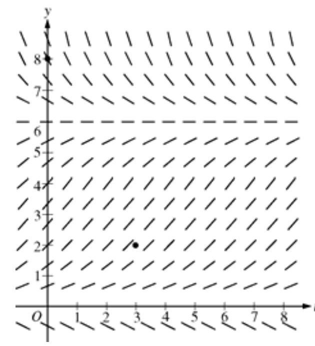
Question 6

Consider the logistic differential equation $\frac{dy}{dt} = \frac{y}{8}(6 - y)$. Let $y = f(t)$ be the particular solution to the differential equation with $f(0) = 8$.

(a) A slope field for this differential equation is given below. Sketch possible solution curves through the points $(3, 2)$ and $(0, 8)$.
(Note: Use the axes provided in the exam booklet.)

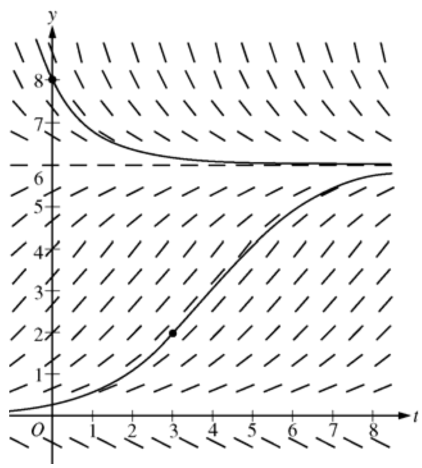


(b) Use Euler's method, starting at $t = 0$ with two steps of equal size, to approximate $f(1)$.

(c) Write the second-degree Taylor polynomial for f about $t = 0$, and use it to approximate $f(1)$.

(d) What is the range of f for $t \geq 0$?

(a)



$2 : \begin{cases} 1\text{: solution curve through } (0,8) \\ 1\text{: solution curve through } (3,2) \end{cases}$

(b) $f\left(\frac{1}{2}\right) \approx 8 + (-2)\left(\frac{1}{2}\right) = 7$

$f(1) \approx 7 + \left(-\frac{7}{8}\right)\left(\frac{1}{2}\right) = \frac{105}{16}$

$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{approximation of } f(1) \end{cases}$

(c) $\frac{d^2y}{dt^2} = \frac{1}{8}\frac{dy}{dt}(6 - y) + \frac{y}{8}\left(-\frac{dy}{dt}\right)$

$f(0) = 8;\ f'(0) = \left.\frac{dy}{dt}\right|_{t=0} = \frac{8}{8}(6 - 8) = -2;$ and

$f''(0) = \left.\frac{d^2y}{dt^2}\right|_{t=0} = \frac{1}{8}(-2)(-2) + \frac{8}{8}(2) = \frac{5}{2}$

The second-degree Taylor polynomial for f about $t = 0$ is $P_2(t) = 8 - 2t + \frac{5}{4}t^2$.

$f(1) \approx P_2(1) = \frac{29}{4}$

$4 : \begin{cases} 2 : \frac{d^2y}{dt^2} \\ 1 : \text{second-degree Taylor polynomial} \\ 1 : \text{approximation of } f(1) \end{cases}$

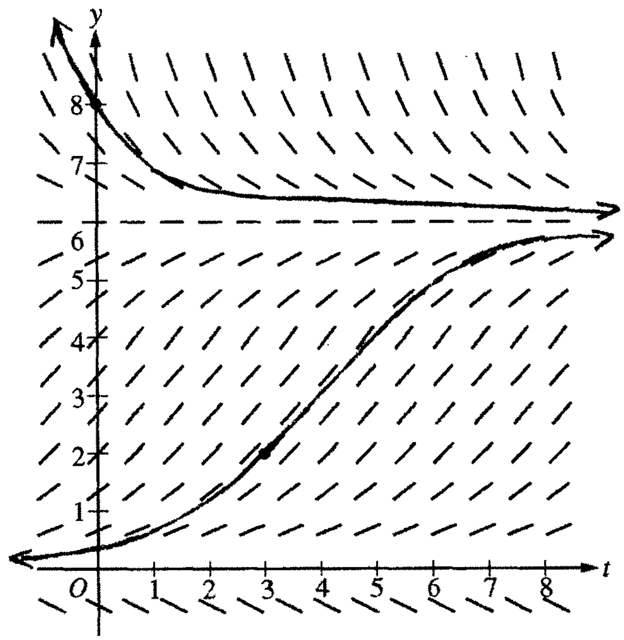
(d) The range of f for $t \geq 0$ is $6 < y \leq 8$.

