CollegeBoard AP

Student Performance Q&A:

2009 AP® Calculus AB and Calculus BC Free-Response Questions

The following comments on the 2009 free-response questions for AP® Calculus AB and Calculus BC were written by the Chief Reader, Michael Boardman of Pacific University in Forest Grove, Oregon. They give an overview of each free-response question and of how students performed on the question, including typical student errors. General comments regarding the skills and content that students frequently have the most problems with are included. Some suggestions for improving student performance in these areas are also provided. Teachers are encouraged to attend a College Board workshop to learn strategies for improving student performance in specific areas.

Question AB1/BC1

What was the intent of this question?

This problem opened with a piecewise-linear graph. The graph models the velocity function $v(t)$ for bicycle rider Caren during a 12-minute period in which she travels along a straight road, starting at home at time $t = 0$ and arriving at school at time $t = 12.$ Part (a) asked for Caren's acceleration at a particular time during her trip, which required students to recognize that acceleration is the derivative of velocity and to acquire the value of this derivative from the slope of the appropriate line segment on the given velocity graph. Part (b) asked for an interpretation of $\int_0^{12} |v(t)|\, dt$ in terms of Caren's trip, as well as for the value of this integral. Part (c) provided the additional information that Caren needed to return home to retrieve her homework shortly after starting her journey. Students needed to associate Caren's direction of motion with the sign of her velocity to determine at what time she turned around. (Students were not required to observe that the distances traveled in each direction match.) In part (d) the velocity function for another bicycle rider, Larry, was modeled by $w(t) = \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)$ for the same 12-minute period, $0 \le t \le 12.$ This part asked who lives closer to school, Caren or Larry. To respond, students needed to compute the two home-to-school distances, $\int_0^{12} v(t)\, dt$ (which equals $\int_5^{12} v(t)\, dt$) and $\int_0^{12} w(t)\, dt.$

How well did students perform on this question?

Most students performed well. The mean score was 4.67 for AB students and 6.15 for BC students out of a possible 9 points. About 4.6 percent of AB students and 9.4 percent of BC students earned all 9 points. About 9.6 percent of AB students and 1.9 percent of BC students did not earn any points.

In part (a) a large majority of students showed their work by writing a difference quotient for $a(7.5)$ and correctly evaluating it using values read from the graph. Students were successful at earning the units point.

The vast majority of students earned the answer point in part (c), and in part (d) most earned at least the 2 points for finding Larry's distance from school.

Some students had difficulty distinguishing between total distance and displacement. These students generally had trouble earning the 2 points in part (b) and the last point in part (d).

What were common student errors or omissions?

In part (b) some students struggled with the difference between distance and displacement. Typically, those students used words like *displacement* or *net change* when trying to explain the meaning of the integral, incorrectly calculated the value of the integral to be 1.4 miles, or made both errors.

In part (c) some students argued that they knew Caren turned around at time $t = 2$ since the areas between the t-axis and the graph of Caren's velocity on the intervals $0 \le t \le 2$ and $2 \le t \le 4$ are equal. This argument was insufficient as an explanation.

In part (d) some students used the same value for Caren's distance from school as they did for the value of the integral in part (b).

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Students need practice with describing in words the meaning of integral expressions involved in a context.
- Students should be reminded to show their work for all problems.

Question AB2/BC2

What was the intent of this question?

This problem presented students with a polynomial $R(t) = 1380t^2 - 675t^3$ that modeled the rate, in people per hour, at which people enter an auditorium during the two hours ($0 \le t \le 2$) prior to the start of a rock concert. It was stated that the auditorium was empty at time $t = 0,$ and part (a) asked for the number of people in the auditorium at time $t = 2,$ which required computation of the definite integral $\int_0^2 R(t)\,dt.$ In part (b) students needed to find the time t that maximizes $R(t).$ Part (c) defined the total wait time for all the people in the auditorium and stated that a function w that models the total wait time for all the people who entered the auditorium by time t has derivative $w'(t) = (2 - t)R(t).$ Students were asked to evaluate $w(2) - w(1)$ and should have recognized that this is computed by $\int_1^2 w'(t)\,dt.$

Part (d) asked for the average amount of time that a concertgoer spent waiting for the concert to begin after entering the auditorium. Students needed to compute the total wait time, $\int_0^2 w'(t)\,dt$, for all people attending the concert and divide this by the number of people in the auditorium at the start of the concert as found in part (a).

How well did students perform on this question?

