Question 4

Consider the differential equation $\frac{dy}{dx} = 6x^2 - x^2 y$. Let $y = f(x)$ be a particular solution to this differential equation with the initial condition $f(-1) = 2$.

(a) Use Euler's method with two steps of equal size, starting at $x = -1$, to approximate $f(0)$. Show the work that leads to your answer.

(b) At the point $(-1, 2)$, the value of $\frac{d^2y}{dx^2}$ is -12. Find the second-degree Taylor polynomial for f about $x = -1$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(-1) = 2$.

(a) $$f\left(-\frac{1}{2}\right) \approx f(-1) + \left(\left.\frac{dy}{dx}\right|_{(-1,\,2)}\right) \cdot \Delta x$$
$$= 2 + 4 \cdot \frac{1}{2} = 4$$

$$f(0) \approx f\left(-\frac{1}{2}\right) + \left(\left.\frac{dy}{dx}\right|_{\left(-\frac{1}{2},\,4\right)}\right) \cdot \Delta x$$
$$\approx 4 + \frac{1}{2} \cdot \frac{1}{2} = \frac{17}{4}$$

$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{answer} \end{cases}$

(b) $P_2(x) = 2 + 4(x+1) - 6(x+1)^2$

1 : answer

(c) $$\frac{dy}{dx} = x^2(6 - y)$$
$$\int \frac{1}{6-y}\,dy = \int x^2\,dx$$
$$-\ln|6 - y| = \frac{1}{3}x^3 + C$$
$$-\ln 4 = -\frac{1}{3} + C$$
$$C = \frac{1}{3} - \ln 4$$
$$\ln|6 - y| = -\frac{1}{3}x^3 - \left(\frac{1}{3} - \ln 4\right)$$
$$|6 - y| = 4e^{-\frac{1}{3}\left(x^3+1\right)}$$
$$y = 6 - 4e^{-\frac{1}{3}\left(x^3+1\right)}$$

$6 : \begin{cases} 1 : \text{separation of variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration
Note: $0/6$ if no separation of variables

NO CALCULATOR ALLOWED

CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

Work for problem 4(a)

Let Δx = step size = .5

$f(-.5) = f(-1) + (6(-1)^2 - (-1)^2(2)).5$

$= 2 + (6-2).5 = 2+2 = 4 = f(-.5)$

$f(0) \approx f(-.5) + (6(-.5)^2 - (-.5)^2(4)).5$

$= 4 + (6(\frac{1}{4}) - (\frac{1}{4})(4))(.5)$

$= 4 + (\frac{6}{4} - \frac{4}{4})(\frac{1}{2}) = 4 + (\frac{1}{2})(\frac{1}{2}) = \boxed{4 + \frac{1}{4}}$

Work for problem 4(b)

$\frac{dy}{dx}\Big]_{(-1,2)} = 6(1) - (1)(2) = 4$

$\frac{d^2y}{dx^2}\Big]_{(-1,2)} = -12$

$f(-1) = 2$

$\boxed{P(x) = 2 + 4(x+1) - \frac{12(x+1)^2}{2}} = 2 + 4(x+1) - 6(x+1)^2$

Do not write beyond this border.

Continue problem 4 on page

NO CALCULATOR ALLOWED

Work for problem 4(c)

$\frac{dy}{dx} = 6x^2 - x^2y$

$\frac{dy}{dx} = x^2(6-y)$

$\int \frac{dy}{(6-y)} = \int x^2 dx$

$-\ln|6-y| = \frac{x^3}{3} + C$

$\ln|6-y| = \frac{-x^3}{3} + C$

Using point $(-1, 2)$

$\ln(4) = \frac{1}{3} + C$

$C = \ln 4 - \frac{1}{3}$

$\ln|6-y| = \frac{-x^3}{3} + \ln 4 - \frac{1}{3}$

$6 - y = e^{-x^3/3 + \ln 4 - \frac{1}{3}}$

$\boxed{y = -e^{-\frac{x^3}{3} + \ln 4 - \frac{1}{3}} + 6}$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

4B.

4 [redacted] 4 [redacted] 4 [redacted] 4 [redacted] 4 [redacted]

NO CALCULATOR ALLOWED

CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

Work for problem 4(a)

$\frac{dy}{dx} = 6x^2 - x^2y \quad f(x) = y \quad f(-1) = 2$

$y_{new} = \left(\frac{dy}{dx}\right)(\text{step size}) + y_{old} \quad$ step size $= .5$

$(6x^2 - x^2y)(.5) + y_{old}$

so $\boxed{\cancel{so}\ f(0) = 4.25}$

step 1 $\quad 6(-1)^2 - (-1)^2(2)(.5) + 2 = 4$

$(6-2)(.5)$

$2 + 2 = 4$

so at $x = -.5 \quad y = 4$

step 2 $\quad (6(-.5)^2 - (-.5)^2(4))(.5) + 4 =$

$(6(.25) - (.25)(4))(.5) + 4$

$\left(\frac{6}{4} - \cancel{\frac{4}{4}}\right)(.5) + 4 =$

$\frac{1}{2}(.5) + 4 =$

$.25 + 4 = 4.25$

Work for problem 4(b)

$\frac{dy}{dx} = 6x^2 - x^2y$

$x^2(6-y)$

$\frac{dy}{6-y} = x^2\,dx$

$-\int\frac{-dy}{-y+6} = \int x^2 dx$

$-\ln|y+6| = \frac{x^3}{3} + C$

$y+6 = e^{-\frac{x^3}{3}} + C$

$y = e^{-\frac{x^3}{3}} + C$

$f(-1) = 2$

$f'(-1) = 4$

$f''(-1) = -12$

$\frac{f^{(n)}(c)}{n!}(x-c)^{(n)}$

$6(-1)^2 - (-1)^2(2) = f'(-1)$

$6 - 2 = 4$

$P(x) = 4(x+1)^1 - 6(x+1)^2$

Do not write beyond this border.

