Question 3

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

(a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

(b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

(a) $P_1(x) = 80 + 128(x - 2)$, so $h(1.9) \approx P_1(1.9) = 67.2$

$P_1(1.9) < h(1.9)$ since h' is increasing on the interval $1 \le x \le 3$.

$$4 : \begin{cases} 2 : P_1(x) \\ 1 : P_1(1.9) \\ 1 : P_1(1.9) < h(1.9) \text{ with reason} \end{cases}$$

(b) $P_3(x) = 80 + 128(x - 2) + \frac{488}{6}(x - 2)^2 + \frac{448}{18}(x - 2)^3$

$h(1.9) \approx P_3(1.9) = 67.988$

$$3 : \begin{cases} 2 : P_3(x) \\ 1 : P_3(1.9) \end{cases}$$

(c) The fourth derivative of h is increasing on the interval $1 \le x \le 3$, so $\max\limits_{1.9 \le x \le 2} \left| h^{(4)}(x) \right| = \frac{584}{9}$.

Therefore, $\left| h(1.9) - P_3(1.9) \right| \le \frac{584}{9} \frac{|1.9 - 2|^4}{4!}$

$$= 2.7037 \times 10^{-4}$$
$$< 3 \times 10^{-4}$$

$$2 : \begin{cases} 1 : \text{form of Lagrange error estimate} \\ 1 : \text{reasoning} \end{cases}$$

BC
3A, 1

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
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Work for problem 3(a)

$$P(x) = h(2) + \frac{h'(2)}{1!}(x-2)^1$$

$$P(x) = 80 + 128(x-2)$$

$$h(1.9) \approx P(1.9) = 67.2000$$

$$h(1.9) \approx 67.2000$$

This approximation is less than $h(1.9)$ because since h and h' are increasing, h is concave up and the linear approximation line lies below graph of h $\therefore$ the approximation is less than $h(1.9)$

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Continue problem 3 on page 9.

Work for problem 3(b)

$$P(x) = h(2) + \frac{h'(2)}{1!}(x-2)^1 + \frac{h''(2)}{2!}(x-2)^2 + \frac{h'''(2)}{3!}(x-2)^3$$

$$P(x) = 80 + 128(x-2)^1 + \frac{\frac{488}{3}}{2}(x-2)^2 + \frac{\frac{448}{3}}{6}(x-2)^3$$

$$h(1.9) \approx P(1.9) = 67.9884$$

$$h(1.9) \approx 67.9884$$

Work for problem 3(c)

$$|\text{error}| < 3 \times 10^{-4} \qquad 3 \times 10^{-4} = .0003$$

$$\text{Lagrange Error} = \left| \frac{\text{Max deriv}}{(n+1)!}(x-c)^{n+1} \right| < 3 \times 10^{-4}$$

$$\left| \frac{h^4(x)}{4!}(1.9-2)^4 \right| < 3 \times 10^{-4}$$

$$\left| \frac{\frac{584}{9}}{4!}(1.9-2)^4 \right| < 3 \times 10^{-4}$$

$$.00027 < .0003$$

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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0th d. 1st d. 2nd ds 3rd ds 4th d.

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
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Work for problem 3(a)

$$T(x) = \frac{f^{(0)}(x-a)^0}{0!} + \frac{f^{(1)}(x-a)^1}{1!}$$

$$T_1(x) = 80 + 128(x-2)$$

$$h(1.9) \approx 67.2$$

The first-degree Taylor approximation is an underapproximation of $h(1.9)$ because the double-derivative of h, $h''(x)$, at 2 is positive ∴ the function is concave up around $x=2$ and the values will be increasing.

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Continue problem 3 on page 9.

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Work for problem 3(b)

$$T_3^{(2)} = \frac{f^{(0)}(x-a)^0}{0!} + \frac{f^{(1)}(x-a)^1}{1!} + \frac{f^{(2)}(x-a)^2}{2!} + \frac{f^{(3)}(x-a)^3}{3!}$$

$$T_3^{(2)} = 80 + 128(x-2) + \frac{488}{3}\cdot\frac{1}{2}(x-2)^2 + \frac{448}{3}\cdot\frac{1}{6}(x-2)^3$$

$$T_3^{(2)} = 80 + 128(x-2) + \frac{244(x-2)^2}{3} + \frac{224(x-2)^3}{9}$$

$h(1.9) \approx 67.986$

Work for problem 3(c)

Langrange error bound for n^{th} degree Taylor series is the $(n+1)^{th}$ term

4th term is $\dfrac{f^{(4)}(x-a)^4}{4!} = \dfrac{584(x-2)^4}{9\cdot 4!} = \dfrac{73}{27}(x-2)^4$

plug in 1.9 $= 2.704 \times 10^{-4}$

yes, the langrange error bound for the third degree Taylor polynomial is $< 3\times 10^{-4}$ and thus the error is $< 3\times 10^{-4}$.