Overall, students performed well. The mean score was 4.20 for AB students and 5.94 for BC students out of a possible 9 points. About 1.7 percent of AB students and 5.1 percent of BC students earned all 9 points. About 12.9 percent of AB students and 2.6 percent of BC students did not earn any points.

Most students set up and evaluated the correct integral for part (a).

Some students had difficulty with part (b), where they were asked to maximize a rate function.

Although the context for part (c) was rather difficult, most students recognized the need to compute a definite integral. Some students, however, had difficulty computing this integral.

Many students misunderstood what they were required to calculate in part (d).

What were common student errors or omissions?

In part (a) some students had difficulty showing that they were calculating a definite integral. These students wrote an indefinite integral but then proceeded to apply the Fundamental Theorem of Calculus later in their work. They would have been better served by writing proper notation for a definite integral from the start.

In part (b) some students set the function, rather than its derivative, equal to 0 in order to find critical points.

In this problem some students evaluated the definite integrals by hand rather than with a graphing calculator. Many of these students made mistakes in algebra or in antidifferentiation.

In part (d) many students did not understand the average they were to compute. A common error was to find the average value of $w'(t)$ rather than the average wait time per person.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- When the value of an integral is needed as part of an answer, students should always write the definite integral, using proper notation, as part of the required setup.
- Teachers should encourage students to include intermediate steps in their solutions to problems.
- Students need practice with a variety of methods for determining the location of a global maximum.

- Students need practice with communicating mathematics using clear, mathematically precise language.

Question AB3

What was the intent of this question?

This problem provided the context of a company, Mighty Cable, that manufactures and sells cables. Mighty sells cable for $120 per meter, and the cost of producing the portion of a length of cable that is x meters from the beginning of the cable is reported to be $6\sqrt{x}$ dollars per meter. Part (a) asked for Mighty's profit on the sale of a 25-meter cable, which is defined to be the difference between the revenue from selling the cable and the cost to produce it. To calculate the cost to produce the cable, a student should have recognized that $6\sqrt{x}$ represents the rate of change of production cost for the cable with respect to the distance x from the beginning of the cable and that integrating this rate of change of cost gives the total cost to produce the cable. Part (b) asked students to interpret the definite integral $\int_{25}^{30} 6\sqrt{x}\, dx$ in the context of the problem. In part (c) students were asked to write an expression involving an integral that represents Mighty's profit on the sale of a k-meter cable, thus generalizing part (a) with the parameter k in place of the constant 25. Part (d) asked for the length k that maximizes profit, which required students either to apply the Fundamental Theorem of Calculus to a qualifying answer from part (d) or to recognize that the rate of change of profit with respect to length k is the difference between rates of change of income ($120 per meter) and of production cost ($6\sqrt{k}$ dollars per meter).

How well did students perform on this question?

Overall, student performance was poor. The mean score was 1.92 out of a possible 9 points. Only 0.2 percent of students earned all 9 points. About 36.9 percent of students did not earn any points.

Many students earned all the points in part (a). Most students correctly interpreted the integral in part (b).

However, most students had difficulty with the general expression for profit in part (c). Many of the students who obtained an expression for profit in part (c) had difficulty justifying the global maximum in part (d).

What were common student errors or omissions?

In part (a) some students misinterpreted the cost function as the cost for a cable of length x rather than the marginal cost per meter of cable.

In part (b) some students did not give a precise, correct meaning of the integral.

In part (c) some students used the variable k both as a limit on the definite integral and as the variable of integration. Some students used $120x - 6\sqrt{x}$ as the integrand.

Many students had difficulty applying the Fundamental Theorem of Calculus in part (d). Many did not correctly justify that their maximum value was the global maximum value for profit.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Students need practice with problems set in a variety of different contexts.
- Students should understand a variety of calculus methods for determining a global maximum, including the case of an unbounded interval.
- Students should be advised to use calculus methods for determining extrema. It is not sufficient to view the graph of a function to determine the extrema.

Question AB4

What was the intent of this question?

Students were given the graph of a region R bounded by two curves in the *xy*-plane, $y = 2x$ and $y = x^2$. The points of intersection of the two curves were shown on the supplied graph. In part (a) students were asked to find the area of R, which required an appropriate integral (or difference of integrals), antiderivative, and evaluation. Part (b) asked students to find the volume of a solid whose cross-sectional area (perpendicular to the *x*-axis) at each x is given by $A(x) = \sin\left(\frac{\pi}{2}\right)$. Students had to set up the appropriate integral and find an antiderivative to evaluate the integral. Part (c) asked students to write, but not evaluate, an integral expression for the volume of a solid whose base is the region R and whose cross sections perpendicular to the *y*-axis are squares.

