AP® CALCULUS BC
2008 SCORING GUIDELINES

Question 5

The derivative of a function f is given by $f'(x) = (x-3)e^x$ for $x > 0$, and $f(1) = 7$.

(a) The function f has a critical point at $x = 3$. At this point, does f have a relative minimum, a relative maximum, or neither? Justify your answer.

(b) On what intervals, if any, is the graph of f both decreasing and concave up? Explain your reasoning.

(c) Find the value of $f(3)$.

(a) $f'(x) < 0$ for $0 < x < 3$ and $f'(x) > 0$ for $x > 3$

Therefore, f has a relative minimum at $x = 3$.

$2 : \begin{cases} 1\text{: minimum at } x = 3 \\ 1\text{: justification} \end{cases}$

(b) $f''(x) = e^x + (x-3)e^x = (x-2)e^x$

$f''(x) > 0$ for $x > 2$

$f'(x) < 0$ for $0 < x < 3$

Therefore, the graph of f is both decreasing and concave up on the interval $2 < x < 3$.

$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with reason} \end{cases}$

(c) $f(3) = f(1) + \int_1^3 f'(x)\,dx = 7 + \int_1^3 (x-3)e^x\,dx$

$$u = x - 3 \quad dv = e^x\,dx$$
$$du = dx \quad v = e^x$$

$$f(3) = 7 + (x-3)e^x\Big|_1^3 - \int_1^3 e^x\,dx$$
$$= 7 + \left((x-3)e^x - e^x\right)\Big|_1^3$$
$$= 7 + 3e - e^3$$

$4 : \begin{cases} 1 : \text{uses initial condition} \\ 2 : \text{integration by parts} \\ 1 : \text{answer} \end{cases}$

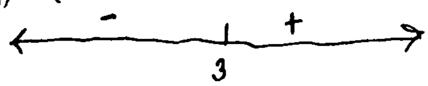
BC

NO CALCULATOR ALLOWED

5A$_1$

Work for problem 5(a)

$f'(x) = (x-3)e^x$



$f'(0) = -3(e^0) = -3$

$f'(5) = 2e^5$

f has a rel minimum b/c on the interval ~~of~~ of X from (0,3) the function is decreasing, or $f'(x)<0$, and on the interval of x from $(3,\infty)$ the function is increasing or $f'(x)>0$, which means there is a relative minimum when $x=3$.

Do not write beyond this border.

Do not write beyond this border.

Work for problem 5(b)

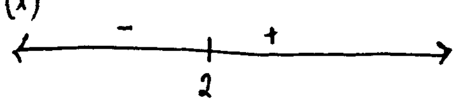
$f''(x) = e^x + (x-3)e^x$

$0 = e^x(1+x-3)$

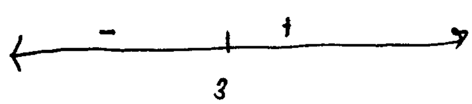
$0 = e^x(x-2)$

critical value: $x=2$

$f''(x)$



$f'(x)$



On the x interval ~~(...)~~ (2,3) the graph is both decreasing and concave up b/c $f'(x)<0$ on $x\in(0,3)$ and $f''(x)>0$ when $x\in(2,\infty)$ and therefore the graph of f is concave up + decreasing when $x\in(2,3)$.

Continue problem 5 on page 13.

NO CALCULATOR ALLOWED

5A2

Work for problem 5(c)

~~f(1)~~

$f(1)=7$

$f(3)>7$ b/c $f(x)$ is increasing from $x\in(1,\infty)$.

$f(x)=\int(x-3)e^x\,dx$ $\qquad dv=e^x \quad u=x-3$

$\qquad v=e^x \quad du=1$

$\int u\,dv=uv-\int v\,du$

$= (x-3)e^x-\int e^x\,dx \rightarrow e^x x-3e^x-e^x$

$e^x x-4e^x$

$=\boxed{(x-3)e^x-e^x+C} \Rightarrow e^x(x-4)+C$

$\cancel{2e^1-e^1+C=7}$ $\qquad e^1(-3)+C=7$

$\cancel{e+e=7}$ $\qquad -3e+C=7$

$\cancel{C=7e}$ $\qquad 7+3e=C$

$e^x(x-3)-e^x+7-e$ $\qquad \boxed{(x-3)e^x-e^x+7+3e=f(x)}$

$e^3(3-3)-e^3+7-e$ $\qquad (0)e^3-e^3+7+3e=f(3)$

$0+7-e-e^3=f(3)$ $\qquad \boxed{7+3e-e^3=f(3)}$

$\cancel{\boxed{7-e-e^3}=f(3)}$

GO ON TO THE NEXT PAGE.

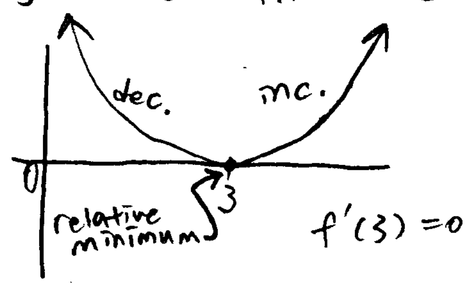
Do not write beyond this border.

