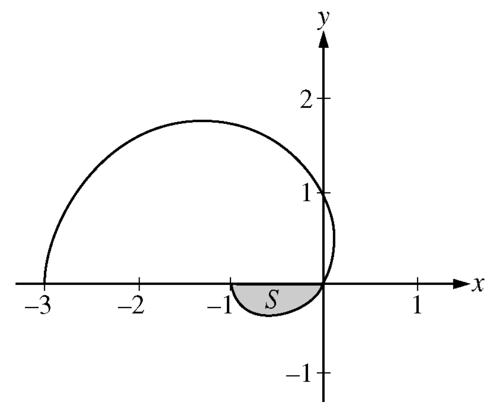
AP® CALCULUS BC
2009 SCORING GUIDELINES (Form B)

Question 4



The graph of the polar curve $r = 1 - 2\cos\theta$ for $0 \le \theta \le \pi$ is shown above. Let S be the shaded region in the third quadrant bounded by the curve and the x-axis.

(a) Write an integral expression for the area of S.

(b) Write expressions for $\dfrac{dx}{d\theta}$ and $\dfrac{dy}{d\theta}$ in terms of θ.

(c) Write an equation in terms of x and y for the line tangent to the graph of the polar curve at the point where $\theta = \dfrac{\pi}{2}$. Show the computations that lead to your answer.

(a) $r(0) = -1$; $r(\theta) = 0$ when $\theta = \dfrac{\pi}{3}$.

Area of $S = \dfrac{1}{2}\displaystyle\int_0^{\pi/3} (1 - 2\cos\theta)^2 \, d\theta$

$2 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \end{cases}$

(b) $x = r\cos\theta$ and $y = r\sin\theta$

$$\frac{dr}{d\theta} = 2\sin\theta$$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta = 4\sin\theta\cos\theta - \sin\theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta = 2\sin^2\theta + (1 - 2\cos\theta)\cos\theta$$

$4 : \begin{cases} 1 : \text{uses } x = r\cos\theta \text{ and } y = r\sin\theta \\ 1 : \dfrac{dr}{d\theta} \\ 2 : \text{answer} \end{cases}$

(c) When $\theta = \dfrac{\pi}{2}$, we have $x = 0$, $y = 1$.

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\frac{\pi}{2}} = -2$$

The tangent line is given by $y = 1 - 2x$.

$3 : \begin{cases} 1 : \text{values for } x \text{ and } y \\ 1 : \text{expression for } \dfrac{dy}{dx} \\ 1 : \text{tangent line equation} \end{cases}$

NO CALCULATOR ALLOWED

CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

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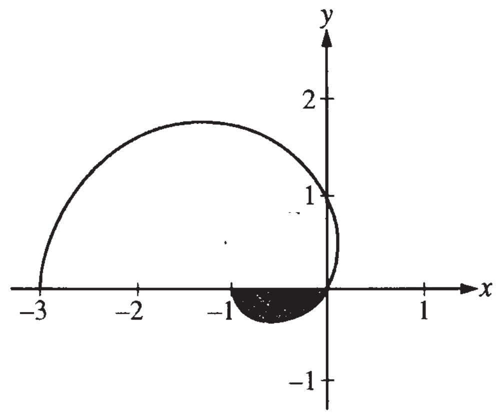
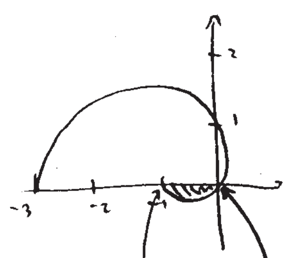


Work for problem 4(a)

$$\int_0^{\frac{1}{3}\pi} \frac{1}{2} r^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{1}{3}\pi} (1-2\cos\theta)^2 \, d\theta$$



$r=-1$ $r=0$.

$1-2\cos\theta=-1$ $\therefore 1-2\cos\theta=0$

$\cos\theta=1$ $\cos\theta=\frac{1}{2}$

$\therefore \theta=0$, $\theta=\frac{1}{3}\pi$

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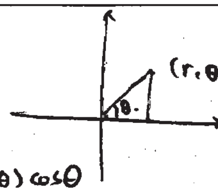
Work for problem 4(b)



$r = f(\theta)$

$(r, \theta) \Rightarrow (r\cos\theta,\ r\sin\theta)$
$= (f(\theta)\cos\theta,\ f(\theta)\sin\theta)$

$r = 1 - 2\cos\theta = f(\theta)$

$x = r \cdot \cos\theta = f(\theta)\cos\theta$

$y = r \cdot \sin\theta = f(\theta)\sin\theta$

$f(\theta) = 1 - 2\cos\theta \quad f'(\theta) = 2\sin\theta$

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta = 2\sin\theta\cos\theta - (1-2\cos\theta)\sin\theta$$
$$= 2\sin 2\theta - \sin\theta \qquad (\because 2\sin\theta\cos\theta = \sin 2\theta)$$

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta = 2\sin^2\theta + (1-2\cos\theta)\cos\theta$$
$$= 2\sin^2\theta + \cos\theta - 2\cos^2\theta$$

Work for problem 4(c)

$\theta = \frac{\pi}{2}$ $\left(\begin{array}{l} x = r\cdot\cos\theta = (1-2\cos\theta)\cos\theta \\ \quad = 0 \\ y = r\cdot\sin\theta = (1-2\cos\theta)\sin\theta \\ \quad = 1 \end{array} \right.$

line tangent:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2\sin^2\theta + \cos\theta - 2\cos^2\theta}{2\sin 2\theta - \sin\theta}.$$