How well did students perform on this question?

Students performed well, particularly given that an area/volume problem has not appeared on the noncalculator portion of the free-response section in some time. The mean score was 4.07 out of a possible 9 points. About 5.5 percent of students earned all 9 points. About 9.8 percent of students did not earn any points.

Most students correctly computed the area in part (a). Many students did well in part (b), but many had difficulty with part (c).

What were common student errors or omissions?

In part (b) some students mistakenly multiplied the correct integral by π or by $\frac{4}{3}$, the area found in part (a). Other students misunderstood $A(x)$ and used $\left(2x - x^2\right)A(x)$ as the integrand. In both parts (a) and (b) some students had difficulty correctly finding an antiderivative.

In part (c) many students wrote an expression that involved cross sections perpendicular to the *x*-axis rather than perpendicular to the *y*-axis.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Students continue to need practice with paper-and-pencil techniques for evaluating definite integrals, as well as with using technology to evaluate them.

- When the value of an integral is needed as part of an answer, students should always write the definite integral, using proper notation, as part of the required setup.

- Teachers should emphasize a conceptual understanding of how integrating cross-sectional area leads to volume. This will help students tackle problems that go beyond volumes of revolution.

Question AB5/BC5

What was the intent of this question?

This problem presented students with a table of values for a function f sampled at five values of x. It was also stated that f is twice differentiable for all real numbers. Part (a) asked for an estimate for $f'(4)$. Since $x = 4$ falls between the values sampled on the table, students should have calculated the slope of the secant line to the graph of f corresponding to the closest pair of points in the supplied data that brackets $x = 4$. Part (b) tested students' ability to apply properties of the definite integral to evaluate $\int_2^{13} (3 - 5f'(x))\,dx$. Part (c) asked for an approximation to $\int_2^{13} f(x)\,dx$ using the subintervals of $[2, 13]$ indicated by the data in the table. In part (d) it was also stated that $f'(5) = 3$ and $f''(x) < 0$ for all x in $[5, 8]$. Students were asked to use the line tangent to the graph of f at $x = 5$ to show that $f(7) \le 4$ and to use the secant line for the graph of f on $5 \le x \le 8$ to show that $f(7) \ge \frac{4}{3}$. For the former inequality, students should have used the fact that f'' is negative (so f' is decreasing) on $[5, 8]$ so that the tangent line at $x = 5$ lies above the graph of f throughout $(5, 8]$. For the latter inequality, students should have used the sign of f'' to conclude that the indicated secant line lies below the graph of f for $5 < x < 8$; in particular, the point on the graph of the secant line corresponding to $x = 7$ is below the corresponding point on the graph of f.

How well did students perform on this question?

Overall, student performance was disappointing. Students who had difficulty with algebra and/or arithmetic performed poorly on all parts. The mean score was 2.74 for AB students and 4.71 for BC students out of a possible 9 points. About 1.7 percent of AB students and 9.0 percent of BC students earned all 9 points. About 26.5 percent of AB students and 7.9 percent of BC students did not earn any points.

Most students correctly estimated $f'(4)$ in part (a) and earned at least 1 point in part (b), but many students did not show sufficient work to earn both points.

Many students had difficulty with part (c). Some students correctly found the tangent line in part (d), but most students had difficulty with the secant line.

What were common student errors or omissions?

In each part of the problem many students did not show enough work to earn all the possible points. Many students had algebraic or arithmetic errors.

Some students had difficulty setting up the right Riemann sum in part (c). It was common for many students to include an extra term in their sum.

In part (d) some students correctly found the tangent and secant lines but had difficulty with their explanations as to why the inequalities held. A correct explanation needed to refer either directly to $f''(x) > 0$ on the interval or to the fact that the graph of $y = f(x)$ is concave down on the interval.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Students should be encouraged to check their algebraic and arithmetic calculations.
- Students need practice with writing explanations and justifications.
- Students should be able to construct various integral estimates from given tabular data for a function.
- Teachers should remind students to show intermediate steps that lead to their answers.

Question AB6

What was the intent of this question?

