)$ does not switch signs.

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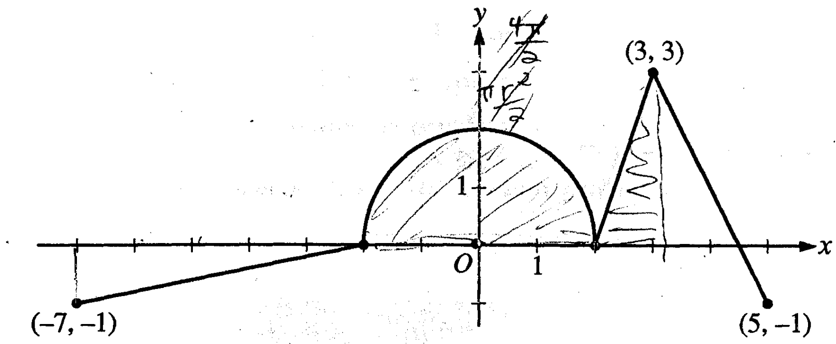

Graph of g'

Work for problem 5(a)

$g(3) = 5 + \int_0^3 g'(x)\,dx$

$= 5 + \left[\left(\frac{1}{2}(3)(1)\right) + \left(\frac{\pi \cancel{4}}{\cancel{4}}\right)\right]$

$= 5 + \frac{3}{2} + \pi$

$g(3) = \frac{13}{2} + \pi$

$A = \frac{\pi r^2}{4}$

$= \frac{2^2 \pi}{4}$

$= \pi$

$g(-2) = 5 + \int_0^{-2} g'(x)\,dx$

$= 5 - \int_{-2}^{0} g'(x)\,dx$

$g(-2) = 5 - \pi$

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Continue problem 5 on page 13.

Work for problem 5(b)

$g''(x) = 0$ @ $x = 0$

$g''(x) = \text{DNE}$ @ $x = 2$, $x = 3$, $x = -2$

$+ \nearrow \quad + \nearrow \quad - \searrow \quad + \nearrow \quad - \searrow$

$-7 \quad -2 \quad 0 \quad 2 \quad 3 \quad 5$

g has a pt of inflection @ $x = 0$, $x = 2$, $x = 3$ b/c g' goes from inc to dec, or vice versa, at those pts.

$t - 4$

Work for problem 5(c)

$g'(x) = x$

$h'(x) = g'(x) - x$

$0 = g'(x) - x$

$h'(x) = 0$ @ $x = 3$

$+ \nearrow \quad - \searrow$

$-7 \quad 6 \quad 3 \quad 4 \quad 5$

h has a critical pt @ $x=3$. At $x=3$ h has a rel. max b/c h' goes from pos to neg at that pt.

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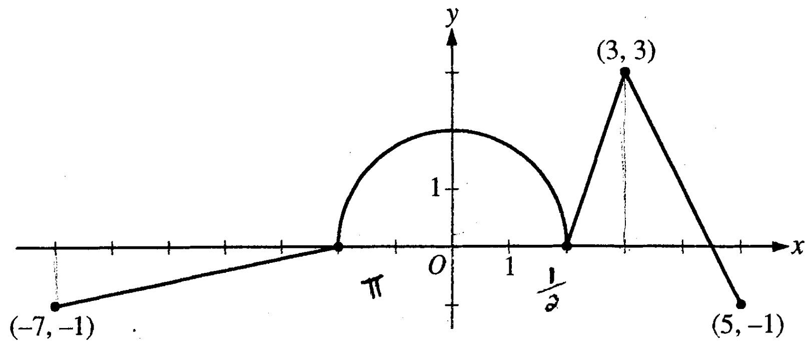

Graph of g'

Work for problem 5(a)

$$g(3) = 5 + \int_0^3 g'(x)\,dx$$

$$5 + \left(\tfrac{1}{2} \cdot 3 \cdot 1\right) + \tfrac{1}{4}\pi 2^2$$

$$g(3) = 5 + \tfrac{3}{2} + \pi$$

$$g(-2) = 5 - \int_{-2}^{0} g'(x)\,dx$$

$$\pi + \left(7 \cdot 1 \cdot \tfrac{1}{2}\right)$$

$$g(-2) = 5 - \tfrac{7}{2} + \pi$$

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Work for problem 5(b)

point of Inf. when g'' has a zero or when g' changes sign

g' changes sign at $x=0$, $x=2$ and $x=3$

$\therefore$ g has a point of inf. at $x=0$, $x=2$ and $x=3$

Work for problem 5(c)

critical pt when $h'(t)=0$

$h'(x) = g'(x) - x$

critical point at $x=3$

h ↗ | ↘
h' + 3 −

$\therefore$ there is a rel. max when $x=3$

Question 5

Overview

This problem described a function g that is defined and differentiable on $[-7, 5]$ and with $g(0) = 5.$ The graph of $y = g'(x)$ on $[-7, 5]$ was given, consisting of three line segments and a semicircle. Part (a) asked for values of $g(3)$ and $g(-2).$ These values are given by $5 + \int_0^3 g'(x)\,dx$ and $5 + \int_0^{-2} g'(x)\,dx,$ respectively, with the definite integrals computed using geometry and properties of definite integrals. Part (b) asked for the x-coordinates of points of inflection for the graph of $y = g(x)$ on the interval $-7 < x < 5.$ Students needed to reason graphically that these occur where the graph of g' changes from increasing to decreasing or vice versa. Part (c) introduced a new function $h(x) = g(x) - \frac{1}{2}x^2$ and asked for x-coordinates of critical points of h and for the classification of each critical point as the location of a relative minimum, relative maximum or neither. Students needed to find that $h'(x) = g'(x) - x$ in order to determine the x-coordinates of critical points and apply a sign analysis of h' to classify these critical points.

Sample: 5A
Score: 9

The student earned all 9 points.

Sample: 5B
Score: 6

The student earned 6 points: 3 points in part (a), 2 points in part (b), and 1 point in part (c). In parts (a) and (b), the student's work is correct. In part (c) the student earned the first point for correctly computing $h'(x).$ Since $x = \sqrt{2}$ is never identified, the student did not earn the second and third points. The fourth point was not earned since the student attempts to justify that $x = 3$ is the x-coordinate of a relative maximum.

Sample: 5C
Score: 4

The student earned 4 points: 2 points in part (a), 1 point in part (b), and 1 point in part (c). In part (a) the student earned the first 2 points for correctly using $g(0) = 5$ and computing $g(3).$ The third point was not earned since the value for $g(-2)$ is incorrect. In part (b) the student earned the first point for correctly identifying all three x-coordinates of the points of inflection. The student's statement that "g' changes sign" is not a justification for a point of inflection, so the second point was not earned. In part (c) the student earned the first point for correctly computing $h'(x).$ Since $x = \sqrt{2}$ is never identified, the student did not earn the second and third points. The fourth point was not earned since the student attempts to justify that $x = 3$ is the x-coordinate of a relative maximum.