$\therefore$ where $\theta = \frac{\pi}{2}$ $\quad \frac{dy}{dx} = \frac{2\sin^2(\frac{\pi}{2}) + \cos\frac{\pi}{2} - 2\cos^2(\frac{\pi}{2})}{2\sin\pi - \sin\frac{\pi}{2}} = \frac{2}{-1} = -2.$

$\therefore$ line tangent : $y - 1 = -2(x - 0)$

$\Rightarrow \quad y = -2x + 1$

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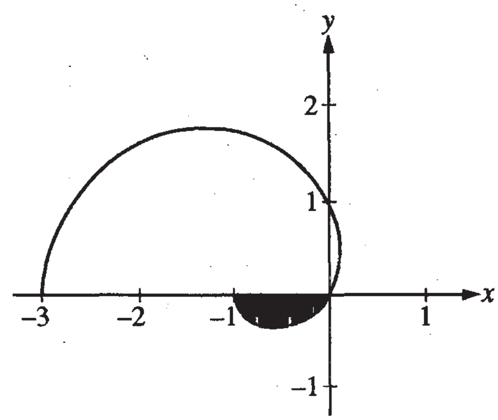

Work for problem 4(a)

$1-2\cos\theta = 0 \quad \cos\theta = \frac{1}{2} \quad \theta = \frac{\pi}{6}$

$1-2\cos\theta = -1 \quad \cos\theta = 1 \quad \theta = 0.$

$$S = \frac{1}{2}\int_0^{\frac{\pi}{6}} r^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} (1-2\cos\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} (1-4\cos\theta + 4\cos^2\theta)\, d\theta.$$

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NO CALCULATOR ALLOWED

Work for problem 4(b)

$x = r\cos\theta$

$= (1-2\cos\theta)\cos\theta$

$= \cos\theta - 2\cos^2\theta$

$2\cos^2\theta = \cos 2\theta + 1$

$\frac{dx}{d\theta} = -\sin\theta + 2\sin 2\theta$

$y = r\sin\theta$

$= (1-2\cos\theta)\sin\theta$

$= \sin\theta - \sin 2\theta$

$\frac{dy}{d\theta} = \cos\theta - 2\cos 2\theta$

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Work for problem 4(c)

$r = 1 - 2\cos\frac{\pi}{2} = 1$

when $\theta = \frac{\pi}{2}$ the coordinate of the point is $(0, 1)$

the coordinate of the center of the circle that pass through point $(0,1)$ and $(0,0)$ is $(-\sqrt{2}, 0)$.

$(x+\sqrt{2})^2 + (y+\sqrt{2})^2 = 1$

$x^2 + 2\sqrt{2}x + 2 + y^2 + 2\sqrt{2}y + 2 = 1$

$x^2 + y^2 + 2\sqrt{2}(x+y) = -3$

$2x + 2y\frac{dy}{dx} + 2\sqrt{2} + 2\sqrt{2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{2x+2\sqrt{2}}{2y+2\sqrt{2}} = -\frac{2\sqrt{2}}{2+2\sqrt{2}}$

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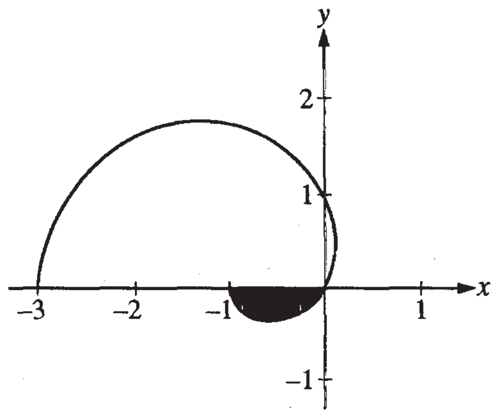

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Work for problem 4(a)

$1-2\cos\theta=0$ when $\cos\theta=\frac{1}{2}$. $\therefore \theta = \frac{\pi}{4}$ when $r=0$.

~~Area~~ $\int_0^{\frac{\pi}{4}}(1-2\cos\theta)\,d\theta$ is negative though, because the graph is under the x-axis.

therefore, $S = -\int_0^{\frac{\pi}{4}}(1-2\cos\theta)\,d\theta$

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Continue problem 4 on page 11.

NO CALCULATOR ALLOWED

Work for problem 4(b)

$x = r\cos\theta$

$\therefore \frac{dx}{d\theta} = -r\sin\theta$

$y = r\sin\theta$

$\therefore \frac{dy}{d\theta} = r\cos\theta$

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Work for problem 4(c)

~~tangent line =~~

when $\theta = \frac{\pi}{2}$, $\cos\theta = 0$.

$\therefore r = 1$.

$\therefore (x, y) = (0, 1)$.

$\therefore$ tangent line: $(y-1) = \frac{dy}{dx}(x-0)$.

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r\cos\theta}{-r\sin\theta} = -\cot\theta$.

$\therefore y-1 = -\cot\theta \cdot x$

GO ON TO THE NEXT PAGE.

Question 4

Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: 1 point in part (a), 4 points in part (b), and 1 point in part (c). In part (a) the student earned the integrand point, but the student's limits are incorrect. In part (b) the student's work is correct. Prior to differentiating, the student uses a trigonometric identity to rewrite the expression for x in terms of θ. Although $\frac{dr}{d\theta}$ is not explicitly stated, the student earned the $\frac{dr}{d\theta}$ point. In part (c) the student earned the first point for the coordinates of the point of tangency.

Sample: 4C
Score: 3

The student earned 3 points: no points in part (a), 1 point in (b), and 2 points in part (c). In part (a) the student's work is incorrect. In part (b) the student earned the first point. In part (c) the student's third line earned the first point. The second point is conceptual. The student earned the point by importing incorrect derivatives from part (b) and combining them correctly to form $\frac{dy}{dx}$. The student did not earn the third point in part (c).