NO CALCULATOR ALLOWED

Work for problem 5(a)

The function has a relative minimum because $f'(x) < 0$ on $0 < x < 3$ and $f'(x) > 0$ on $3 < x$. This means that the function f has a relative minimum since f is decreasing on $0 < x < 3$ and increasing on $3 < x$.

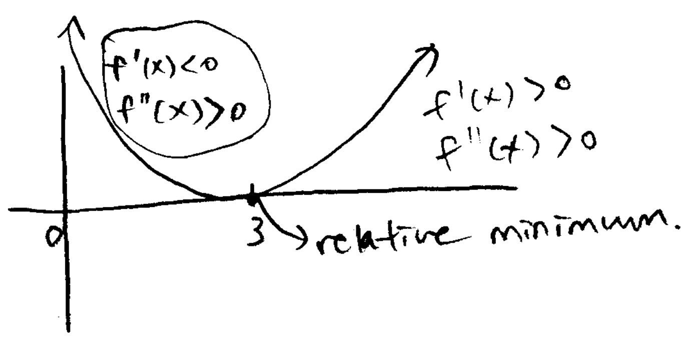
ex)



Work for problem 5(b)

The graph of f is both decreasing and concave up on $0 < x < 3$ because $f'(x) < 0$ and $f''(x) > 0$ on $0 < x < 3$.

ex)



Do not write beyond this border.

5B2

5 5 5 5 5

NO CALCULATOR ALLOWED

Work for problem 5(c)

$f(x) = \int (x-3)e^x\,dx$

let $u = x-3 \qquad dv = e^x\,dx$

$du = dx \qquad v = e^x$

$f(x) = e^x(x-3) - \int e^x\,dx + C$

$= e^x(x-3) - e^x + C$

$f(1) = e(-2) - e + C = 1$

$C = 1 + 3e$

$f(3) = e^3\cancel{(3-3)}^{0} - e^3 + (1+3e)$

$= \boxed{1 + 3e - e^3}$

Do not write beyond this border.

Do not write beyond this border.

NO CALCULATOR ALLOWED

5C1

Work for problem 5(a)

$- \mid +$

$\overline{\;\;3\;\;}$

$\searrow \quad \nearrow$

MIN

AT $x=3$ There is A RELATIVE MIN BECAUSE THE FUNCTION goes From Decreasing TO Increasing.

Work for problem 5(b)

Dec $(0,3)$

$y'' = (x-3)(e^x) + e^x$

$- \mid +$

$\overline{\cap \; 3 \; \cup}$

The Graph is Never Decreasing AND Concave up

Continue problem 5 on page 13.

NO CALCULATOR ALLOWED

5C2

Work for problem 5(c)

$\int (x-3)e^x \, dx$

$U = x-3 \quad x = U+3 \quad dU = dx$

$\int_1^7 (U) e^{U+3} \, dU$

$\frac{1}{2}U^2 e^{U+3} \Big]_1^7$

$\frac{1}{2} 49 e^{10} - \frac{1}{2}(1)e^4 =$

$\underline{\frac{49}{2}e^{10} - \frac{1}{2}e^4}$

Do not write beyond this border.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Question 5

Overview

In this problem, students were told that a function f has derivative $f'(x) = (x - 3)e^x$ and that $f(1) = 7.$ In part (a) students needed to determine with justification the character of the critical point for f at $x = 3.$ Part (b) asked for the intervals on which the graph of f is both decreasing and concave up. For this, students had to apply the product rule to obtain $f''(x).$ In part (c) students needed to solve the initial value problem to find $f(3),$ employing integration by parts along the way.

Sample: 5A
Score: 9

The student earned all 9 points. In part (a) the student correctly identifies $x = 3$ as the relative minimum and gives a correct justification. It is important to note that had the student only written that the function changes from decreasing to increasing at $x = 3,$ the response would not have earned the justification point. The student makes the connection between the derivative being negative and then positive on either side of the critical point to earn the justification point. In part (b) the student has a correct second derivative using the product rule. The student also finds the correct interval and explains the reason for the choice. In part (c) the student uses integration by parts to find the correct antiderivative. The student uses the initial condition and gives a correct answer.

Sample: 5B
Score: 6

The student earned 6 points: 2 points in part (a), no points in part (b), and 4 points in part (c). In part (a) the student correctly identifies $x = 3$ as the relative minimum and gives a correct justification. In part (b) the student does not have a correct second derivative or the correct answer. In part (c) the student uses integration by parts to find the correct antiderivative. The student uses the initial condition and gives a correct answer.

Sample: 5C
Score: 3

The student earned 3 points: 1 point in part (a), 2 points in part (b), and no points in part (c). In part (a) the student correctly identifies $x = 3$ as the relative minimum but does not give a correct justification. In part (b) the student has a correct second derivative but does not give a correct answer. In part (c) the student chooses to integrate using substitution instead of integration by parts so did not earn any points.