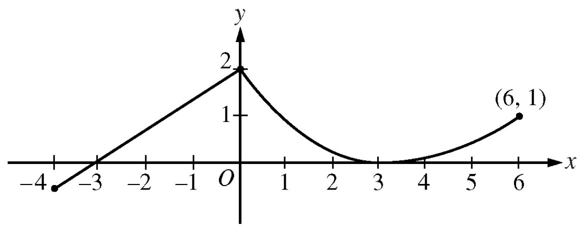
AP® CALCULUS AB
2009 SCORING GUIDELINES (Form B)

Question 3



Graph of f

A continuous function f is defined on the closed interval $-4 \le x \le 6$. The graph of f consists of a line segment and a curve that is tangent to the x-axis at $x = 3$, as shown in the figure above. On the interval $0 < x < 6$, the function f is twice differentiable, with $f''(x) > 0$.

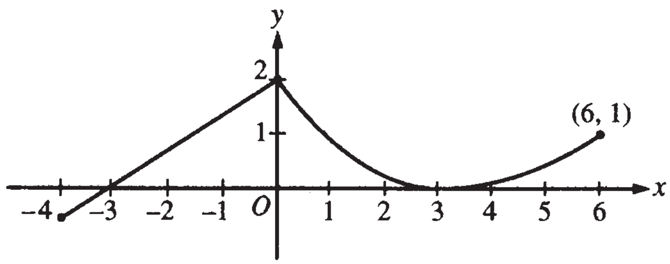
(a) Is f differentiable at $x = 0$? Use the definition of the derivative with one-sided limits to justify your answer.

(b) For how many values of a, $-4 \le a < 6$, is the average rate of change of f on the interval $[a, 6]$ equal to 0? Give a reason for your answer.

(c) Is there a value of a, $-4 \le a < 6$, for which the Mean Value Theorem, applied to the interval $[a, 6]$, guarantees a value c, $a < c < 6$, at which $f'(c) = \frac{1}{3}$? Justify your answer.

(d) The function g is defined by $g(x) = \int_0^x f(t)\,dt$ for $-4 \le x \le 6$. On what intervals contained in $[-4, 6]$ is the graph of g concave up? Explain your reasoning.

(a) $\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \frac{2}{3}$ $\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} < 0$ Since the one-sided limits do not agree, f is not differentiable at $x = 0$.	$2 : \begin{cases} 1 : \text{sets up difference quotient at } x = 0 \\ 1 : \text{answer with justification} \end{cases}$
(b) $\frac{f(6) - f(a)}{6 - a} = 0$ when $f(a) = f(6)$. There are two values of a for which this is true.	$2 : \begin{cases} 1 : \text{expression for average rate of change} \\ 1 : \text{answer with reason} \end{cases}$
(c) Yes, $a = 3$. The function f is differentiable on the interval $3 < x < 6$ and continuous on $3 \le x \le 6$. Also, $\frac{f(6) - f(3)}{6 - 3} = \frac{1 - 0}{6 - 3} = \frac{1}{3}$. By the Mean Value Theorem, there is a value c, $3 < c < 6$, such that $f'(c) = \frac{1}{3}$.	$2 : \begin{cases} 1 : \text{answers "yes" and identifies } a = 3 \\ 1 : \text{justification} \end{cases}$
(d) $g'(x) = f(x)$, $g''(x) = f'(x)$ $g''(x) > 0$ when $f'(x) > 0$ This is true for $-4 < x < 0$ and $3 < x < 6$.	$3 : \begin{cases} 1 : g'(x) = f(x) \\ 1 : \text{considers } g''(x) > 0 \\ 1 : \text{answer} \end{cases}$



Graph of f

Work for problem 3(a)

f is differentiable at $x=0 \iff \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ exists

Now $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x} < 0$ because $f(x)<f(0)$ at $x>0$

$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x} > 0$ because $f(x)<f(0)$ at $x<0$

so $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x} \neq \lim_{x\to 0^-} \frac{f(x)-f(0)}{x}$

$\Rightarrow \lim_{x\to 0} \frac{f(x)-f(0)}{x}$ does not exist $\Rightarrow f$ is not differentiable at $x=0$

Work for problem 3(b)

Average rate of change of f on $[a, 6] = 0$

$\Rightarrow \frac{f(6)-f(a)}{6-a} = 0 \Rightarrow f(a) = f(6)$, $a \neq 6$

$\Rightarrow$ # of possible values of $a = 2$

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Work for problem 3(c)

Choose $a = 3$.

f is differentiable on $(3,6)$ and continuous on $[3,6]$.

By Mean Value Theorem,

$\exists\, c \in [3,6]$ such that $f'(c) = \dfrac{f(6)-f(3)}{6-3}$

$= \dfrac{1-0}{3}$

$= \dfrac{1}{3}$

Thus there is a value of a that satisfies the condition, which is 3.

