Question 2

t (hours)	0	2	5	7	8
$E(t)$ (hundreds of entries)	0	4	13	21	23

A zoo sponsored a one-day contest to name a new baby elephant. Zoo visitors deposited entries in a special box between noon $(t = 0)$ and 8 P.M. $(t = 8)$. The number of entries in the box t hours after noon is modeled by a differentiable function E for $0 \le t \le 8$. Values of $E(t)$, in hundreds of entries, at various times t are shown in the table above.

(a) Use the data in the table to approximate the rate, in hundreds of entries per hour, at which entries were being deposited at time $t = 6$. Show the computations that lead to your answer.

(b) Use a trapezoidal sum with the four subintervals given by the table to approximate the value of $\frac{1}{8}\int_0^8 E(t)\,dt$.

Using correct units, explain the meaning of $\frac{1}{8}\int_0^8 E(t)\,dt$ in terms of the number of entries.

(c) At 8 P.M., volunteers began to process the entries. They processed the entries at a rate modeled by the function P, where $P(t) = t^3 - 30t^2 + 298t - 976$ hundreds of entries per hour for $8 \le t \le 12$. According to the model, how many entries had not yet been processed by midnight $(t = 12)$?

(d) According to the model from part (c), at what time were the entries being processed most quickly? Justify your answer.

(a) $E'(6) \approx \dfrac{E(7) - E(5)}{7 - 5} = 4$ hundred entries per hour

1 : answer

(b) $\frac{1}{8}\int_0^8 E(t)\,dt \approx$

$$\frac{1}{8}\left(2\cdot\frac{E(0)+E(2)}{2} + 3\cdot\frac{E(2)+E(5)}{2} + 2\cdot\frac{E(5)+E(7)}{2} + 1\cdot\frac{E(7)+E(8)}{2}\right)$$

$= 10.687$ or 10.688

$\frac{1}{8}\int_0^8 E(t)\,dt$ is the average number of hundreds of entries in the box between noon and 8 P.M.

$3 : \begin{cases} 1 : \text{trapezoidal sum} \\ 1 : \text{approximation} \\ 1 : \text{meaning} \end{cases}$

(c) $23 - \int_8^{12} P(t)\,dt = 23 - 16 = 7$ hundred entries

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(d) $P'(t) = 0$ when $t = 9.183503$ and $t = 10.816497$.

t	$P(t)$
8	0
9.183503	5.088662
10.816497	2.911338
12	8

Entries are being processed most quickly at time $t = 12$.

$3 : \begin{cases} 1 : \text{considers } P'(t) = 0 \\ 1 : \text{identifies candidates} \\ 1 : \text{answer with justification} \end{cases}$

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Work for problem 2(a)

$$\frac{dE(6)}{dt} \approx \frac{E(7)-E(5)}{7-5}$$

$$= \frac{21-13}{2}$$

$$= \frac{8}{2}$$

$$= 4$$

about 400 entries per hour at $t=6$.

Work for problem 2(b)

$$\frac{1}{8}\int_0^8 E(t)\,dt \approx \frac{1}{8}\left[\frac{1}{2}(2-0)(4+0)+\frac{1}{2}(5-2)(4+13)+\frac{1}{2}(7-5)(21+13)+\frac{1}{2}(8-7)(23+21)\right]$$

$$= \frac{1}{8}\left[\cancel{\tfrac{1}{2}}\cancel{(2)}(4)+\frac{1}{2}(3)(17)+\cancel{\tfrac{1}{2}}\cancel{(2)}(34)+\frac{1}{2}(1)(44)\right]$$

$$= \frac{1}{8}\left(4+\frac{51}{2}+34+22\right) = \frac{1}{8}\left(\frac{171}{2}\right) = \frac{171}{16} \text{ (or } 10.6875)$$

$\frac{1}{8}\int_0^8 E(t)\,dt$ is approximately 10.6875 [hundred entries].

$\frac{1}{8}\int_0^8 E(t)\,dt$ signifies the average value of the number of entries over the interval $0 \le t \le 8$ in hundreds

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Work for problem 2(c)

$\#$ of processed entries $= \int_8^{12} p(t)\,dt$

$= \int_8^{12} (t^3 - 30t^2 + 298t - 976)\,dt = 16$

$\therefore$ the # of unprocessed entries is $(E(8) - 16)$,* which is $23 - 16 = 7$ *in hundreds

$\boxed{\text{7 hundred entries had not yet been processed}}$

Work for problem 2(d)

when $P(t)$ is at maximum, ~~the~~ rate of process is the ~~fastest~~

$\rightarrow$ $P(t)$ @ local max, when $P'(t) = 0$ and $P''(t) < 0$

$P'(t) = 3t^2 - 60t + 2\cancel{95} = 0 \rightarrow t = 9.1835$ or $t = 10.8165$

$P''(t) = 6t - 60$

$P''(9.1835) = 6(9.1835) - 60 < 0$

$P''(10.8165) = 6(10.8165) - 60 > 0$ $\quad \therefore t = 10.8165$ is not valid

$\rightarrow$ $P(t)$ at end points (aka $t = 8$ or $t = 12$)

$P(8) = 0$

$P(12) = 8$ $\qquad P(12) > P(9.1835)$

and $P(9.1835) = 5.08866$

$\therefore P(t)$ is at maximum at $t = 12$.

