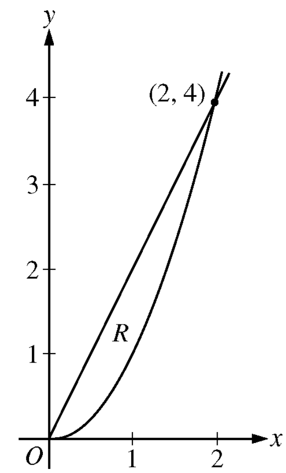
Question 4

Let R be the region in the first quadrant enclosed by the graphs of $y = 2x$ and $y = x^2$, as shown in the figure above.



(a) Find the area of R.

(b) The region R is the base of a solid. For this solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \sin\left(\frac{\pi}{2}x\right)$. Find the volume of the solid.

(c) Another solid has the same base R. For this solid, the cross sections perpendicular to the y-axis are squares. Write, but do not evaluate, an integral expression for the volume of the solid.

(a) Area $= \int_0^2 \left(2x - x^2\right) dx$

$$= x^2 - \frac{1}{3}x^3 \Big|_{x=0}^{x=2}$$

$$= \frac{4}{3}$$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \int_0^2 \sin\left(\frac{\pi}{2}x\right) dx$

$$= -\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)\Big|_{x=0}^{x=2}$$

$$= \frac{4}{\pi}$$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(c) Volume $= \int_0^4 \left(\sqrt{y} - \frac{y}{2}\right)^2 dy$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits} \end{cases}$

NO CALCULATOR ALLOWED

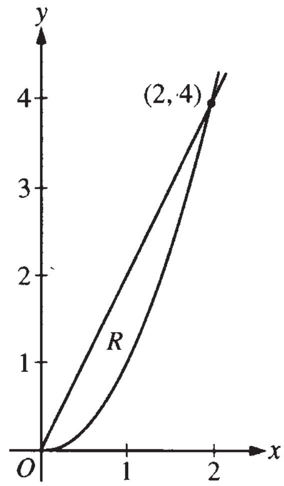
CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



Work for problem 4(a)

$y = 2x \quad y = x^2$

$A = \int_0^2 (2x) - (x^2)\,dx$

$= x^2 - \frac{1}{3}x^3 \Big]_0^2$

$= 4 - \frac{1}{3}(8)$

$= 4 - \frac{8}{3} = \frac{12}{3} - \frac{8}{3} = \boxed{\frac{4}{3}}$

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Continue problem 4 on page 11.

NO CALCULATOR ALLOWED

Work for problem 4(b)

$V = \int A(x)$

$V = \int_0^2 \left[\sin\left(\frac{\pi}{2}x\right)\right] dx \qquad u = \frac{\pi}{2}x$

$\frac{2}{\pi} du = \cancel{\frac{\pi}{2}} du$

$V = \frac{2}{\pi}\left[-\cos\frac{\pi}{2}x\right]_0^2$

$= \frac{2}{\pi}\left[-\cos\pi + \cos 0\right)$

$= \frac{2}{\pi}\left[1 + 1\right] = \boxed{\frac{4}{\pi}}$

Work for problem 4(c)

$V = \int A(x) \qquad A(x) = s^2$

$s = \sqrt{y} - \frac{1}{2}y$

$\boxed{V = \int_0^4 \left[\sqrt{y} - \frac{1}{2}y\right]^2 dy}$

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NO CALCULATOR ALLOWED

CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

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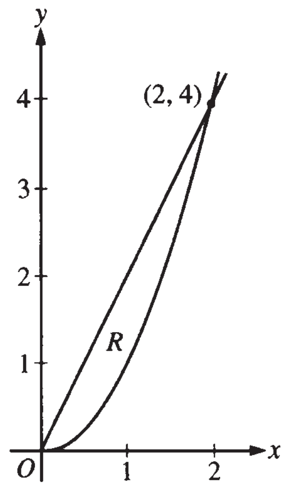

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Work for problem 4(a)

$$\int_0^2 2x - x^2\,dx$$

$$x^2 - \frac{1}{3}x^3\Big]_0^2$$

$$\left(2^2 - \frac{1}{3}2^3\right) - \left(0^2 - \frac{1}{3}0^3\right)$$

$$4 - \frac{1}{3}\cdot 8$$

$$\frac{12}{3} - \frac{8}{3}$$

$$\boxed{\frac{4}{3}\text{ units}^2}$$

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Continue problem 4 on page 11.

