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Let $R$ be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$, as shown in the figure above.

(a) Find the area of $R$.

(b) The horizontal line $y = -2$ splits the region $R$ into two parts. Write, but do not evaluate, an integral expression for the area of the part of $R$ that is below this horizontal line.

(c) The region $R$ is the base of a solid. For this solid, each cross section perpendicular to the $x$-axis is a square. Find the volume of this solid.

(d) The region $R$ models the surface of a small pond. At all points in $R$ at a distance $x$ from the $y$-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

(a) $\sin(\pi x) = x^3 - 4x$ at $x = 0$ and $x = 2$

$$\text{Area} = \int_0^2 (\sin(\pi x) - (x^3 - 4x)) \, dx = 4$$

(b) $x^3 - 4x = -2$ at $r = 0.5391889$ and $s = 1.6751309$

The area of the stated region is

$$\int_r^s (-2 - (x^3 - 4x)) \, dx$$

(c) Volume $= \int_0^2 (\sin(\pi x) - (x^3 - 4x))^2 \, dx = 9.978$

(d) Volume $= \int_0^2 (3 - x)(\sin(\pi x) - (x^3 - 4x)) \, dx = 8.369$ or $8.370$
Concert tickets went on sale at noon \((t = 0)\) and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time \(t\) is modeled by a twice-differentiable function \(L\) for \(0 \leq t \leq 9\). Values of \(L(t)\) at various times \(t\) are shown in the table above.

(a) Use the data in the table to estimate the rate at which the number of people waiting in line was changing at 5:30 P.M. \((t = 5.5)\). Show the computations that lead to your answer. Indicate units of measure.

(b) Use a trapezoidal sum with three subintervals to estimate the average number of people waiting in line during the first 4 hours that tickets were on sale.

(c) For \(0 \leq t \leq 9\), what is the fewest number of times at which \(L'(t)\) must equal 0? Give a reason for your answer.

(d) The rate at which tickets were sold for \(0 \leq t \leq 9\) is modeled by \(r(t) = 550e^{-t/2}\) tickets per hour. Based on the model, how many tickets were sold by 3 P.M. \((t = 3)\) to the nearest whole number?

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\text{t (hours)} & 0 & 1 & 3 & 4 & 7 & 8 & 9 \\
\hline
L(t) \text{ (people)} & 120 & 156 & 176 & 126 & 150 & 80 & 0 \\
\hline
\end{array}
\]

(a) \(L'(5.5) = \frac{L(7) - L(4)}{7 - 4} = \frac{150 - 126}{3} = 8\) people per hour

(b) The average number of people waiting in line during the first 4 hours is approximately

\[
\begin{aligned}
\frac{1}{4} & \left( \frac{L(0) + L(1)}{2} \right)(1 - 0) + \frac{L(1) + L(3)}{2}(3 - 1) + \frac{L(3) + L(4)}{2}(4 - 3) \\
& = 155.25 \text{ people}
\end{aligned}
\]

(c) \(L\) is differentiable on \([0, 9]\) so the Mean Value Theorem implies \(L'(t) > 0\) for some \(t\) in \((1, 3)\) and some \(t\) in \((4, 7)\). Similarly, \(L'(t) < 0\) for some \(t\) in \((3, 4)\) and some \(t\) in \((7, 8)\). Then, since \(L'\) is continuous on \([0, 9]\), the Intermediate Value Theorem implies that \(L'(t) = 0\) for at least three values of \(t\) in \([0, 9]\).

OR

The continuity of \(L\) on \([1, 4]\) implies that \(L\) attains a maximum value there. Since \(L(3) > L(1)\) and \(L(3) > L(4)\), this maximum occurs on \((1, 4)\). Similarly, \(L\) attains a minimum on \((3, 7)\) and a maximum on \((4, 8)\). \(L\) is differentiable, so \(L'(t) = 0\) at each relative extreme point on \((0, 9)\). Therefore \(L'(t) = 0\) for at least three values of \(t\) in \([0, 9]\).

\[\text{Note: There is a function } L \text{ that satisfies the given conditions with } L'(t) = 0 \text{ for exactly three values of } t.\]

(d) \(\int_0^3 r(t) \, dt = 972.784\)

There were approximately 973 tickets sold by 3 P.M.
Let \( h \) be a function having derivatives of all orders for \( x > 0 \). Selected values of \( h \) and its first four derivatives are indicated in the table above. The function \( h \) and these four derivatives are increasing on the interval \( 1 \leq x \leq 3 \).

(a) Write the first-degree Taylor polynomial for \( h \) about \( x = 2 \) and use it to approximate \( h(1.9) \). Is this approximation greater than or less than \( h(1.9) \)? Explain your reasoning.

(b) Write the third-degree Taylor polynomial for \( h \) about \( x = 2 \) and use it to approximate \( h(1.9) \).

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for \( h \) about \( x = 2 \) approximates \( h(1.9) \) with error less than \( 3 \times 10^{-4} \).

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
x & h(x) & h'(x) & h''(x) & h'''(x) & h^{(4)}(x) \\
\hline
1 & 11 & 30 & 42 & 99 & 18 \\
2 & 80 & 128 & \frac{488}{3} & \frac{448}{3} & \frac{584}{9} \\
3 & 317 & 753 & \frac{1383}{4} & \frac{3483}{16} & \frac{1125}{16} \\
\hline
\end{array}
\]

(a) \( P_1(x) = 80 + 128(x - 2) \), so \( h(1.9) = P_1(1.9) = 67.2 \)

\( P_1(1.9) < h(1.9) \) since \( h' \) is increasing on the interval \( 1 \leq x \leq 3 \).