1 : answer

BC 6A₁

NO CALCULATOR ALLOWED

Work for problem 6(a)



Work for problem 6(b)

$Y_n = y_{n-1} + F(t_{n-1}, y_{n-1})h$

$\frac{dy}{dt} = \frac{y}{8}(6-y)$

$Y_1 = 8 + \left[\frac{8}{8}(6-8)\right](0.5) = 7$

$(0,8)$

$Y_2 = 7 + \left[\frac{7}{8}(6-7)\right](0.5) = 7 - \frac{7}{16} = \frac{105}{16}$

$7 \times 16 = 112$

$112 - 7 = 105$

$f(1) \approx \boxed{\frac{105}{16}}$

Continue problem 6 on page 15.

NO CALCULATOR ALLOWED

Work for problem 6(c)

$$T_n(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$\frac{dy}{dt} = \frac{3}{4}y - \frac{y^2}{8}$ $\quad T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$

$\left.\frac{dy}{dt}\right|_{t=0} = \frac{8}{8}(6-8)$ $\quad T_2(x) = 8 - 2x + \frac{5}{4}x^2$

$= -2$ $\quad T_2(1) = 8 - 2 + \frac{5}{4} = 7\frac{1}{4} = \frac{29}{4}$

$\frac{d^2y}{dt^2} = \frac{3}{4}\frac{dy}{dt} - \frac{2y}{8}\frac{dy}{dt}$ $\quad f(1) \approx \boxed{\frac{29}{4}}$

$\left.\frac{d^2y}{dt^2}\right|_{t=0} = \frac{3}{4}(-2) - 2(-2) = -\frac{3}{2} + 4 = \frac{5}{2}$

Work for problem 6(d)

since $\lim_{t \to \infty} f(t) = 6$ and $f(0) = 8$,

the range for f on $t \geq 0$ is

$\boxed{(6, 8]}$

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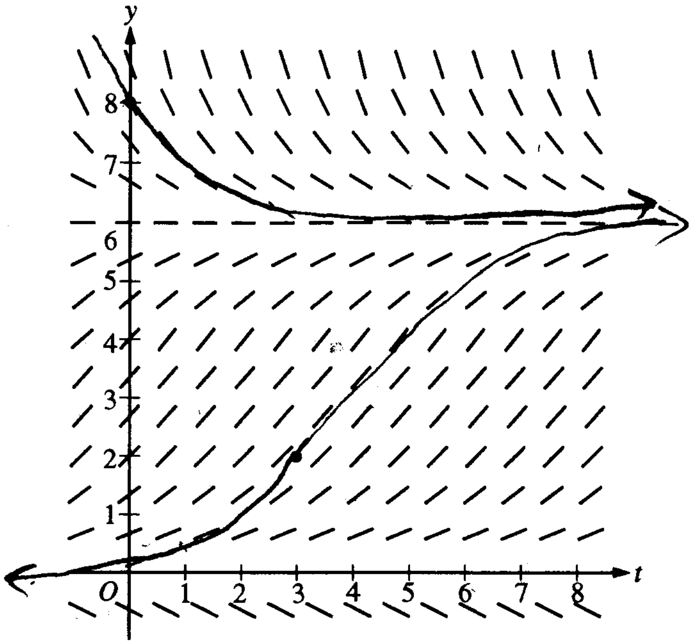
Work for problem 6(a)



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Work for problem 6(b)

t	$f(t)$
0	8
.5	7
1	$7-\frac{9}{16}$

$\frac{dy}{.5} = \frac{8}{8}(6-8)$

$dy = -2(.5)$

$dy = -1$

$\frac{dy}{.5} = \frac{9}{8}(6-7)$

$dy = \frac{-9}{8}\left(\frac{1}{2}\right)$

$dy = \frac{-9}{16}$

$\boxed{f(1) \approx \frac{103}{16}}$

Continue problem 6 on page 15.