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Work for problem 4(b)

$$\int_0^2 \sin\left(\frac{\pi}{2}x\right) dx$$

$$-\cos\left(\frac{\pi}{2}x\right)\cdot\frac{\pi}{2}\Big]_0^2$$

$$-\frac{\pi}{2}\left[\cos\frac{\cancel{2}\pi}{\cancel{2}} - \cos 0\right]$$

$$-\frac{\pi}{2}\left[-1-1\right]$$

$$\cancel{-}\frac{\pi}{\cancel{2}}(\cancel{-}\cancel{2}) = \boxed{\pi \text{ units}^3}$$

Work for problem 4(c)

$$\int_0^2 s^2 dx$$

$$s = 2x - x^2$$

$$\int_0^2 (2x-x^2)^2 dx$$

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NO CALCULATOR ALLOWED

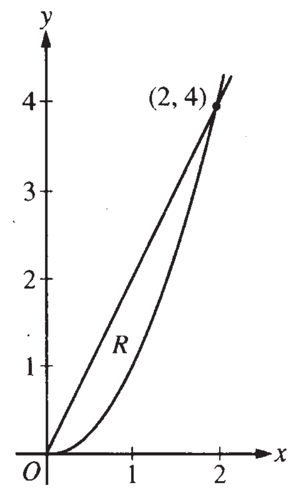
CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



Work for problem 4(a)

Area of R = $\int_0^2 (2x)dx - \int_0^2 (x^2)dx$

$\left[(x^2) - \frac{1}{3}x^3\right]_0^2$

$4 - \frac{8}{3}$

$\frac{12}{3} - \frac{8}{3} = \frac{4}{3}$

$\boxed{\text{Area of R: } 1.667}$

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NO CALCULATOR ALLOWED

Work for problem 4(b)

$A(x) = \sin\left(\frac{\pi}{2}x\right)$

$\int_0^2 (A(x))dx = \text{volume}$

$\int_0^2 \sin\left(\frac{\pi}{2}x\right)dx$

$\left[\frac{\pi}{2}\cos(\frac{\pi}{2}x)\right]_0^2$

$\frac{\pi}{2}\cos(\pi) = \boxed{\frac{\pi}{2}}$

Work for problem 4(c)

$A = s^2$

$s = \sqrt{\sin\left(\frac{\pi}{2}x\right)}$

$V = \left(\sqrt{\sin\left(\frac{\pi}{2}x\right)}\right)^3$

$\int_0^2 \left(\sin\frac{\pi}{2}x\right)^{3/2} dx$

$V = \left(\sqrt{\sin\left(\frac{\pi}{2}x\right)}\right)^3$

$\left(\left(\sin\frac{\pi}{2}x\right)^{\frac{1}{2}}\right)^3$

$\left(\sin\frac{\pi}{2}x\right)^{\frac{3}{2}}$

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AP® CALCULUS AB
2009 SCORING COMMENTARY

Question 4

Overview

Students were given the graph of a region R bounded by two curves in the xy-plane, $y = 2x$ and $y = x^2$. The points of intersection of the two curves were shown on the supplied graph. In part (a) students were asked to find the area of R, which required an appropriate integral (or difference of integrals), antiderivative, and evaluation. Part (b) asked students to find the volume of a solid whose cross-sectional area (perpendicular to the x-axis) at each x is given by $A(x) = \sin\left(\frac{\pi}{2}\right)$. Students had to set up the appropriate integral and find an antiderivative to evaluate the integral. Part (c) asked students to write, but not evaluate, an integral expression for the volume of a solid whose base is the region R and whose cross sections perpendicular to the y-axis are squares.

Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: 3 points in part (a), 2 points in part (b), and 1 point in part (c). In part (a) the student's work is correct. In part (b) the student earned the first point for a correct integrand. The student's u-substitution is incorrect. The student was eligible for and earned the answer point. In part (c) the student's answer is correct for the volume of the solid with square cross sections perpendicular to the $\underline{x\text{-axis}}$. This special case of $\int_0^2 \left(2x - x^2\right)^2\, dx$ earned 1 point.

Sample: 4C
Score: 3

The student earned 3 points: 2 points in part (a), 1 point in part (b), and no points in part (c). In part (a) the student earned the first 2 points. The student incorrectly reports the correct answer of $\frac{4}{3}$ as 1.667. In part (b) the student earned the first point for a correct integrand. The student's u-substitution is incorrect, and the student was not eligible for the answer point since $\frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right)\Big|_0^2$ is negative. In part (c) the student did not earn any points for the integrand since it is not of the form $(f(y) - g(y))^2$. The student's limits are incorrect.