In this problem a function *f* satisfies $f(0) = 5$ and has continuous first derivative for $-4 \le x \le 4$. The graph of f' was supplied. For $-4 \le x \le 0$, the graph of f' is a semicircle tangent to the *x*-axis at $x = -2$ and tangent to the *y*-axis at $y = 2$. For $0 < x \le 4$, $f'(x) = 5e^{-x/3} - 3$. Part (a) asked for those values of x in the interval $-4 < x < 4$ at which the graph of *f* has a point of inflection; these correspond to points where the graph of f' changes from increasing to decreasing, or vice versa. In part (b) students had to use the given initial value for *f* and the appropriate piece of f' to find $f(-4)$ and $f(4)$. The former value required the evaluation of an integral using geometry, and the latter required the evaluation of an integral via an antiderivative. Part (c) asked for the value of x at which *f* attains its absolute maximum on the interval $[-4, 4]$. Using the derivative of *f*, students should have concluded that *f* is increasing on $\left[-4, 3\ln\left(\frac{5}{3}\right)\right]$ and decreasing on $\left[3\ln\left(\frac{5}{3}\right), 4\right]$, so that the maximum must occur at $x = 3\ln\left(\frac{5}{3}\right)$.

How well did students perform on this question?

Student performance was generally poor. The mean score was 2.07 out of a possible 9 points. Only 0.2 percent of students earned all 9 points. About 23.8 percent of students did not earn any points.

Many students were able to earn both points in part (a).

Part (b) proved difficult for most students, as there were many opportunities for arithmetic and calculus errors.

Many students earned 1 point in part (c) by correctly identifying the location of the absolute maximum for f, but most did not earn the second point.

What were common student errors or omissions?

In part (a) many students mistakenly identified the x-intercepts of f' as the location of inflection points for the graph of f.

Many students used the incorrect antiderivative $-\frac{5}{3}e^{-x/3} - 3x$ in part (b). Some students did not use the initial condition $f(0) = 5$.

Few students were able to give a correct justification for the location of the absolute maximum value of f in part (c). Most students gave a local argument rather than a global argument.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Students should be able to provide justification for global extreme values, as opposed to local extreme values. The fact that the derivative of a function changes from positive to negative or vice versa only indicates a local extreme value.
- Students need practice with writing explanations and justifications.
- Students should have practice and facility with the relationship $f(x) = f(a) + \int_a^x f'(t)\,dt$ for a differentiable function f on an interval containing a.

Question BC3

What was the intent of this question?

This problem described the path of a diver's shoulders during a dive from a platform into a pool, using parametric functions $x(t)$ for the horizontal distance from the edge of the platform and $y(t)$ for the vertical distance from the water. It was stated that the diver's shoulders were 11.4 meters above the water at the start, and that $\frac{dx}{dt} = 0.8$ and $\frac{dy}{dt} = 3.6 - 9.8t$ for $0 \le t \le A$, where A is the time that the diver's shoulders enter the water. Part (a) asked for the maximum height of the diver's shoulders above the water. Students needed to use the given derivative of y to determine when the diver's shoulders were highest and then to combine the initial height of the shoulders above the water with the appropriate integral to determine the maximum height. Part (b) asked for the time A that the diver's shoulders enter the water. This involved integrating to find the height of the shoulders above the water at time A and then solving for when this height was zero. Part (c) asked for the total distance the diver's shoulders traveled during the dive, which required an arclength calculation for the curve described by $x(t)$ and $y(t)$ from time 0 to the time A found in part (b). Part (d) asked for the acute angle between the path of the diver and the water's surface at time A. Students needed to find the slope of the path, $\frac{dy}{dx}$, at time A and then solve for the angle by realizing that its tangent matches the absolute value of this slope.

How well did students perform on this question?

The mean score was 3.70 out of a possible 9 points. About 3.5 percent of students earned all 9 points. About 15.9 percent of students did not earn any points.

Overall, students performed well on parts (a) and (b), with many students earning all the points in these parts.

In part (c) many students had difficulty working with a parametrically defined curve.

Many students earned the first point in part (d) but did not find the correct angle to earn the second point.

What were common student errors or omissions?

In part (a) some students did not show the work that resulted in the *t*-value they used to find the maximum height.

In part (b) many students did not include the initial height of 11.4 meters in their equation.

In part (c) many students tried to use the arclength formula for a path along the graph of $y = f(x)$, $\int_a^b \sqrt{1 + f'(x)^2}\, dx$, instead of using the formula for the arclength of a path presented parametrically, $\int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$.

In general, many students did not show the work that led to their answers.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Students need to show the intermediate steps that lead to their answers.
- Students should be reminded of the standard decimal presentation rule that is described in the instructions for the free-response section of the AP Calculus Exams.