END OF PART A OF SECTION II

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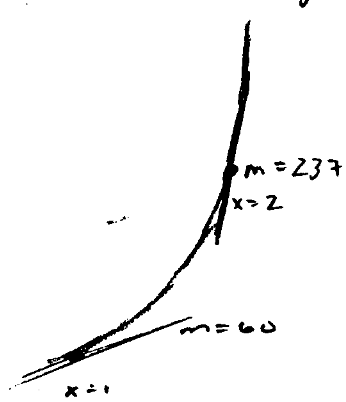
Work for problem 3(a)

a) $h(x) = 80 + 128(x-2)$

$h(1.9) = 80 + 128(1.9-2) = 67.200$

The approximation is less than the actual value because when comparing the slopes between 1 and 2, and, 2 and 3, they are 69 and 237 respectively. Meaning that the slopes are becoming faster, and steeper and because h is increasing, the tangent lines are under the concave up graph



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Continue problem 3 on page

$3C_2$

Work for problem 3(b)

$$h(x) = 80 + 128(x-2) + \frac{\frac{488}{3}}{2!}(x-2)^2 + \frac{\frac{448}{3}}{3!}(x-2)^3$$

$$= 80 + 128(x-2) + \frac{244}{3}(x-2)^2 + \frac{224}{9}(x-2)^3$$

$$h(1.9) = 80 + 128(1.9-2) + \frac{244}{3}(1.9-2)^2 + \frac{244}{9}(1.9-2)^3$$

$$= 67.986$$

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Work for problem 3(c)

approximate $h(1.9) = 67.986$

. error less than $.0003 = 3 \cdot 10^{-4}$

~~Hde~~

H has an error less than 3×10^{-4}.

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END OF PART A OF SECTION II

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Question 3

Overview

This problem presented students with a table of values for a function h and its derivatives up to the fourth order at $x = 1$, $x = 2$, and $x = 3$. The question stated that h has derivatives of all orders, and that the first four derivatives are increasing on $1 \le x \le 3$. Part (a) asked for the first-degree Taylor polynomial about $x = 2$ and the approximation for $h(1.9)$ given by this polynomial. Students needed to use the given information to determine that the graph of h is concave up between $x = 1.9$ and $x = 2$ to conclude that this approximation is less than the value of $h(1.9)$. Part (b) asked for the third-degree Taylor polynomial about $x = 2$ and the approximation for $h(1.9)$ given by this polynomial. In part (c) students were expected to observe that the given conditions imply that $\left|h^{(4)}(x)\right|$ is bounded above by $h^{(4)}(2)$ on $1.9 \le x \le 2$ and apply this to the Lagrange error bound to show that the estimate in part (b) has error less than 3×10^{-4}.

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

The student earned 6 points: 3 points in part (a), 2 points in part (b), and 1 point in part (c). In part (a) the student gives a correct linear polynomial and a correct evaluation of $P_1(1.9)$. The student earned the first 3 points. The student states that the approximation is an underapproximation but only mentions that $h''(2) > 0$. An argument at a point was not sufficient to earn the last point. In part (b) the student's polynomial is correct and earned both points. The student incorrectly evaluates $P_3(1.9)$ so did not earn the last point. In part (c) the student has the proper form for the Lagrange error term and earned the first point. The student never bounds the fourth derivative so did not earn the last point.

Sample: 3C
Score: 4

The student earned 4 points: 2 points in part (a), 2 points in part (b), and no points in part (c). In part (a) the student gives a linear polynomial that is correctly centered but equates the polynomial to $h(x)$ and earned only 1 point. The student correctly evaluates $P_1(1.9)$ and earned 1 point. The student states that the "approximation is less than the actual value" but provides an argument that is not sufficient to earn the last point. In part (b) the student's polynomial is correct and earned both points. The student incorrectly evaluates $P_3(1.9)$ so did not earn the last point. The student was not penalized a second time for equating $h(x)$ to a polynomial. In part (c) the student does not have the proper form for the Lagrange error term so did not earn either point.