$\boxed{\text{at midnight, the entries are being processed most quickly}}$

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t (hours)	0	2	5	7	8
$E(t)$ (hundreds of entries)	0	4	13	21	23

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Work for problem 2(a)

$$\frac{E(7)-E(5)}{7-5} = \frac{21-13}{2} = 4 \text{ hundreds/hr}$$

Work for problem 2(b)

$$\frac{1}{8}\int_0^8 E(t)\,dt$$

$$= \frac{1}{8}\left[\frac{(0+4)(2-0)}{2} + \frac{(4+13)(5-2)}{2} + \frac{(13+21)(7-5)}{2} + \frac{(21+23)(8-7)}{2}\right]$$

≈ 10.688 hundreds of entries

The average numbers of entries is about 10.688 hundreds

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Work for problem 2(c)

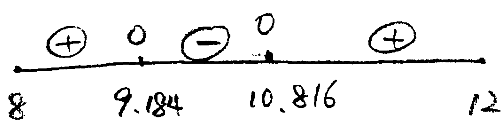
$\int_8^{12} P(t)\,dt = 16$ hundreds

$23 - 16 = 7$ hundreds

Work for problem 2(d)

$P'(t) = 3t^2 - 60t + 298$

$P'(t) = 0 \Rightarrow t \approx 9.184 \quad t \approx 10.816$



At $t = 9.184$, the entries were being processed most quickly because $P'(t) = 0$ at $t = 9.184$ and $P'(t)$ changes from positive to negative, which means local maximum occurs at $t = 9.184$, which was the time that entries processed most quickly

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Work for problem 2(a)

$$\frac{E(7)-E(5)}{7-5} = \frac{21-13}{7-5} = 4$$

$$\boxed{\approx 400 \text{ entries per hour}}$$

Work for problem 2(b)

$$\frac{(2-0)\left(\frac{4-0}{2}\right)+(5-2)\left(\frac{13-4}{2}\right)+(7-5)\left(\frac{21-13}{2}\right)+(8-7)\left(\frac{23-21}{2}\right)}{8}$$

$$\boxed{\approx 331.25 \text{ entries per hour}}$$

$\frac{1}{8}\int_0^8 E(t)\,dt$ is the average rate of entries per hour.

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Work for problem 2(c)

$$23 - \int_8^{12} P(t)\,dt = \boxed{700 \text{ entries}}$$

Work for problem 2(d)

$t=8$ $P(8)=0$

$t=9.1835$ $P(9.1835)=5.087$

$t=12$ $P(12)=8$

At $t=12$, the entries were being processed the quickest because the rate of change is largest there

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Question 2

Overview

This problem involved a zoo's contest to name a baby elephant. Students were presented with a table of values indicating the number of entries $E(t)$, measured in hundreds, received in a special box and recorded at various times t during an eight-hour period. Part (a) asked for an estimate of the rate of deposit of entries into the box at time $t = 6.$ Students needed to recognize this rate to be the derivative value $E'(6).$ Since $t = 6$ falls between the time values specified in the table, students needed to calculate the average rate of change of E across the smallest time subinterval from the table that brackets $t = 6.$ Part (b) asked for an approximation to $\frac{1}{8}\int_0^8 E(t)\,dt$ using a trapezoidal sum and the subintervals of $[0, 8]$ indicated by the data in the table. Students were further asked to interpret this expression in context, with the expectation that they would recognize that it gives the average number of hundreds of entries in the box during the eight-hour period. In part (c) a function P was supplied that models the rate at which entries from the box were processed, by the hundred, during a four-hour period ($8 \le t \le 12$) that began after all entries had been received. This part asked for the number of entries that remained to be processed after the four hours. Students needed to recognize that the number of entries processed is given by $\int_8^{12} P(t)\,dt,$ so that the number remaining to be processed, in hundreds of entries, is given by the difference between the total number of entries in the box, $E(8),$ as given by the table, and the value of this integral. Part (d) cited the model $P(t)$ introduced in the previous part and asked for the time at which the entries were being processed most quickly. Students should have recognized this as asking for the time corresponding to the maximum value of $P(t)$ on the interval $8 \le t \le 12$ and applied a standard process for optimization on a closed interval.

Sample: 2A
Score: 9

The student earned all 9 points.

Sample: 2B
Score: 6

The student earned 6 points: 1 point in part (a), 2 points in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the student sets up a correct difference quotient based on the values in the table and correctly evaluates for the numerical answer. In part (b) the student sets up a correct trapezoidal sum and evaluates it based on the data in the table to obtain a correct approximation. The student did not earn the third point in part (b) because the meaning given does not address the time interval over which the average was computed. In part (c) the student earned both points. The first point was earned for correctly providing the definite integral that represents the number of hundreds of entries processed between 8 P.M. and midnight. The second point was earned for subtracting that value from the initial condition of 23 hundred entries in the box at 8 P.M. to obtain the answer. In part (d) the student earned the first point for setting $P'(t) = 0.$ The student does not consider the endpoints as candidates, so no additional points were earned.

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Question 2 (continued)

Sample: 2C
Score: 3

The student earned 3 points: 1 point in part (a), no points in part (b), 2 points in part (c), and no points in part (d). In part (a) the student sets up a correct difference quotient based on the values in the table and correctly evaluates for the numerical answer. In part (b) the student subtracts the function values at endpoints of the subintervals rather than adding them. The student interprets the integral expression as an average rate rather than an average number. In part (c) the student's work is correct. "700" was accepted because of the units in this question. In part (d) the student never considers $P'(t)$, so no points were earned.

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