6B2

Work for problem 6(c)

$P(t) = \frac{f(0)(x)^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!}$

$P(t) = 8 + -2(x) + \frac{-3}{2}\left(\frac{x^2}{2}\right)$

$P(t) = 8 - 2x - \frac{3x^2}{4}$

$f(1) \approx P(1) = 8 - 2(1) - \frac{3(1)^2}{4}$

$f(1) \approx 8 - 2 - \frac{3}{4}$

$f(1) \approx \frac{32}{4} - \frac{8}{4} - \frac{3}{4}$

$\boxed{f(1) \approx \frac{21}{4}}$

$f(0) = 8$
$f'(0) = -2$

$f''(t) = \frac{d}{dt}\left[\frac{1}{8}y(6-y)\right]$

$f''(t) = \frac{1}{8}\frac{d}{dt}\left[6y - y^2\right]$

$f''(t) = \frac{1}{8}\left(6\frac{dy}{dt} - 2y\frac{dy}{dt}\right)$

$f''(0) = \frac{1}{8}\left[6(f'(0)) - 2(0)(f'(0))\right]$

$f''(0) = \frac{-12}{8} = -\frac{3}{2}$

Work for problem 6(d)

$\frac{dy}{dt} = \frac{y}{8}(6-y)$

$\frac{dy}{dt} = \frac{1}{8}(6y - y^2)$

$\int \frac{8}{(6-y)y}\,dy = \int dt$

$\frac{8}{(6-y)y} = \frac{A}{6-y} + \frac{B}{y}$

$8 = Ay - By + 6B$

$A - B = 0 \quad 6B = 8$

$A - \frac{4}{3} = 0 \quad B = \frac{4}{3}$

$A = \frac{4}{3}$

$\frac{4}{3}\left[\int \frac{1}{6-y}\,dy + \int \frac{1}{y}\,dy\right] = t + C$

$\frac{4}{3}\left[\ln|6-y| + \ln|y|\right] = \frac{3}{4}(t + C)$

$e^{\left[\ln|6y-y^2|\right]} = e^{\frac{3}{4}t + C_1}$

$-y^2 + 6y = C_1 e^{-\frac{3}{4}t}$

$(y-3)^2 = C_1 e^{-\frac{3}{4}t} + 9$

$y = \sqrt{C_1 e^{-\frac{3}{4}t} + 9} + 3$

$\lim_{t\to\infty} \sqrt{C_1 e^{-\frac{3}{4}t} + 9} + 3$

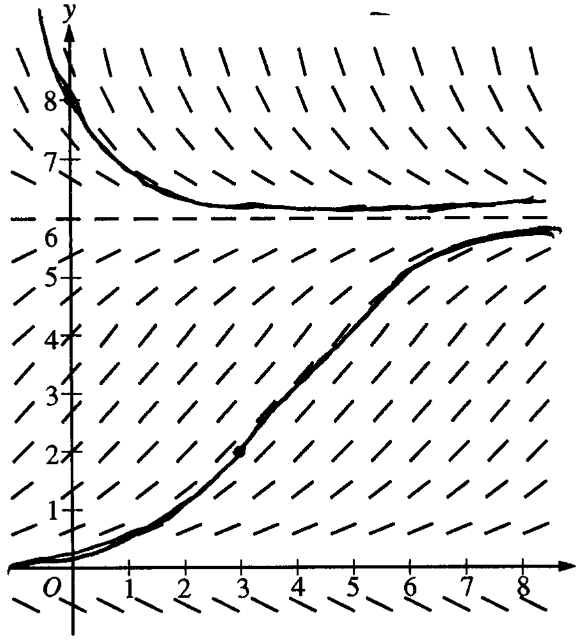
$\lim_{t\to\infty} \sqrt{9} + 3 = 6$

range $= [8, 6]$

 6C1

NO CALCULATOR ALLOWED

Work for problem 6(a)



Work for problem 6(b)

