AP® CALCULUS BC
2009 SCORING GUIDELINES

Question 6

The Maclaurin series for e^x is $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$. The continuous function f is defined by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1$. The function f has derivatives of all orders at $x = 1$.

(a) Write the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$.

(b) Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(c) Use the ratio test to find the interval of convergence for the Taylor series found in part (b).

(d) Use the Taylor series for f about $x = 1$ to determine whether the graph of f has any points of inflection.

(a) $1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-1)^6}{6} + \cdots + \frac{(x-1)^{2n}}{n!} + \cdots$

$2: \begin{cases} 1: \text{first four terms} \\ 1: \text{general term} \end{cases}$

(b) $1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{6} + \frac{(x-1)^6}{24} + \cdots + \frac{(x-1)^{2n}}{(n+1)!} + \cdots$

$2: \begin{cases} 1: \text{first four terms} \\ 1: \text{general term} \end{cases}$

(c) $\lim\limits_{n\to\infty} \left| \dfrac{\dfrac{(x-1)^{2n+2}}{(n+2)!}}{\dfrac{(x-1)^{2n}}{(n+1)!}} \right| = \lim\limits_{n\to\infty} \frac{(n+1)!}{(n+2)!}(x-1)^2 = \lim\limits_{n\to\infty} \frac{(x-1)^2}{n+2} = 0$

Therefore, the interval of convergence is $(-\infty, \infty)$.

$3: \begin{cases} 1: \text{sets up ratio} \\ 1: \text{computes limit of ratio} \\ 1: \text{answer} \end{cases}$

(d) $$f''(x) = 1 + \frac{4 \cdot 3}{6}(x-1)^2 + \frac{6 \cdot 5}{24}(x-1)^4 + \cdots + \frac{2n(2n-1)}{(n+1)!}(x-1)^{2n-2} + \cdots$$

Since every term of this series is nonnegative, $f''(x) \geq 0$ for all x. Therefore, the graph of f has no points of inflection.

$2: \begin{cases} 1: f''(x) \\ 1: \text{answer} \end{cases}$

NO CALCULATOR ALLOWED

Work for problem 6(a)

$$1+(x-1)^2+\frac{(x-1)^4}{2}+\frac{(x-1)^6}{6}+\cdots+\frac{(x-1)^{2n}}{n!}+\cdots$$

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Work for problem 6(b)

$(x-1)^2$

$$1+\frac{(x-1)^2}{2}+\frac{(x-1)^4}{6}+\frac{(x-1)^6}{24}+\cdots+\frac{(x-1)^{2n}}{(n+1)!}+\cdots$$

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NO CALCULATOR ALLOWED

Work for problem 6(c)

$$\lim_{n\to\infty}\left|\frac{(x-1)^{2(n+1)}}{(n+2)!}\cdot\frac{(n+1)!}{(x-1)^{2n}}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{2n+2}}{(n+2)(x-1)^{2n}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(x-1)^2}{(n+2)}\right| = \lim_{n\to\infty}\left|\frac{x^2-2x+1}{n+2}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\frac{x^2}{n}-\frac{2x}{n}+\frac{1}{n}}{1+\frac{2}{n}}\right| = \lim_{n\to\infty}\frac{0}{1} = 0<1 \qquad \boxed{(-\infty,\infty)}$$

converges for all x.

Work for problem 6(d)

$$f'(x) = (x-1)+\frac{2}{3}(x-1)^3+\frac{1}{4}(x-1)^5+\cdots+\frac{2n-1}{(n+1)!}(x-1)^{2n-1}+\cdots$$

$$f''(x) = 1+2(x-1)^2+\frac{5}{4}(x-1)^4+\cdots+\frac{2n(2n-1)}{(n+1)!}(x-1)^{2n-2}+\cdots$$

$f''(x)$ is always positive so f has no points of inflection.

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NO CALCULATOR ALLOWED

Work for problem 6(a)

$$P_4(x) = 1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-1)^6}{6} + \frac{(x-1)^8}{24}$$