- Students need practice with various types of contextual problems. Students should also be encouraged to reflect on the reasonableness of their answers in the context of a given problem.

Question BC4

What was the intent of this question?

This problem presented the differential equation $\frac{dy}{dx} = 6x^2 - x^2 y$ and a particular solution $y = f(x)$ satisfying $f(-1) = 5.$ Part (a) asked students to use Euler's method with two steps of equal size to approximate $f(0).$ In part (b) it was stated that $\left.\frac{d^2y}{dx^2}\right|_{(-1,\,2)} = -12,$ and students were asked to provide the second-degree Taylor polynomial for f about $x = -1.$ Part (c) asked for the particular solution $y = f(x).$

How well did students perform on this question?

The mean score was 4.70 out of a possible 9 points. About 9.5 percent of students earned all 9 points. About 17.3 percent of students did not earn any points.

Most students did well on part (a). Some students did not label their work in a way that made it clear they were using Euler's method, or they had an incorrect answer, thereby failing to earn points in part (a).

Many students earned the point in part (b).

Many students earned 3–5 points in part (c). Some students did not recognize that the equation was separable by factoring the right-hand side and as a result did not earn any points in this part.

What were common student errors or omissions?

In part (a) some students used a step-size of $-\frac{1}{2}$ in Euler's method rather than $\frac{1}{2}.$ Some students presented their work for Euler's method in a table but did not label the quantities. When there was an error, it was difficult to ascertain what those quantities represented and whether or not the student was correctly applying Euler's method.

In part (b) some students mistakenly centered the Taylor polynomial at either $x = 0$ or $x = 1.$

In part (c) some students did not recognize that the equation was separable. Many students incorrectly gave $\ln|6 - y|$ as an antiderivative of $\frac{1}{6 - y},$ leaving off the negative sign, and many students, after antidifferentiating, had difficulty correctly solving for $y.$

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Although the tabular approach can provide a useful framework for a student to work through the steps of Euler's method, students who use this approach must also be able to show how they obtained the values they entered into the table. Students should be encouraged to show their work in calculating values in the table regardless of the relative difficulty of the calculations.

- Students need practice and facility with simplifying exponential and logarithmic expressions.

Question BC6

What was the intent of this question?

This problem reminded students of the Maclaurin series for e^x and defined a function f by $f(x) = \dfrac{e^{(x-1)^2} - 1}{(x-1)^2}$ if $x \neq 1$ and $f(1) = 1.$ It was noted that f is continuous and has derivatives of all orders at $x = 1.$ Part (a) asked for the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$. Students could have found this by substituting $(x-1)^2$ for x in the Maclaurin series for $e^x.$ In part (b) students needed to manipulate the Taylor series from part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1.$ Part (c) asked students to apply the ratio test to determine the interval of convergence for the Taylor series found in part (b). Part (d) asked students to use the Taylor series from part (c) to determine whether f has any points of inflection. Students needed to differentiate the Taylor series term-by-term twice; then they should have concluded that the resulting series for f'' is nonnegative for all x, from which it follows that the graph of f has no points of inflection.

How well did students perform on this question?

Most students performed poorly. The mean score was 1.79 out of a possible 9 points. Only 0.8 percent of students earned all 9 points. About 38.7 percent of students did not earn any points.

Although many students were able to earn both points in part (a), many had difficulty earning points in part (b).

Since both parts (c) and (d) relied on the student's answer to part (b), many students had difficulty earning points in parts (c) and (d) because of the complexity of their incorrect answers in part (b).

What were common student errors or omissions?

Some students tried to find the Taylor polynomial in part (a) by computing derivatives of $e^{(x-1)^2}$ rather than using the given series for $e^x.$ These students usually were unsuccessful.

In part (b) many students had difficulty using the series for $e^{(x-1)^2}$ to find the series for $\frac{e^{(x-1)^2} - 1}{(x-1)^2}$. In particular, some students, rather than dropping the constant term 1 from the series for $e^{(x-1)^2}$, subtracted 1 from each term of their series for $e^{(x-1)^2}$. They then went on to divide each resulting term by $(x-1)^2$. This resulted in a very complicated series that was undefined at $x = 1$.

In part (d) many students included only the first few terms of the series for f in their reasoning as to why the graph of this function has no points of inflection.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Students need practice with creating new series from a given series, using techniques like substitution and algebraic manipulation.

- The topics listed in the *AP Calculus Course Description* for polynomial approximations and series are an important part of the Calculus BC course. Teachers should devote sufficient time to covering series topics, which provide the most challenging content of the BC course.