AP® CALCULUS AB
2009 SCORING GUIDELINES

Question 3

Mighty Cable Company manufactures cable that sells for \$120 per meter. For a cable of fixed length, the cost of producing a portion of the cable varies with its distance from the beginning of the cable. Mighty reports that the cost to produce a portion of a cable that is x meters from the beginning of the cable is $6\sqrt{x}$ dollars per meter. (Note: Profit is defined to be the difference between the amount of money received by the company for selling the cable and the company's cost of producing the cable.)

(a) Find Mighty's profit on the sale of a 25-meter cable.

(b) Using correct units, explain the meaning of $\int_{25}^{30} 6\sqrt{x}\, dx$ in the context of this problem.

(c) Write an expression, involving an integral, that represents Mighty's profit on the sale of a cable that is k meters long.

(d) Find the maximum profit that Mighty could earn on the sale of one cable. Justify your answer.

(a) Profit $= 120 \cdot 25 - \int_0^{25} 6\sqrt{x}\, dx = 2500$ dollars	$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$
(b) $\int_{25}^{30} 6\sqrt{x}\, dx$ is the difference in cost, in dollars, of producing a cable of length 30 meters and a cable of length 25 meters.	1 : answer with units
(c) Profit $= 120k - \int_0^{k} 6\sqrt{x}\, dx$ dollars	$2: \begin{cases} 1: \text{integral} \\ 1: \text{expression} \end{cases}$
(d) Let $P(k)$ be the profit for a cable of length k. $P'(k) = 120 - 6\sqrt{k} = 0$ when $k = 400$. This is the only critical point for P, and P' changes from positive to negative at $k = 400$. Therefore, the maximum profit is $P(400) = 16{,}000$ dollars.	$4: \begin{cases} 1: P'(k) = 0 \\ 1: k = 400 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$

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Work for problem 3(a)

For a 25-meter cable,

Total cost = $\int_0^{25} 6\sqrt{x}\, dx$

= \$500.

Total sell = $120 \cdot 25$ = \$3000

Profit = 3000 − 500

= $\boxed{\$2500}$

Work for problem 3(b)

According to the problem, $\int_{25}^{30} 6\sqrt{x}\, dx$ represents the total cost in dollars. of producing the part of a cable starting from 25 meters from the beginning of the cable to 30 meters from the beginning of the cable

Continue problem 3 on page 9.

Work for problem 3(c)

Profit =

$$P(k) = 120k - \int_0^k 6\sqrt{x}\,dx$$

$$= 120k - 6\int_0^k x^{\frac{1}{2}}\,dx$$

$$= 120k - 6\left(\frac{1}{\frac{3}{2}}x^{\frac{3}{2}} + C\right)$$

$$= 120k - \left(4x^{\frac{3}{2}} + C\right)$$

$\because$ when $x = 0$, total cost $= 0$, $\therefore C = 0$

$\therefore \boxed{P(k) = 120k - 4k^{\frac{3}{2}} \text{ is the equation}}$

Work for problem 3(d)

According to the profit equation,

$P'(k) = 120 - 6k^{\frac{1}{2}}$, when $k = 400$m, $P'(k) = 0$ which is the only critical pt

Interval	$(0, 400)$	$(400, +\infty)$
Sign of p'	$+$	$-$
Behavior	inc.	dec.

Since p' changes sign from positive to negative at $k = 400$ m, it is where the maximum occurs

when $k = 400$m, $P(k) = \boxed{\$16000}$

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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3B, 1

Work for problem 3(a)

revenue = \$120(25) = \$3,000

Cost = \$6 $\sqrt{25}$ (25) = \$750

profit = revenue - cost = \$3,000 - \$750 = $\boxed{\$2,250}$

Work for problem 3(b)

$\int_{25}^{30} 6\sqrt{x}\,dx$ is the total number of dollars it costs the company to produce the portion of a cable from 25 meters to 30 meters.

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Continue problem 3 on page 9.

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Work for problem 3(c)

$P(k) = 120k - \int_0^k (6\sqrt{x})\,dx$

Work for problem 3(d)

$P'(k) = 120 - 6\sqrt{k} = 0$

$120 = 6\sqrt{k}$

$20 = \sqrt{k}$

$400 = k$

$P(400) = 120(400) - \int_0^{400} 6\sqrt{x}\,dx$

$\boxed{\text{max profit} = \$16,000}$

$P'(k)$: + | − at 400

This is the max profit because the derivative of profit equals zero here and is changing from positive to negative.