$(0,8) \qquad (0.5,\ 0.5\cdot\frac{dy}{dt}+8)$

$\frac{dy}{dt}$ @ $(0,8) = \frac{8}{8}(6-8) = -2$

$(0.5, -1+8) = (0.5, 7)$

$\frac{dy}{dt}$ @ $(0.5, 7) = \frac{7}{8}(6-7) = -\frac{7}{8}$

$(1, 7+0.5\cdot\frac{-7}{8}) = (1, \frac{105}{16})$

$f(1) \approx \frac{105}{16} = 6.5625$

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Work for problem 6(c)

$f(c) + f'(c)(\cancel{t}-c) + \frac{f''(c)(\cancel{t}-c)^2}{2}$ @ $c=0$

$f(0) + f'(0)(t) + \frac{f''(0)\cancel{t}^2}{2}$

$f''(\cancel{y}) = \frac{d}{dt}\frac{y}{8}(6-y) = \frac{-y}{8} + \frac{6-y}{8} = \frac{3-y}{4}$

$8 - 2x - \frac{3-y}{4} = y$

$6 - \frac{3-y}{4} = y$

$24 - 3 + y = 4y$

$21 = 3y$

$y = 7 = f(1)$

Work for problem 6(d)

Asymptotes at: $\frac{y}{8}(6-y)=0$, $y=0$, $y=6$. All other values are achievable.

Thus the range is $\{y \mid y \in \mathbb{R}, y \neq 0, 6\}$

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Question 6

Overview

This problem presented students with a logistic differential equation and the initial value $f(0) = 8$ of a particular solution $y = f(t)$. In part (a) a slope field for the differential equation was given, and students were asked to sketch solution curves through two specified points. In particular, students should have demonstrated appropriate behavior for these curves for $t \geq 0$, especially with regard to the horizontal lines $y = 0$ and $y = 6$. For part (b) students needed to use the given initial value for the solution f and a two-step Euler's method to approximate $f(1)$. In part (c) students were directed to find the second-degree Taylor polynomial for f about $t = 0$ and use it to approximate $f(1)$. Part (d) asked for the range of the particular solution $y = f(t)$.

Sample: 6A
Score: 9

The student earned all 9 points. Note that the reason provided by the student in part (d) was not required.

Sample: 6B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), 3 points in part (c), and no points in part (d). In part (a) the student presents a solution curve through the point $(0, 8)$ and a solution curve through the point $(3, 2)$. Both curves presented have the general form required relative to the slope field, and the student earned both points. In part (b) the student earned the first point for correctly demonstrating two steps of Euler's method using the initial value $f(0) = 8$ and a step size of $\Delta t = \frac{1}{2}$ to approximate $f(1)$. The student makes a copy error in the second step of Euler's method when $y = 9$ is used instead of $y = 7$ in the first term of $\frac{dy}{dt}$; therefore, the response did not earn the second point. In part (c) the first 2 points were earned when the student correctly finds $\frac{d^2y}{dt^2}$. The student goes on to evaluate $f'(0)$ and $f''(0)$. The student earned the third point when using the declared values of $f'(0)$ and $f''(0)$ along with $f(0) = 8$ to correctly construct a second-degree Taylor polynomial for f about $t = 0$. The student makes an arithmetic error in the calculation of $f''(0)$ so did not earn the fourth point. In part (d) the student presents an incorrect range for f so did not earn the point.

Sample: 6C
Score: 4

The student earned 4 points: 2 points in part (a), 2 points in part (b), no points in part (c), and no points in part (d). In part (a) the student presents a solution curve through the point $(0, 8)$ and a solution curve through the point $(3, 2)$. Both curves presented have the general form required relative to the slope field, and the student earned both points. In part (b) the student correctly demonstrates two steps of Euler's method using the initial value $f(0) = 8$ and a step size of $\Delta t = \frac{1}{2}$ to approximate $f(1)$. The student earned both points. Note that the student was not required to provide a decimal presentation of the approximation to $f(1)$ but does so correctly. In part (c) the student

Question 6 (continued)

does not use the chain rule to find $\frac{d^2y}{dt^2}$ and did not earn the points. The student provides a formula for the second-degree Taylor polynomial but does not calculate values of $f'(0)$ and $f''(0)$ to construct a Taylor polynomial that could be used to approximate $f(1)$. The student was not eligible for the third point and, as a result, was not eligible for the fourth point. In part (d) the student presents an incorrect range for f so did not earn the point.