4B2

4 4 4 4 4 4 4 4 4 4

NO CALCULATOR ALLOWED

Work for problem 4(c)

$y = f(x)$

$$\frac{dy}{dx} = 6x^2 - x^2 y$$

$$= x^2(6-y)$$

$$\frac{dy}{6-y} = x^2\,dx$$

$$-\int \frac{dy}{-y+6} = \int x^2\,dx$$

$$-\ln|y+6| = \frac{x^3}{3} + C$$

$$y+6 = e^{-\frac{x^3}{3}} + C$$

$$y = e^{-\frac{x^3}{3}} + C$$

$$2 = e^{-\frac{(-1)^3}{3}} + C$$

$$2 = e^{\frac{1}{3}} + C$$

$$\ln 2 = \frac{1}{3} + C$$

$$\ln 2 - \frac{1}{3} = C$$

or

$$2 - e^{\frac{1}{3}} = C$$

$$y = e^{-\frac{x^3}{3}} + \left(\ln 2 - \frac{1}{3}\right)$$

or

$$y = e^{-\frac{x^3}{3}} + \left(2 - e^{\frac{1}{3}}\right)$$

Do not write beyond this border.

GO ON TO THE NEXT PAGE

4C1

4 4 4 4 4 4 4 4 4 4

NO CALCULATOR ALLOWED

CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

Do not write beyond this border.

Work for problem 4(a)

$\frac{dy}{dx} = 6x^2 - x^2y$

$f(-1) = 2$

$6(-1)^2 - (-1)^2(2)$

$(6) - (2) = 4$

$.25 \times 6 = 1.50$ $.5 \times .5 = .25$ $.25 \times 2 = .50$

$6(-0.5)^2 - (-0.5)^2(2)$

$(6)(0.25) - (0.25)(2)$

$(1.50) - (.50)$

$= 1$

$y - y_1 = m(x - x_1)$

$y - 2 = 4(-.5 + 1)$

$y - 2 = 2$

$y = 2$

$(-0.5, 2)$

$y - 2 = 1(0 + (-0.5))$

$y - 2 = 0.5$

$+2 \quad +2$

$y = 2.5$

$\boxed{f(0) = 2.5}$

Work for problem 4(b)

$(-1, 2)$ $\frac{d^2y}{dx^2} = -12$

Continue problem 4 on page 1

4C2

NO CALCULATOR ALLOWED

Work for problem 4(c)

$x \quad y$
$f(-1)=2$

$$\frac{dy}{dx} = 6x^2 - x^2 y$$

$$dy = 6x^2 - x^2 y \, dx$$

$$\frac{dy}{y} = 6x^2 - x^2 dx$$

$$\int \frac{dy}{y} = \int 6x^2 - x^2 \, dx$$

$$\ln y = \frac{6x^3}{3} - \frac{x^3}{3} + C$$

$$e^{\ln y} = e^{2x^3 - \frac{x^3}{3}} + C$$

$$y = e^{2x^3 - \frac{x^3}{3}} + C$$

$$y = e^{2x^3 - \frac{x^3}{3}} \cdot e^C$$

$-1 \cdot -1 = 1 \cdot -1 = -1$

$$y = C \cdot e^{2x^3 - x^3/3}$$

$$2 = C \cdot e^{2(-1)^3 - (-1)^3/3}$$

$\frac{3(-2)}{3 \cdot 1} - \left(\frac{-1}{3}\right)$

$$2 = C \cdot e^{(-2) - (-1/3)}$$

$\frac{(-6) - (-1)}{3}$

$$2 = Ce^{-5/3}$$

$\frac{-6+1}{3} = \frac{-5}{3}$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Question 4

Overview

This problem presented the differential equation $\frac{dy}{dx} = 6x^2 - x^2 y$ and a particular solution $y = f(x)$ satisfying $f(-1) = 5$. Part (a) asked students to use Euler's method with two steps of equal size to approximate $f(0)$. In part (b) it was stated that $\left.\frac{d^2y}{dx^2}\right|_{(-1,\,2)} = -12$, and students were asked to provide the second-degree Taylor polynomial for f about $x = -1$. Part (c) asked for the particular solution $y = f(x)$.

Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: 2 points in part (a), no points in part (b), and 4 points in part (c). In part (a) the student's work is correct. In part (b) the student neglects to include the constant term in the polynomial. In part (c) the student earned the first point for separating the variables correctly. The student earned only 1 of the antiderivative points since the antidifferentiation of the expression that results in the logarithm is incorrect. The student earned the constant of integration and initial condition points. The student makes errors in attempting to find the specific constant of integration and was not eligible for the last point since the logarithm of a difference is not included (either $\ln|6 - y|$, $\ln(6 - y)$, or $\ln|y - 6|$.)

Sample: 4C
Score: 4

The student earned 4 points: 1 point in part (a), no points in part (b), and 3 points in part (c). In part (a) the student earned the first point for indicating two steps of Euler's method. The student makes an arithmetic error in calculating the y-coordinate in the first step and did not earn the answer point. In part (c) the student did not earn the separation of variables point. Because the separation results in the reciprocal of a linear function in y and a nontrivial function of x, the student was eligible for both antiderivative points. In this case, the student earned only 1 of the antiderivative points since the absolute value is required for all other arguments of the logarithmic function except $6 - y$. The student earned the constant of integration and initial condition points. The student was not eligible for the last point since the antiderivative in y does not result in $\ln(6 - y)$ or $\ln(y - 6)$.