Work for problem 6(b)

$$f(x) = \frac{1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-\cancel{6})^6}{6} + \frac{(x-1)^8}{24} - 1}{(x-1)^2}$$

$$= \frac{(x-1)^2}{(x-1)^2} + \frac{(x-1)^4}{2(x-1)^2} + \frac{(x-1)^6}{6(x-1)^2} + \frac{(x-1)^8}{24(x-1)^2}$$

$$= 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{6} + \frac{(x-1)^6}{24}$$

general term: $\sum_{n=1}^{\infty} \frac{(x-1)^{2n-2}}{n!}$

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Work for problem 6(c)

$$\lim_{n\to\infty}\left|\frac{(x-1)^{2n}}{(n+1)!}\cdot\frac{n!}{(x-1)^{2n-2}}\right| = (x-1)^2\lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\right|$$

$$= \left|(x-1)^2\right|\lim_{n\to\infty}\left|\frac{n!}{(n+1)n!}\right|$$

$$= \left|(x-1)^2\right|\lim_{n\to\infty}\frac{1}{n+1}$$

$$= \left|(x-1)^2\right| \cdot 0$$

interval of convergence $(-\infty,\infty)$

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Work for problem 6(d)

$$P_4(x) = 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{6} + \frac{(x-1)^6}{24}$$

$$P_4'(x) = \frac{2(x-1)}{2} + \frac{4(x-1)^3}{6} + \frac{6(x-1)^5}{24} + 8$$

$$= (x-1) + \frac{2(x-1)^3}{3} + \frac{(x-1)^5}{4}$$

$$P_4''(x) = \frac{2}{3} + \frac{6(x-1)^2}{3} + \frac{5(x-1)^4}{4}$$

$$0 = \frac{2}{3} + 2(x-1)^2 + \frac{5(x-1)^4}{4}$$

$$-\frac{2}{3} = (x-1)^2(2 + \tfrac{5}{4}(x-1)^2)$$

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NO CALCULATOR ALLOWED

Work for problem 6(a)

$$e^x = \sum \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$e^{(x-1)^2} = 1 + (x-1)^2 + \frac{((x-1)^2)^2}{2!} + \frac{((x-1)^2)^3}{3!} + \ldots$$

$$\boxed{e^{(x-1)^2} = 1 + (x-1)^2 + \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \ldots \frac{(x-1)^{2n}}{n!} + \ldots}$$

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Work for problem 6(b)

$$\frac{\left(\cancel{1} + (x-1)^2 + \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \ldots\right) \cancel{-1}}{(x-1)^2} = \frac{\cancel{(x-1)^2} + \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \ldots}{\cancel{(x-1)^2}}$$

$$= \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \frac{(x-1)^8}{4!} + \ldots$$

$$T(x) = \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \frac{(x-1)^8}{4!} + \ldots \frac{(x-1)^{2n+2}}{(n+2)!} + \ldots$$

NO CALCULATOR ALLOWED

Work for problem 6(c)

$2n+2+2 \quad 2n+4$

$$\left| \frac{(x-1)^{2(n+1)+2}}{(\cancel{n+3})!} \cdot \frac{(\cancel{n+2})!}{(x-1)^{2n+2}} \right|$$

$$= (x-1)^2 \cdot \lim_{n\to\infty} \frac{1}{n+3}$$

$$= (x-1)^2 \cdot \infty$$

interval = $\boxed{-\infty < x < \infty}$

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Work for problem 6(d)

$f''(1) > 0$ and $f'(1)$ is zero based on taylor series

∴ since $f''(1) > 0$ and $f'(1) = 0$, there is a point of inflection at $x=1$ by second derivative test.

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Question 6

Overview

This problem reminded students of the Maclaurin series for e^x and defined a function f by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ if $x \neq 1$ and $f(1) = 1$. It was noted that f is continuous and has derivatives of all orders at $x = 1$. Part (a) asked for the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$. Students could have found this by substituting $(x-1)^2$ for x in the Maclaurin series for e^x. In part (b) students needed to manipulate the Taylor series from part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$. Part (c) asked students to apply the ratio test to determine the interval of convergence for the Taylor series found in part (b). Part (d) asked students to use the Taylor series from part (c) to determine whether f has any points of inflection. Students needed to differentiate the Taylor series term-by-term twice; then they should have concluded that the resulting series for f'' is nonnegative for all x, from which it follows that the graph of f has no points of inflection.

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 1 point in part (a), 2 points in part (b), 3 points in part (c), and no points in part (d). In part (a) the student earned the first point since the first five terms are correct. The student does not include a general term. In parts (b) and (c) the student's work is correct. In part (d) the student's second derivative is incorrect.

Sample: 6C
Score: 4

The student earned 4 points: 2 points in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the response is missing the first term of the first four nonzero terms of the series, so the first point was not earned. The student's general term is correct and earned the point. In part (c) the student earned the first point for the correct ratio. The limit is computed incorrectly. Although the interval of convergence provided is the correct answer, it does not follow from the student's limit. In part (d) the student does not find a series for $f''(x)$.