END OF PART A OF SECTION II

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Work for problem 3(a)

production cost $= \$\,6\sqrt{x}/m$ $\quad x =$ distance from end

price $= \$120/m$

production(25) $= \int_0^{25} 6\sqrt{x}\,dx$

$\boxed{\approx 500 \text{ dollars}}$

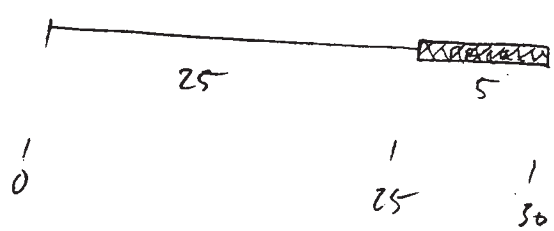
price(25) $= 120/m = \$120 \cdot 25m$

$\boxed{\approx 3000 \text{ dollars}}$

$\boxed{\text{profit} = \$2500}$

Work for problem 3(b)

$\int_{25}^{30} 6\sqrt{x}\,dx$ is the price of producing a 5m length of cable, starting 25m from the end (the shaded portion).



Continue problem 3 on page 9

Work for problem 3(c)

$$\boxed{\int_0^K 120x - 6\sqrt{x}\,dx}$$

4.

$.7 \times \cancel{70}$

Work for problem 3(d)

$P(t) = \int_0^K 120x - 6\sqrt{x}\,dx$ $\qquad x^{1/2}$

$P'(t) = 120K - 6\sqrt{K} = 0$

$6\sqrt{K} = 120K$

$\frac{6}{120} = \frac{K}{\sqrt{K}}$

$\frac{1}{2} = K^1(K^{.5})$

$\frac{1}{2} = K^{1/2}$

$\boxed{K \approx .7}$

$P'(t) = 120K - 6\sqrt{K}$

$P''(t) = 120 - 3/\sqrt{K}$

When $P'(t) = 0$ + $P''(t)$ is negative that is profit.

$P''(.7) = 120 - 3/\sqrt{\cancel{.7}}$

$120 - 3/$

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END OF PART A OF SECTION II

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Question 3

Overview

This problem provided the context of a company, Mighty Cable, that manufactures and sells cables. Mighty sells cable for $120 per meter, and the cost of producing the portion of a length of cable that is x meters from the beginning of the cable is reported to be $6\sqrt{x}$ dollars per meter. Part (a) asked for Mighty's profit on the sale of a 25-meter cable, which is defined to be the difference between the revenue from selling the cable and the cost to produce it. To calculate the cost to produce the cable, a student should have recognized that $6\sqrt{x}$ represents the rate of change of production cost for the cable with respect to the distance x from the beginning of the cable and that integrating this rate of change of cost gives the total cost to produce the cable. Part (b) asked students to interpret the definite integral $\int_{25}^{30} 6\sqrt{x}\, dx$ in the context of the problem. In part (c) students were asked to write an expression involving an integral that represents Mighty's profit on the sale of a k-meter cable, thus generalizing part (a) with the parameter k in place of the constant 25. Part (d) asked for the length k that maximizes profit, which required students either to apply the Fundamental Theorem of Calculus to a qualifying answer from part (d) or to recognize that the rate of change of profit with respect to length k is the difference of rates of change of income ($120 per meter) and of production cost ($6\sqrt{k}$ dollars per meter).

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

The student earned 6 points: no points in part (a), 1 point in part (b), 2 points in part (c), and 3 points in part (d). In part (a) the student makes no reference to an integral. In parts (b) and (c) the student's work is correct. In part (d) the student earned the first point with the equation $P'(k) = 120 - 6\sqrt{k} = 0.$ The student earned the second and third points by correctly solving for k and finding the maximum profit of $16,000. The justification point was not earned because the student uses a local argument.

Sample: 3C
Score: 4

The student earned 4 points: 2 points in part (a), no points in part (b), 1 point in part (c), and 1 point in part (d). In part (a) the student's work is correct. The student earned the answer point but does not explicitly state $3000 - 500 = 2500.$ In part (b) the student does not use the unit of dollars. In part (c) the student earned the first point for presenting an expression involving $\int_0^k 6\sqrt{x}\, dx.$ The student incorrectly uses $120x$ in the integrand and did not earn the second point. In part (d) the student earned the first point with the equation $P'(k) = 120k - 6\sqrt{k} = 0.$ Although this profit equation is incorrect, the student was rewarded for correctly handling the imported incorrect expression from part (c). The student solves the equation incorrectly. The student's function does not have an absolute maximum, so the student was not eligible for additional points.