Work for problem 3(d)

g is concave up on (a,b)

$\Leftrightarrow g''(x) = \dfrac{d}{dx} g'(x) = \dfrac{d}{dx} f(x) = f'(x) \geq 0$ on (a,b)

$\Leftrightarrow f(x)$ is increasing on (a,b)

$f(x)$ is increasing on $(-4,0)$ and $(3,6)$

Thus g is concave up on the intervals $(-4,0)$ and $(3,6)$

END OF PART A OF SECTION II
IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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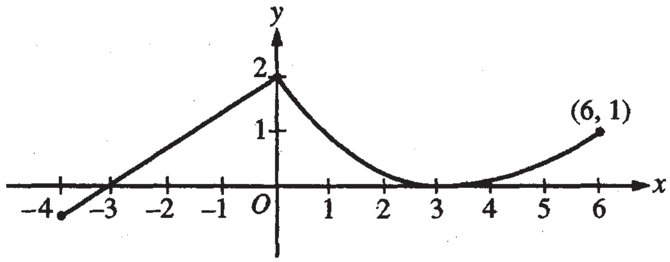

Graph of f

Work for problem 3(a)

No, f is not differentiable.

For f(0) to be differentiable, $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f(x)$.

$f'(0^-) = \frac{2}{3}$, but $f'(0^+) = 1$.

Work for problem 3(b)

There are two values where the average rate of change of f on [a,6] equals 0. Average rate, or slope of the secant line, must equal to zero: Average rate $= \frac{1 - f(a)}{6-a}$.

For the slope to be zero, f(a)=1. There are two x values in the graph with a corresponding y value of 1.

Continue problem 3 on page 9.

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Work for problem 3(c)

Yes, there is. For the Mean value theorem, $f(x)$ must be continuous and differentiable at $[a, b]$. $f(x)$ with endpoint is continuous and differentiable at points from $x=0$ to $x=6$.
Mean Value Theorem states the following:

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{1}{3}$$

$$= \frac{1-f(a)}{6-a} = \frac{1}{3}$$

At $a=3$, $\frac{1-0}{6-3} = \frac{1}{3}$.

Work for problem 3(d)

For $g(x)$ to be concave up, $g''(x) > 0$.

$g''(x) = f'(x) > 0$.

$f'(x) > 0$ on the intervals $[-4, 2]$ and $[3, 6]$.

END OF PART A OF SECTION II
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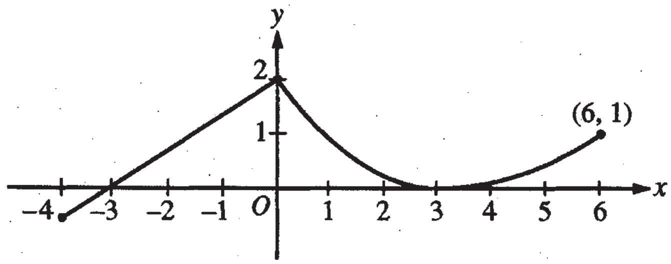

Graph of f

Work for problem 3(a)

No. $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \Rightarrow$ nonexistent.

Work for problem 3(b)

$\frac{\int_a^6 f'(x)\,dx}{6-a} = \frac{f(6)-f(a)}{6-a} = 0.$

$f(6)=f(a)=1$, $a\neq 6$

$\therefore$ 2.

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 3C$_2$

Work for problem 3(c)

$$\frac{f(6)-f(a)}{6-a} = f'(c) = \frac{1}{3}$$

Yes. f is differentiable at all points of $0<x<6$. $-4<x<0$

∴ There exists a "c" ~~tease~~ at which point $f'(c)=\frac{1}{3}$

Work for problem 3(d)

$g''(x)>0$

$g'(x) = f(x)$

$g''(x) = f'(x)$

$f'(x)>0$.

$\therefore -3 \leq x < 0$, $3 < x \leq 6$.

END OF PART A OF SECTION II

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Question 3

Sample: 3A
Score: 9

The student earned all 9 points. Note that in part (c) the student affirms the hypotheses of the Mean Value Theorem, but generally that was not required to earn the second point. In part (d) the student earned the first point implicitly via $g''(x) = f'(x)$.

Sample: 3B
Score: 6

The student earned 6 points: no points in part (a), 2 points in part (b), 2 points in part (c), and 2 points in part (d). In part (a) the student is not working with a difference quotient. The answer is correct, but the justification is insufficient. In part (b) the student's work is correct. In part (c) the student earned both points even though the statement that "there exists a c with $3 < c < 6$" is not included *and* the student may be implying that f is differentiable at $x = 0$. In part (d) the student earned the first 2 points. The student implicitly connects g' and f via $g''(x) = f'(x)$. The student makes the common error of using $f(0)$, instead of 0, as the right-hand endpoint of one of the intervals.

Sample: 3C
Score: 4

The student earned 4 points: no points in part (a), 2 points in part (b), no points in part (c), and 2 points in part (d). In part (a) the student is working with a difference quotient but not at $x = 0$. The answer is correct, but the justification is insufficient. In part (b) the student's work is correct. In part (c) the student never identifies $a = 3$. In part (d) the student earned the first 2 points, but the answer is not correct. Note that students were not penalized for including the endpoints in the correct intervals.