(b) \( P_3(x) = 80 + 128(x - 2) + \frac{488}{6}(x - 2)^2 + \frac{448}{18}(x - 2)^3 \)

\( h(1.9) = P_3(1.9) = 67.988 \)

(c) The fourth derivative of \( h \) is increasing on the interval \( 1 \leq x \leq 3 \), so 

\[
\max_{1 \leq x \leq 3} |h^{(4)}(x)| = \frac{584}{9}.
\]

Therefore, 

\[
|h(1.9) - P_3(1.9)| \leq \frac{584}{9} |1.9 - 2|^4
\]

\[
= \frac{584}{9} \times 1\times 1\times 1\times 1
\]

\[
= 2.7037 \times 10^{-4}
\]

\[
< 3 \times 10^{-4}
\]
A particle moves along the x-axis so that its velocity at time \( t \), for \( 0 \leq t \leq 6 \), is given by a differentiable function \( v \) whose graph is shown above. The velocity is 0 at \( t = 0 \), \( t = 3 \), and \( t = 5 \), and the graph has horizontal tangents at \( t = 1 \) and \( t = 4 \). The areas of the regions bounded by the t-axis and the graph of \( v \) on the intervals \([0, 3]\), \([3, 5]\), and \([5, 6]\) are 8, 3, and 2, respectively. At time \( t = 0 \), the particle is at \( x = -2 \).

(a) For \( 0 \leq t \leq 6 \), find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.

(b) For how many values of \( t \), where \( 0 \leq t \leq 6 \), is the particle at \( x = -8 \)? Explain your reasoning.

(c) On the interval \( 2 < t < 3 \), is the speed of the particle increasing or decreasing? Give a reason for your answer.

(d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

(a) Since \( v(t) < 0 \) for \( 0 < t < 3 \) and \( 5 < t < 6 \), and \( v(t) > 0 \) for \( 3 < t < 5 \), we consider \( t = 3 \) and \( t = 6 \).

\[
x(3) = -2 + \int_{0}^{3} v(t) \, dt = -2 - 8 = -10
\]

\[
x(6) = -2 + \int_{0}^{6} v(t) \, dt = -2 - 8 + 3 - 2 = -9
\]

Therefore, the particle is farthest left at time \( t = 3 \) when its position is \( x(3) = -10 \).

(b) The particle moves continuously and monotonically from \( x(0) = -2 \) to \( x(3) = -10 \). Similarly, the particle moves continuously and monotonically from \( x(3) = -10 \) to \( x(5) = -7 \) and also from \( x(5) = -7 \) to \( x(6) = -9 \).

By the Intermediate Value Theorem, there are three values of \( t \) for which the particle is at \( x(t) = -8 \).

(c) The speed is decreasing on the interval \( 2 < t < 3 \) since on this interval \( v < 0 \) and \( v \) is increasing.

(d) The acceleration is negative on the intervals \( 0 < t < 1 \) and \( 4 < t < 6 \) since velocity is decreasing on these intervals.
The derivative of a function $f$ is given by $f'(x) = (x - 3)e^x$ for $x > 0$, and $f(1) = 7$.

(a) The function $f$ has a critical point at $x = 3$. At this point, does $f$ have a relative minimum, a relative maximum, or neither? Justify your answer.

(b) On what intervals, if any, is the graph of $f$ both decreasing and concave up? Explain your reasoning.

(c) Find the value of $f(3)$.

(a) $f'(x) < 0$ for $0 < x < 3$ and $f'(x) > 0$ for $x > 3$

Therefore, $f$ has a relative minimum at $x = 3$.

(b) $f''(x) = e^x + (x - 3)e^x = (x - 2)e^x$

$f''(x) > 0$ for $x > 2$

$f'(x) < 0$ for $0 < x < 3$

Therefore, the graph of $f$ is both decreasing and concave up on the interval $2 < x < 3$.

(c) $f(3) = f(1) + \int_1^3 f'(x) \, dx = 7 + \int_1^3 (x - 3)e^x \, dx$

$u = x - 3 \quad dv = e^x \, dx$

$du = dx \quad v = e^x$

$f(3) = 7 + (x - 3)e^x \bigg|_1^3 - \int_1^3 e^x \, dx$

$= 7 + ((x - 3)e^x - e^x) \bigg|_1^3$

$= 7 + 3e - e^3$
Consider the logistic differential equation \( \frac{dy}{dt} = \frac{y}{8}(6 - y) \). Let \( y = f(t) \) be the particular solution to the differential equation with \( f(0) = 8 \).

(a) A slope field for this differential equation is given below. Sketch possible solution curves through the points (3, 2) and (0, 8).

(Note: Use the axes provided in the exam booklet.)

(b) Use Euler’s method, starting at \( t = 0 \) with two steps of equal size, to approximate \( f(1) \).

(c) Write the second-degree Taylor polynomial for \( f \) about \( t = 0 \), and use it to approximate \( f(1) \).

(d) What is the range of \( f \) for \( t \geq 0 \) ?

\[ f\left(\frac{1}{2}\right) \approx 8 + (-2)\left(\frac{1}{2}\right) = 7 \]
\[ f(1) = 7 + \left(-\frac{7}{8}\right)\left(\frac{1}{2}\right) = \frac{105}{16} \]

\( f''(0) = \frac{d^2y}{dt^2} \bigg|_{t=0} = \frac{1}{8}(-2)(-2) + \frac{8}{8}(2) = \frac{5}{2} \)

The second-degree Taylor polynomial for \( f \) about \( t = 0 \) is \( P_2(t) = 8 - 2t + \frac{5}{4}t^2 \).

\[ f(1) = P_2(1) = \frac{29}{4} \]

The range of \( f \) for \( t \geq 0 \) is \( 6 < y \leq 8 \).